MULTIPLE POSITIVE SOLUTIONS FOR A FOURTH-ORDER NONLINEAR EIGENVALUE PROBLEM

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Abstract

In this paper, by using the Guo-Krasnoselskii’s fixed-point theorem, we establish the existence and multiplicity of positive solutions for a fourth-order nonlinear eigenvalue problem. The corresponding examples are also included to demonstrate the results we obtained.

Keywords positive solutions; eigenvalue problem; fixed point

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1 Introduction

In the past decades, an increasing interest in the existence and multiplicity of positive solutions for boundary value problems has been evolved by using some fixed-point theorems, for example, by the Krasnoselskii’s fixed-point theorem, Ma [1] and Li [2] respectively established the existence and multiplicity of positive solutions for some fourth-order boundary value problems. Zhong [3] established the existence of at least one positive solution for the following four-point boundary value problem

\[\begin{align*}
& y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 \leq t \leq 1, \\
& y(0) = y(1) = 0, \\
& ay''(\xi_1) - by'''(\xi_1) = 0, \quad cy''(\xi_2) + dy'''(\xi_2) = 0.
\end{align*}\]

In 2015, Wu [4] obtained some new results on the existence of at least one positive solution for the following fourth-order three-point nonlinear eigenvalue problem

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Preliminaries

We need. Throughout this paper, we make the following assumptions:

In Section 3, we state and prove our main results on the existence and multiplicity of positive solutions for the following fourth-order three-point eigenvalue problem

where \( \lambda \) is a positive parameter, \( 0 < \frac{1}{4} < \xi < \frac{2}{3} < 1 \), \( a, b, c, d \) are nonnegative constants satisfying \( ad + bc + ac > 0 \), \( b - a \xi \geq 0 \), \( h(t) \in C([0, 1], f \in C([0, 1] \times [0, +\infty) \times (0, 0], [0, +\infty)) \).

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we state and prove our main results on the existence and multiplicity of positive solutions for (1.1). At the same time, the corresponding examples are also included to demonstrate the results we obtained.

2 Preliminaries

For convenience, we first state some definitions and preliminary results which we need. Throughout this paper, we make the following assumptions:

(H1) \( f \in C([0, 1] \times [0, +\infty) \times (0, 0], [0, +\infty)) \) is continuous;

(H2) \( h(t) \in C([0, 1], h(t) \leq 0 \) for all \( t \in [0, \xi] \), \( h(t) \geq 0 \) for all \( t \in [\xi, 1] \), where \( 0 < \frac{1}{4} < \xi < \frac{2}{3} < 1 \); and \( h(t) \neq 0 \) for any subinterval of \( [0, 1] \).

Denote

and

\[
A = \int_{\xi}^{1} G_2(s, s) h(s) ds,
\]

\[
B = \min \left\{ \frac{a}{\Delta} \left( \frac{1}{4} + \frac{3\xi}{4} \right) \left( \frac{c}{2} + d \right) \right\} \frac{3}{1 + \frac{3\xi}{4}} h(s) ds,
\]

\[
\frac{a}{2\Delta} \left( \frac{c}{4} + d \right) \frac{3}{1 + \frac{3\xi}{4}} h(s) ds.
\]
where $\Delta = ad + bc + ac(1 - \xi) > 0$.

Our main results in this paper mainly depend on the following Guo-Krasnoselskii’s fixed-point theorem.

**Theorem 2.1** [9] Let $E$ be a Banach space, $K \subset E$ be a cone in $E$. Assume that $\Omega_1$ and $\Omega_2$ are bounded open subsets of $E$ with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is a completely continuous operator such that either

(i) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial \Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial \Omega_2$; or

(ii) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial \Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial \Omega_2$

holds, then $T$ has a fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$.

Let $C[0, 1]$ be endowed with the maximum norm

$$
\|u\| = \max_{t \in [0, 1]} |u(t)|,
$$

and $C^2[0, 1]$ be endowed with the norm

$$
\|u\|_2 = \|u\| + \|u''\| = \max_{t \in [0, 1]} |u(t)| + \max_{t \in [0, 1]} |u''(t)|.
$$

Let $G_1(t, s)$ and $G_2(t, s)$ be the Green’s functions of the following boundary value problems

$$
\begin{cases}
- u'' = 0, & t \in (0, 1), \\
u(0) = u(1) = 0,
\end{cases}
$$

and

$$
\begin{cases}
- v'' = 0, & t \in (0, 1), \\
av(\xi) - bv'(\xi) = 0, \\
cv(1) + dv'(1) = 0,
\end{cases}
$$

respectively. In particular, $G_1(t, s)$ and $G_2(t, s)$ are given by

$$
G_1(t, s) = \begin{cases}
(1 - s)t, & 0 \leq t \leq s \leq 1, \\
(1 - t)s, & 0 \leq s \leq t \leq 1;
\end{cases}
$$

and

$$
G_2(t, s) = \begin{cases}
\frac{1}{\Delta} (a(t - \xi) + b)(c(1 - s) + d), & 0 \leq t \leq s \leq 1, \xi \leq s \leq 1, \\
\frac{1}{\Delta} (a(s - \xi) + b)(c(1 - t) + d), & 0 \leq s \leq t \leq 1, \xi \leq s \leq 1.
\end{cases}
$$

It is easy to check that

$$
0 \leq G_1(t, s) \leq G_1(s, s), \quad 0 \leq t \leq s \leq 1, \quad (2.3)
$$

$$
G_1(t, s) \geq \frac{1}{4} G_1(s, s), \quad s \in [0, 1], \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad (2.4)
$$

$$
0 \leq G_2(t, s) \leq G_2(s, s), \quad (t, s) \in [0, 1] \times [\xi, 1], \quad (2.5)
$$

$$
G_2(t, s) \geq \frac{1}{4} G_2(s, s), \quad s \in [\xi, 1], \quad t \in \left[\frac{1}{4}, \frac{3\xi}{4}\right], \quad (2.6)
$$

Define a cone $K$ in $C^2[0, 1]$ by
\[ K = \left\{ u \in C^2[0, 1] : u \geq 0, \ u'' \leq 0, \ \min_{t \in [1/4, 3/4]} u(t) \geq \frac{1}{4} \| u \|, \ \min_{t \in [1/4, 3/4]} [-u''(t)] \geq \frac{1}{4} \| u'' \| \right\}. \]  

(2.7)

Define an integral operator \( T : K \rightarrow C^2[0, 1] \) by

\[
(Tu)(t) = \int_0^1 \int_0^1 G_1(t, s)G_2(s, \tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau ds.
\]

(2.8)

Therefore,

\[
(Tu)''(t) = -\int_0^1 G_2(t, \tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau \leq 0,
\]

(2.9)

and

\[
|(Tu)''(t)| = \int_0^1 G_2(t, \tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau.
\]

It is easy to check that

\[
|u(t)| + |u''(t)| \geq \frac{1}{4} \| u \|_2, \quad u \in K, \ t \in \left[ \frac{1}{4} + \frac{3\xi}{4}, \frac{3}{4} \right].
\]

(2.10)

Obviously, \( u(t) \) is a solution for the BVP (1.1) if and only if \( u(t) \) is a fixed point of the operator \( T \).

**Lemma 2.1** Assume that (H1) and (H2) hold. If \( b \geq a\xi \), then \( T : K \rightarrow K \) is completely continuous.

**Proof** Denote

\[
(Tu)(t) = \int_0^1 \int_0^1 G_1(t, s)G_2(s, \tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau ds
\]

\[
= \int_0^1 G_1(t, s)\left[ \int_\xi^s (s-\tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau + \frac{1}{\Delta} \int_\xi^1 (b-a(\xi-s))(c(1-\tau) + d)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau \right] ds
\]

\[
= \int_0^1 G_1(t, s)(Qu)(s)ds,
\]

(2.11)

where

\[
(Qu)(s) = \int_\xi^s (s-\tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau
\]

\[
+ \frac{1}{\Delta} \int_\xi^1 (b-a(\xi-s))(c(1-\tau) + d)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau.
\]

(2.12)

Next, for each \( t \in [0, 1] \), we consider the following two cases:

**Case 1** When \( t \in [0, \xi] \), for any \( u \in K \), from (2.12), (H1), (H2) and \( b \geq a\xi \), we have
\[(Qu)(t) = \int_{t}^{\xi} (t - \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau + \frac{1}{\Delta} \int_{\xi}^{1} (b - a(\xi - t))(c(1 - \tau) + d)\lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \geq 0. \] (2.13)

**Case 2** When \( t \in [\xi, 1] \), for any \( u \in K \), from (2.12), (H1), (H2) and \( b \geq a \xi \), we have

\[(Qu)(t) = \int_{\xi}^{t} (t - \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau + \frac{1}{\Delta} \int_{\xi}^{t} (b - a(\xi - \tau))(c(1 - \tau) + d)\lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \geq 0. \] (2.14)

From (2.13) and (2.14), we get

\[ (Qu)(t) \geq 0 \text{ and } (Tu)''(t) = -(Qu)(t) \leq 0, \quad t \in [0, 1]. \]

Moreover, for any \( u \in K \), from (2.13), (2.14), (H1), (H2), and \( b \geq a \xi \), we have

\[ \|Tu\| = \max_{t \in [0,1]} |(Tu)(t)| \]
\[ = \max_{t \in [0,1]} \left| \int_{0}^{1} \left[ \int_{\xi}^{1} G_{1}(t, s)G_{2}(s, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \right] ds \right| \]
\[ \leq \int_{0}^{1} \left[ \int_{\xi}^{1} G_{1}(s, \tau)G_{2}(s, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \right] ds, \]

and

\[ \| (Tu)'' \| \leq \int_{\xi}^{1} G_{2}(\tau, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau. \]

On the other hand,

\[ \min_{t \in [\frac{1}{4}, \frac{3}{4}]} (Tu)(t) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{0}^{1} \left[ \int_{\xi}^{1} G_{1}(t, s)G_{2}(s, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \right] ds \]
\[ \geq \frac{1}{4} \int_{0}^{1} \left[ \int_{\xi}^{1} G_{1}(s, \tau)G_{2}(s, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \right] ds \geq \frac{1}{4} \| Tu \|, \]
By (2.5) and (2.8), we have
\[ \| Tu \| = \min_{t \in \left[ \frac{1}{4} + \frac{3\delta}{4}, \frac{1}{2} - \frac{\delta}{2} \right]} -(Tu)''(t) = \min_{t \in \left[ \frac{1}{4} + \frac{3\delta}{4}, \frac{1}{2} - \frac{\delta}{2} \right]} (Qu)(t) \]
\[ \geq \frac{1}{4} \int_{\xi}^{1} G_{2}(t, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau \geq \frac{1}{4} \| (Tu)'' \|. \]
Consequently \( T : K \rightarrow K \). Furthermore, it is not difficult to check that the operator \( T \) is completely continuous by Arzela-Ascoli theorem. This completes the proof.

Lemma 2.2 Suppose that (H1) and (H2) hold. If \( b \geq \alpha \xi \), then for the operator \( T : K \rightarrow K \), the following conclusions hold:

(i) If \( \frac{f_{0}}{\alpha} < \frac{6}{3\lambda A} \), \( r > 0 \) is small enough, then \( \| Tu \| _{2} \leq \| u \| _{2} \) for any \( u \in K \) with \( \| u \| _{2} = r \);

(ii) If \( \frac{f_{0}}{\alpha} > \frac{4}{3\lambda} \), \( r > 0 \) is small enough, then \( \| Tu \| _{2} \geq \| u \| _{2} \) for any \( u \in K \) with \( \| u \| _{2} = r \);

(iii) If \( \frac{f_{\infty}}{\alpha} < \frac{6}{3\lambda A} \), \( R > 0 \) is sufficiently large, then \( \| Tu \| _{2} \leq \| u \| _{2} \) for any \( u \in K \) with \( \| u \| _{2} = R \);

(iv) If \( \frac{f_{\infty}}{\alpha} > \frac{4}{3\lambda} \), \( R > 0 \) is sufficiently large, then \( \| Tu \| _{2} \geq \| u \| _{2} \) for any \( u \in K \) with \( \| u \| _{2} = R \).

Proof We only prove (iii) and (iv), since the proofs of (i) and (ii) are similar to those of (iii) and (iv) respectively.

(iii) If \( \frac{f_{\infty}}{\alpha} < \frac{6}{3\lambda A} \), we obtain from (2.1) that there exists a number \( R_{0} > 0 \) satisfying \( |u| + |v| \geq R_{0} \), such that \( f(t, u, v) \leq \frac{6}{3\lambda A} (|u| + |v|) \). Let \( R_{1} \gg R_{0} \), then \( \max \{ f(t, u, v) : |u| + |v| \leq R_{0} \} \leq \frac{6}{3\lambda A} R_{1} \). Choose a \( R > R_{1} \) satisfying \( |u| + |v| \leq R \) such that \( f(t, u, v) \leq \frac{6}{3\lambda A} R \).

For any \( t \in [\xi, 1] \), \( u \in K \), \( \| u \| _{2} = R \), we get \( f(t, u, v) \leq \frac{6}{3\lambda A} R = \frac{6}{3\lambda A} \| u \| _{2} \). By (2.3), (2.5) and (2.8), we have
\[ \| Tu \| \leq \int_{0}^{1} \int_{\xi}^{1} G_{1}(s, s) G_{2}(s, \tau) \lambda h(\tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \]
\[ \leq \frac{6}{7\lambda A} \| u \| _{2} \int_{0}^{1} \int_{\xi}^{1} G_{1}(s, s) G_{2}(s, \tau) h(\tau) d\tau ds \]
\[ \leq \frac{6}{7\lambda A} \| u \| _{2} \int_{0}^{1} \int_{\xi}^{1} G_{1}(s, s) G_{2}(\tau, \tau) h(\tau) d\tau ds \]
\[ = \frac{6}{7\lambda A} \| u \| _{2} \int_{\xi}^{1} G_{2}(\tau, \tau) h(\tau) d\tau = \frac{1}{7} \| u \| _{2}. \]
\[(Tu)^{\prime\prime}\| \leq \int_{\xi}^{1} G_2(s, s)\lambda h(s) f(s, u(s), u''(s)) ds \]
\[\leq \frac{6}{17\lambda A} \|u\|_2 \int_{\xi}^{1} G_2(s, s) h(s) ds \leq \frac{6}{7} \|u\|_2.\]

Therefore, \(\|Tu\|_2 = \|Tu\| + \|(Tu)^{\prime\prime}\| \leq \frac{1}{4} \|u\|_2 + \frac{9}{4} \|u\|_2 = \|u\|_2.\)

(iv) If \(f_{\infty} > \frac{1}{4\beta}\), we obtain from (2.2) that there exists a number \(R_0 > 0\), such that \(f(t, u, v) \geq \frac{1}{4\beta}(|u| + |v|)\) for any \(t \in [0, 1]\) and \(|u| + |v| \geq R_0\). For any \(t \in [\frac{1}{4} + \frac{3\varepsilon}{4}, \frac{3}{4}]\), \(u \in K, R \geq 4R_0\), and \(\|u\|_2 = R\), we have \(|u(t)| + |u''(t)| \geq \frac{1}{3} \|u\|_2 \geq R_0\). Thus, \(f(t, u(t), u''(t)) \geq \frac{4}{17\beta}(|u(t)| + |u''(t)|) \geq \frac{1}{17\beta} \|u\|_2\) for \(t \in [\frac{1}{4} + \frac{3\varepsilon}{4}, \frac{3}{4}]\). Thus, we consider the following two cases:

Case 1 For \(0 \leq s \leq t \leq 1\), we have
\[
\left|(Tu)^{\prime\prime}\left(\frac{1}{2}\right)\right| = \int_{\xi}^{1} G_2\left(\frac{1}{2}, s\right)\lambda h(s) f(s, u(s), u''(s)) ds \\
\geq \int_{\frac{1}{4} + \frac{3\varepsilon}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, s\right)\lambda h(s) f(s, u(s), u''(s)) ds \\
\geq \frac{1}{\lambda B} \int_{\frac{1}{4} + \frac{3\varepsilon}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, s\right)\lambda h(s) ds \cdot \|u\|_2 \\
\geq \frac{\lambda}{\lambda B} \left(\frac{1}{4} + \frac{3\varepsilon}{4}\right) \left(\frac{c}{2} + d\right) \int_{\frac{1}{4} + \frac{3\varepsilon}{4}}^{\frac{3}{4}} h(s) ds \cdot \|u\|_2 \geq \|u\|_2.
\]

Case 2 For \(0 \leq t \leq s \leq 1\), we have
\[
\left|(Tu)^{\prime\prime}\left(\frac{1}{2}\right)\right| = \int_{\xi}^{1} G_2\left(\frac{1}{2}, s\right)\lambda h(s) f(s, u(s), u''(s)) ds \\
\geq \int_{\frac{1}{4} + \frac{3\varepsilon}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, s\right)\lambda h(s) f(s, u(s), u''(s)) ds \\
\geq \frac{1}{\lambda B} \int_{\frac{1}{4} + \frac{3\varepsilon}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, s\right)\lambda h(s) ds \cdot \|u\|_2 \\
\geq \frac{1}{B} \left(\frac{c}{4} + d\right) \frac{a}{2} \int_{\frac{1}{4} + \frac{3\varepsilon}{4}}^{\frac{3}{4}} h(s) ds \cdot \|u\|_2 \geq \|u\|_2.
\]

Therefore, \(\|Tu\|_2 \geq \|(Tu)^{\prime\prime}(\frac{1}{2})\| \geq \|u\|_2\). The proof is completed.

3 Main Results

Theorem 3.1 Suppose (H1) and (H2) hold. Furthermore, \(f\) satisfies either
(i) \(f_0 < \frac{6}{7\lambda A}, f_{\infty} > \frac{4}{\beta}\), for any \(\lambda \in \left(\frac{4}{17\beta}, \frac{6}{17\beta}\right);\) or
(ii) \( f_0 > \frac{4}{AB}, \overline{f_\infty} < \frac{6}{7A}, \) for any \( \lambda \in (\frac{4}{BF_0}, \frac{6}{7A_f_\infty}) \), then the BVP (1.1) has at least one positive solution \( u = u(t) \).

**Proof** Let \( E = C^2[0,1], \Omega_1 = \{ u \in E : \|u\|_2 < r \}, \Omega_2 = \{ u \in E : \|u\|_2 < R \} \), where \( 0 < r < R \). By Lemma 2.1, we know that the operator \( T : K \cap (\Omega_2 \setminus \Omega_1) \to K \) is completely continuous, then the condition (i) or (ii) of Lemma 2.2 is satisfied. Applying Theorem 2.1, it follows that \( T \) has a fixed point \( u_0 \in K \cap (\Omega_2 \setminus \Omega_1) \). Thus \( u_0 \) is the solution for the BVP (1.1) and satisfies

\[
0 \leq u_0(t) \leq \|u\|_2 \leq R.
\]

For any \( t \in (0,1) \), take \( \varepsilon \in (0, \frac{1}{2}) \), such that when \( t \in [\varepsilon, 1 - \varepsilon] \),

\[
G_1(t, s) \geq \begin{cases}
(1 - s)\varepsilon, & t \leq s \leq 1, \\
\varepsilon s, & 0 \leq s \leq t.
\end{cases}
\]

Therefore, \( G_1(t, s) \geq \varepsilon s(1 - s) \) for any \( s \in [0,1] \).

By (2.8), we have

\[
u_0(t) = (Tu_0)(t) = \int_0^1 \left[ \int_0^1 G_1(t, s)G_2(s, \tau)\lambda h(\tau)f(\tau, u_0(\tau), u_0''(\tau))\,d\tau \right] \,ds \\
\geq \varepsilon \int_0^1 \left[ \int_0^1 G_1(s, s)G_2(s, \tau)\lambda h(\tau)f(\tau, u_0(\tau), u_0''(\tau))\,d\tau \right] \,ds \\
\geq \varepsilon \|Tu_0\| = \varepsilon \|u_0\| > 0.
\]

That is \( u_0(t) > 0 \). This completes the proof.

If \( f_0 = \lim_{|u|+|v| \to 0} \max_{t \in [0,1]} \frac{|f(t, u, v)|}{|u| + |v|} \) and \( f_\infty = \lim_{|u|+|v| \to \infty} \max_{t \in [0,1]} \frac{|f(t, u, v)|}{|u| + |v|} \)

exist, then \( \overline{f_0} = f_0, \overline{f_\infty} = f_\infty \).

**Corollary 3.1** Suppose (H1) and (H2) hold. Suppose further that \( f \) satisfies either

(i) \( f_0 = 0, f_\infty = \infty \) (superlinear); or (ii) \( f_0 = \infty, f_\infty = 0 \) (sublinear), then the BVP (1.1) has at least one positive solution.

**Example 3.1** Consider the following boundary value problem

\[
\begin{align*}
u''(\frac{1}{2}) - 2u''(\frac{1}{2}) &= 0, \\
u''(1) &= u''(1) + u'''(1) = 0,
\end{align*}
\]

where \( h(t) = t - \frac{1}{2}, f = \frac{1000(|u|+|v|+1)(|u|+|v|)}{1+|u|+|v|}, a = 1, b = 2, c = 2, d = 1, \xi = \frac{1}{2}, \) then \( \overline{f_0} = 1, f_\infty = 1000 \) and \( \Delta = 6, A = \frac{\overline{f_0}}{\Delta}, B = \min\{\frac{3}{10}, \frac{5}{10}\} = \frac{3}{10}. \) Therefore, if
$\frac{512}{375} < \lambda < \frac{3456}{259}$, then Theorem 3.1 (i) guarantees the existence of one positive solution for the BVP (3.1).

In order to discuss the multiplicity of positive solutions for the BVP (1.1), we further assume

(H3) $f(t, u, v) > 0$, for any $t \in [0, 1]$ and $|u| + |v| > 0$.

**Theorem 3.2** Suppose (H1) and (H2) hold. If one of the following two conditions holds:

(i) (H3) holds, $\int_0^1 f \leq \frac{6}{r_{\lambda \alpha}}$ and there exists a positive constant $R_0 > 0$ satisfying $r_0 < R_0 < R$ such that $\lambda \geq \frac{R_0}{m(R_0) R}$, where $r > 0$ is small enough, $R > 0$ is large enough and $m(R) = \min \{ f(t, u, v) : \frac{R}{4} \leq |u| + |v| \leq R, t \in [\frac{1}{4} + \frac{3K}{4}, \frac{3}{4}] \}$;

(ii) $\int_0^1 f \geq \frac{4}{\lambda B}$ and there exists a positive constant $R_0 > 0$ satisfying $r_0 < R_0 < R$ such that $\lambda \leq \frac{6(R_0) R}{m(R_0) A}$, where $r > 0$ is small enough, $R > 0$ is large enough and $M(R) = \max \{ f(t, u, v) : |u| + |v| \leq R, t \in [\xi, 1] \}$.

then the BVP (1.1) has at least two positive solutions.

**Proof** (i) Let $E = C^2[0, 1]$, $\Omega_1 = \{ u \in E : \|u\|_2 < 1 \}$, $\Omega_2 = \{ u \in E : \|u\|_2 < R \}$. By the condition (i) or (ii) of Lemma 2.2, we get $\|Tu\|_2 \leq \|u\|_2$ when $u \in K \cap \partial \Omega_1$ or $u \in K \cap \partial \Omega_2$. Choose $\Omega_3 = \{ u \in E : \|u\|_2 < R_0 \}$ such that $\Omega_1 \subset \Omega_3$, $\overline{\Omega_3} \subset \Omega_2$. By (2.10), we know that $\frac{R_0}{4} \leq |u(t)| + |u''(t)| \leq R_0$ for any $t \in [\frac{1}{4} + \frac{3K}{4}, \frac{3}{4}]$ and $u \in K \cap \partial \Omega_3$. Thus, it follows the following two cases:

**Case 1** For $0 \leq s \leq t \leq 1$, we have

$$
\|Tu\|_2 \geq \left| (Tu)'' \left( \frac{1}{2} \right) \right| = \int_\xi^1 G_2 \left( \frac{1}{2}, s \right) \lambda h(s) f(s, u(s), u''(s)) ds \\
\geq \int_\frac{1}{4}^{\frac{3}{4}} G_2 \left( \frac{1}{2}, s \right) \lambda h(s) f(s, u(s), u''(s)) ds \geq \frac{1}{\lambda B} \int_\frac{1}{4}^{\frac{3}{4}} G_2 \left( \frac{1}{2}, s \right) \lambda h(s) ds \cdot \|u\|_2 \\
\geq \frac{\lambda}{\lambda B} a \left( \frac{1}{4} + \frac{3K}{4} \right) \left( \frac{c}{2} + d \right) \int_\frac{1}{4}^{\frac{3}{4}} h(s) ds \cdot \|u\|_2 = \lambda m(R_0) B \geq R_0 = \|u\|_2.
$$

**Case 2** For $0 \leq t \leq s \leq 1$, we have

$$
\|Tu\|_2 \geq \left| (Tu)'' \left( \frac{1}{2} \right) \right| = \int_\xi^1 G_2 \left( \frac{1}{2}, s \right) \lambda h(s) f(s, u(s), u''(s)) ds \\
\geq \int_\frac{1}{4}^{\frac{3}{4}} G_2 \left( \frac{1}{2}, s \right) \lambda h(s) f(s, u(s), u''(s)) ds \geq \frac{1}{\lambda B} \int_\frac{1}{4}^{\frac{3}{4}} G_2 \left( \frac{1}{2}, s \right) \lambda h(s) ds \cdot \|u\|_2 \\
\geq \frac{1}{\lambda B} a \left( \frac{c}{4} + d \right) \frac{1}{2} \int_\frac{1}{4}^{\frac{3}{4}} h(s) ds \cdot \|u\|_2 = \lambda m(R_0) B \geq R_0 = \|u\|_2.
$$

Therefore, $\|Tu\|_2 > \|u\|_2$, $u \in K \cap \partial \Omega_3$. By Theorem 2.1, the BVP (1.1) has two positive solutions $u_1 \in K \cap (\overline{\Omega_3} \setminus \Omega_1)$, $u_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_3)$. Therefore
Together with Theorem 3.1, it follows that the BVP (1.1) has two distinct positive solutions \( u_1 \) and \( u_2 \).

(ii) Let \( E = C^2[0,1] \), \( \Omega_1 = \{ u \in E : \| u \|_2 < r \} \), \( \Omega_2 = \{ u \in E : \| u \|_2 < R \} \). By the condition (ii) or (iv) of Lemma 2.2, we get \( \| Tu \|_2 \geq \| u \|_2 \) when \( u \in K \cap \partial \Omega_1 \) or \( u \in K \cap \partial \Omega_2 \). Choose \( \Omega_3 = \{ u \in E : \| u \|_2 < R_0 \} \) such that \( \overline{\Omega_1} \subset \Omega_3 \), \( \overline{\Omega_2} \subset \Omega_3 \). For any \( t \in [\xi, 1] \), \( u \in K \cap \partial \Omega_3 \), then \( |u(t)| + |u''(t)| \leq R_0 \). By combining (2.8) with (2.9), we have

\[
\| Tu \| = \int_0^1 \left[ \int_\xi^1 G_1(t, s)G_2(s, \tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau \right] ds \\
\leq \int_0^1 \left[ \int_\xi^1 G_1(s, s)G_2(s, \tau)\lambda h(\tau)f(\tau, u(\tau), u''(\tau))d\tau \right] ds \\
\leq \lambda M(R_0) \int_0^1 \left[ \int_\xi^1 s(1-s)G_2(\tau, \tau)h(\tau)d\tau \right] ds \\
\leq \frac{1}{6} \lambda M(R_0) \int_\xi^1 G_2(\tau, \tau)h(\tau)d\tau \\
\leq \frac{1}{6} R_0.
\]

Hence, we get \( \| Tu \|_2 = \| Tu \| + \| (Tu)'' \| < \frac{1}{2} R_0 + \frac{6}{7} R_0 = R_0 = \| u \|_2 \).

Applying Theorem 3.1 again, the operator \( T \) has two fixed points \( u_1 \in K \cap (\overline{\Omega_3} \setminus \Omega_1) \) and \( u_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_3) \) satisfying (3.2), which are the two different positive solutions for the BVP (1.1). The proof is completed.

**Corollary 3.2** Suppose (H1)-(H3) hold. Suppose further that \( f \) and \( \lambda \) satisfies either

(i) \( f_0 = f_\infty = 0 \), \( 0 < \lambda \leq \frac{1}{6m(1)} \); or

(ii) \( f_0 = f_\infty = \infty \), \( \lambda \geq \frac{6}{7M(1)} \),

then BVP (1.1) has at least two positive solutions.

**Example 3.2** Consider the following boundary value problem

\[
\begin{align*}
&u^{(4)}(t) = \lambda h(t)f(t, u(t), u''(t)), \quad t \in (0, 1), \\
u(0) = u(1) = 0, \\
u'\left(\frac{1}{2}\right) - 6u''\left(\frac{1}{2}\right) = 0, \quad u''(1) = 0,
\end{align*}
\]
where $h(t) = t - \frac{1}{2}$, $f = \frac{(|u|+|v|)^{\frac{1}{2}}}{10} + \frac{(|u|+|v|)^{3}}{60}$, $a = 1$, $b = 6$, $c = 0$, $d = 1$, $\xi = \frac{1}{2}$, then we get $f_0 = \infty$, $f_{\infty} = \infty$, $\Delta = 1$, $A = \frac{15}{1024}$, $B = \min\{\frac{10}{1024}, \frac{1}{256}\} = \frac{1}{256}$. Moreover, 

$$f = \frac{(|u|+|v|)^{\frac{1}{2}}}{10} + \frac{(|u|+|v|)^{3}}{60} \leq \frac{(|u|+|v|)^{\frac{1}{2}}}{10} + \frac{(|u|+|v|)^{\frac{3}{2}}}{60} = \frac{7}{60}(|u|+|v|)^{\frac{3}{2}}, \quad t \in \left[\frac{1}{2}, 1\right].$$

Let $R_0 = 1$ such that $|u| + |v| \leq 1$, then $M(R_0) = \max_{t \in \left[\frac{1}{2}, 1\right]} f(t, u, v) = \frac{7}{60}$. Hence, if $0 < \lambda \leq \frac{6b_0}{7M(R_0)^3} = \frac{640}{931}$, then Theorem 3.2 (ii) guarantees the existence of two positive solutions for the BVP (3.3).

References


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