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# BIFURCATIONS AND NEW EXACT TRAVELLING WAVE SOLUTIONS OF THE COUPLED NONLINEAR SCHRÖDINGER-KdV EQUATIONS*

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**Abstract**

By using the method of dynamical system, the exact travelling wave solutions of the coupled nonlinear Schrödinger-KdV equations are studied. Based on this method, all phase portraits of the system in the parametric space are given. All possible bounded travelling wave solutions such as solitary wave solutions and periodic travelling wave solutions are obtained. With the aid of Maple software, the numerical simulations are conducted for solitary wave solutions and periodic travelling wave solutions to the coupled nonlinear Schrödinger-KdV equations. The results show that the presented findings improve the related previous conclusions.

**Keywords** dynamical system method; coupled nonlinear Schrödinger-KdV equations; solitary wave solution; periodic travelling wave solution; numerical simulation

**2000 Mathematics Subject Classification** 35Q51

## 1 Introduction

In recent years, the investigation of the exact travelling wave solutions to nonlinear wave equations plays an important role in nonlinear science, since the exact travelling wave solutions can provide much physical information and more insight of the physical and mathematical aspects of the problem and then lead to further applications. Several effective methods for obtaining exact travelling wave solutions of nonlinear wave equations, such as $(G'/\mathrm{G})$-expansion method [1], the theta function method [2], Darboux and Backlund transform [3], tanh-coth method [4], sine/cosine

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method [5], Jacobi elliptic function expansion method [6], the homogeneous balance method [7], the symmetry method [8], functional variable method [9] have been developed. Among them, the dynamical system method is one of these effective methods which has been applied to many nonlinear wave equations [10,11].

In this paper, we consider the following coupled nonlinear Schrödinger-KdV equations [12]

$$\begin{cases} iu_t + u_{xx} = uv, \\ v_t + \alpha v v_x + \beta v_{xxx} = (|u|^2)_x, \end{cases} \tag{1.1}$$

where $\alpha, \beta$ are real parameters. $u$ is a complex function and $v$ is a real function. The study of coupled nonlinear Schrödinger-KdV equations has attracted extensive interest in physics and mathematics. Many numerical methods have been used to solve numerically the single nonlinear Schrödinger and the single KdV equation using finite element and finite difference methods [13-16]. Analytical solutions of the coupled nonlinear Schrödinger-KdV equations using different methods were given in [17-19]. Here, we shall use the dynamical system method to seek exact travelling wave solutions of (1.1).

In order to find travelling wave solutions of (1.1), we assume that

$$u(x,t) = \phi(\xi)\mathrm{e}^{i\eta}, \quad v(x,t) = \psi(\xi), \quad \xi = kx - ct, \quad \eta = px + lt, \tag{1.2}$$

where $k, c, p$ and $l$ are travelling wave parameters.

Substituting (1.2) into the first equation of (1.1), canceling $\mathrm{e}^{i\eta}$ and separating the real and imaginary parts, we have

$$\begin{cases} \phi'(2kp - c) = 0, \\ k^2\phi'' - (p^2 + l)\phi - \phi\psi = 0. \end{cases} \tag{1.3}$$

Obviously, from (1.3), we know that if $\phi' = 0$, then (1.1) has a trivial solution. Otherwise, (1.3) must be satisfied

$$2kp - c = 0. \tag{1.4}$$

Substituting (1.2) into the second equation of (1.1), and integraling once (integral constant is zero), we have

$$\beta k^3\psi'' - c\psi + \frac{\alpha k}{2}\psi^2 - k\phi^2 = 0. \tag{1.5}$$

Therefore, (1.1) is reduced to

$$\begin{cases} 2kp - c = 0, \\ k^2\phi'' - (p^2 + l)\phi - \phi\psi = 0, \\ \beta k^3\psi'' - c\psi + \frac{\alpha k}{2}\psi^2 - k\phi^2 = 0. \end{cases} \tag{1.6}$$

It is very difficult to solve this equations by some ordinary methods, so we consider the special transformation in subtle ways

$$\phi = m\psi. \tag{1.7}$$

Here, $m$ is a constant to be determined later. Substituting (1.7) into (1.6), the system is changed into

$$\begin{cases} 2kp - c = 0, \\ k^2\psi'' - (p^2 + l)\psi - \psi^2 = 0, \\ \beta k^3\psi'' - c\psi + \frac{\alpha k}{2}\psi^2 - m^2 k\psi^2 = 0. \end{cases} \tag{1.8}$$

Compared the coefficients of the second equation with those of the third equation of (1.8), we have

$$m = \sqrt{\frac{\alpha + 2\beta}{2}}, \quad p^2 + l = \frac{c}{\beta k}, \quad 2kp - c = 0. \tag{1.9}$$

Under condition (1.9), system (1.8) is reduced to the following equation

$$k^2\psi'' - (p^2 + l)\psi - \psi^2 = 0. \tag{1.10}$$

Let $A = \frac{p^2+l}{k^2}$, $B = \frac{1}{k^2}$, $k \neq 0$. Thus, (1.10) has the following form

$$\psi'' - A\psi - B\psi^2 = 0, \tag{1.11}$$

which corresponds to the following two-dimensional Hamiltonian system

$$\frac{\mathrm{d}\psi}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = A\psi + B\psi^2, \tag{1.12}$$

with the Hamiltonian

$$H(\psi, y) = \frac{1}{2}y^2 - \frac{1}{2}A\psi^2 - \frac{1}{3}B\psi^3 = h. \tag{1.13}$$

According to the Hamiltonian, we can get all kinds of phase portraits in the parametric space. Because the phase orbits defined the vector fields of system (1.12) determine all their travelling wave solutions of (1.1), we can investigate the bifurcations of phase portraits of system (1.12) to seek the travelling wave solutions of (1.1).

The rest of this paper is built up as follows. In Section 2, we give all phase portraits of system (1.12) and discuss the bifurcations of phase portraits of system (1.12). In Section 3, according to the dynamics of the phase orbits of system (1.12) given by Section 2, we obtain all possible bounded travelling wave solutions of (1.1). Finally, a conclusion is given in Section 4.

## 2 Bifurcations of Phase Portraits of System (1.12)

In this section, we consider the phase portraits of (1.12). Let right hand terms of system (1.12) be zeros, that is $y = 0$, and $A\psi + B\psi^2 = 0$, then the system (1.12) has two equilibrium points $S(-\frac{A}{B},0)$ and $O(0,0)$. For the Hamiltonian $H(\psi,y)=\frac{1}{2}y^2-\frac{1}{2}A\psi^2-\frac{1}{3}B\psi^3=h$, we write $h_0=H(0,0)=0$, $h_1=H(-\frac{A}{B},0)=-\frac{A^3}{6B^2}$. With the change of the parameter group of $A$ and $B$ ($B=\frac{1}{k^2}>0$), the system has different phase portraits for (1.12) as shown in Figs. 1 and 2.

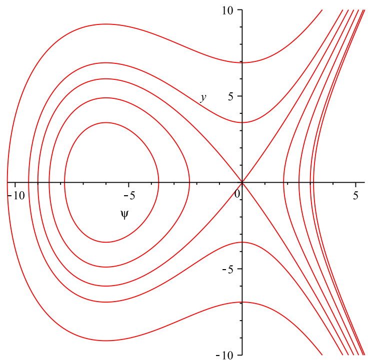


Figure 1: The bifurcations of phase portraits of (1.12) $(A>0,B>0)$.

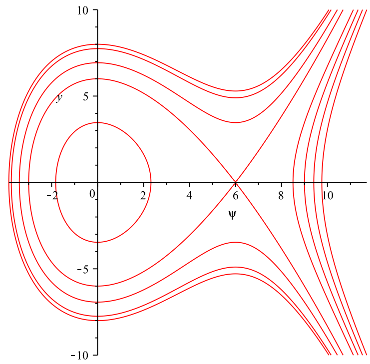


Figure 2: The bifurcations of phase portraits of (1.12) $(A<0,B>0)$.

If $AB\neq 0$, from Figs. 1 and 2, we summarize crucial conclusion as follows.

(1) When $A>0$ $(<0)$, $O$ is a saddle point (center point) and $S$ is a center (saddle point).

(2) System (1.12) has a unique homoclinic orbit $\Gamma$ which is asymptotic to the saddle and enclosing the center. There is a family of periodic orbits which are enclosing the center and filling up the interior of the homoclinic orbit $\Gamma$.

## 3 Exact Travelling Wave Solutions of (1.1)

In this section, we consider the bifurcations of the phase orbits of system (1.12). Because only bounded travelling waves are meaningful to a physical model, we just pay more attention to the bounded solutions of (1.1). In addition, because of $B>0$, we just consider the travelling wave solutions of (1.1) when $B>0$. According to equation (1.13), we have $y=\sqrt{A\psi+\frac{2}{3}B\psi^2+2h}$. Substituting it into $\frac{\mathrm{d}\psi}{\mathrm{d}\xi}=y$, that is, $\frac{\mathrm{d}\psi}{\sqrt{A\psi+\frac{2}{3}B\psi^2+2h}}=\mathrm{d}\xi$. By using the Jacobian elliptic functions [20], integrating $\frac{\mathrm{d}\psi}{\sqrt{A\psi+\frac{2}{3}B\psi^2+2h}}=\mathrm{d}\xi$, we can obtain the exact travelling wave solutions of (1.1).

(1) When $A>0$, $B>0$, there exists a smooth solitary solution which corresponds to a smooth homoclinic orbit $\Gamma$ of (1.12) defined by $H(\psi,y)=h_0=0$. We have the parametric representation

$$\psi(\xi)=\frac{-3A+3A\tanh^2(\frac{\sqrt{A}}{2}\xi)}{2B}. \tag{3.1}$$

(2) When $A<0,\ B>0$, there exists a smooth solitary solution which corresponds to a smooth homoclinic orbit $\Gamma$ of (1.12) defined by $H(\psi,y)=h_1$. We have the parametric representation

$$\psi(\xi)=\frac{|A|}{B}\Big[1-\frac{3}{2}\mathrm{sech}^2\Big(\frac{\sqrt{|A|}}{2}\xi\Big)\Big]. \tag{3.2}$$

(3) When $A>0$ $(A<0)$, $B>0$, similarly, there exist periodic travelling wave solutions which correspond to the family of periodic orbits $\Gamma^h$ of (1.12) defined by $H(\psi,y)=h$, $h\in(h_1,0)$ $(h\in(0,h_1))$. We have the following parametric representation

$$\psi(\xi)=z_3+(z_2-z_3)\mathrm{sn}^2\Big(\frac{\sqrt{6B(z_1-z_3)}}{6}\xi,\sqrt{\frac{z_2-z_3}{z_1-z_3}}\Big), \tag{3.3}$$

where the parameters $z_1,z_2,z_3$ with $z_1>z_2>z_3$ are defined by $y^2=2h+A\psi^2+\frac{2}{3}B\psi^3=\frac{2}{3}B(z_1-\psi)(z_2-\psi)(\psi-z_3)$.

By using the above results and considering conditions (1.9), we obtain the exact travelling wave solutions of (1.1) as follows.

(1) When $A>0,\ B>0$,

$$\begin{cases} u_1(x,t)=m\mathrm{e}^{i(px+lt)}\dfrac{-3A+3A\tanh^2(\frac{\sqrt{A}}{2}(kx-ct))}{2B}, \\ v_1(x,t)=\dfrac{-3A+3A\tanh^2(\frac{\sqrt{A}}{2}(kx-ct))}{2B}. \end{cases} \tag{3.4}$$

(2) When $A<0,\ B>0$,

$$\begin{cases} u_2(x,t)=m\mathrm{e}^{i(px+lt)}\dfrac{|A|}{B}\Big[1-\dfrac{3}{2}\mathrm{sech}^2\Big(\dfrac{\sqrt{|A|}}{2}(kx-ct)\Big)\Big], \\ v_2(x,t)=\dfrac{|A|}{B}\Big[1-\dfrac{3}{2}\mathrm{sech}^2\Big(\dfrac{\sqrt{|A|}}{2}(kx-ct)\Big)\Big]. \end{cases} \tag{3.5}$$

(3) When $A>0$ $(A<0)$, $B>0$,

$$\begin{cases} u_3(x,t)=m\mathrm{e}^{i(px+lt)}\Big[z_3+(z_2-z_3)\mathrm{sn}^2\Big(\dfrac{\sqrt{6B(z_1-z_3)}}{6}(kx-ct),\sqrt{\dfrac{z_2-z_3}{z_1-z_3}}\Big)\Big], \\ v_3(x,t)=z_3+(z_2-z_3)\mathrm{sn}^2\Big(\dfrac{\sqrt{6B(z_1-z_3)}}{6}(kx-ct),\sqrt{\dfrac{z_2-z_3}{z_1-z_3}}\Big). \end{cases} \tag{3.6}$$

Based on the above results, by using the numerical simulation method, the 3D graphics of bounded solutions of (1.1) are shown in Figs. 3-8 (drawn by software Maple).

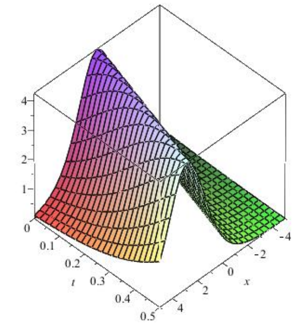


Figure 3: The 3D graphics of $|u_1|$ ($k = p = l = 1, c = 2, \alpha = 2, \beta = 1, -5 \le x \le 5, 0 \le t \le 0.5$).

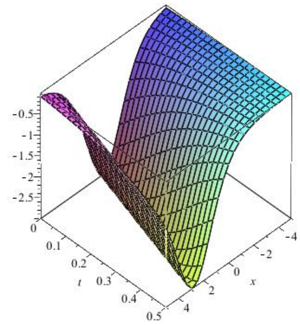


Figure 4: The 3D graphics of $v_1$ ($k = p = l = 1, c = 2, \alpha = 2, \beta = 1, -5 \le x \le 5, 0 \le t \le 0.5$).

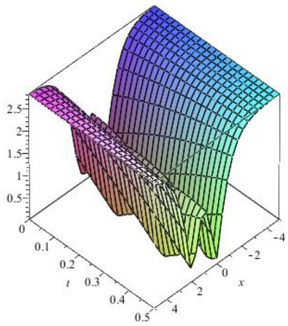


Figure 5: The 3D graphics of $|u_2|$ ($k = 1, p = 1 - \sqrt{3}, l = -1, c = 2 - 2\sqrt{3}, \alpha = 2, \beta = 1, -5 \le x \le 5, 0 \le t \le 0.5$).

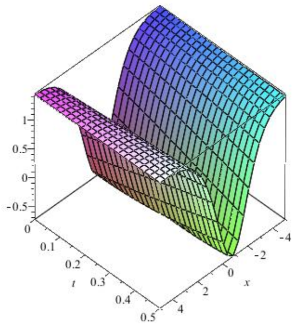


Figure 6: The 3D graphics of $v_2$ ($k = 1, p = 1 - \sqrt{3}, l = -1, c = 2 - 2\sqrt{3}, \alpha = 2, \beta = 1, -5 \le x \le 5, 0 \le t \le 0.5$).

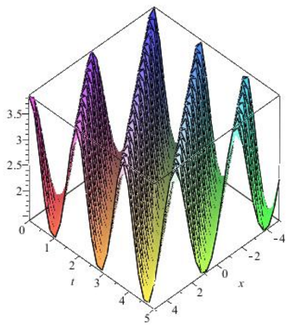


Figure 7: The 3D graphics of $|u_3|$ ($k = p = l = 1, c = 2, h_1 = -\frac{2}{3}, \alpha = 2, \beta = 1, -5 \le x \le 5, 0 \le t \le 5$).

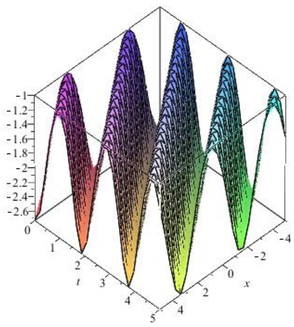


Figure 8: The 3D graphics of $v_3$ ($k = p = l = 1, c = 2, h_1 = -\frac{2}{3}, \alpha = 2, \beta = 1, -5 \le x \le 5, 0 \le t \le 5$).

By using the approach of dynamical system, we obtain the travelling wave solutions of (1.1). Among them, (3.4) and (3.5) are soliton solutions which are expressed by the hyperbolic functions. (3.6) is a periodic solution which is expressed by Jacobian elliptic function. Note that our solutions in this paper are different from the given ones in references [17-19].

# 4 Conclusion

By using the method of dynamical system, the exact explicit travelling wave solutions of (1.1) are shown which have periodic wave solutions and solitary wave solutions. Of course, the dynamical system method is not only able to solve the coupled nonlinear Schrödinger-KdV equations, it can also be applied to some other nonlinear equations.

Furthermore, if without the condition that $B>0$, system (1.12) has another case. If $A>0$ ($A<0$), $B<0$, there exist periodic travelling wave solutions corresponding to the family of periodic orbits $\Gamma^h$ of (1.12) defined by $H(\psi,y)=h$, $h\in(h_1,0)$ ($h\in(0,h_1)$). We have the following parametric representation

$$\psi(\xi)=z_1-(z_1-z_2)\mathrm{sn}^2\Big(\frac{\sqrt{6|B|(z_1-z_3)}}{6}\xi,\sqrt{\frac{z_1-z_2}{z_1-z_3}}\Big),\tag{4.1}$$

where the parameters $z_1,z_2,z_3$ with $z_1>z_2>z_3$ are defined by $y^2=2h+A\psi^2+\frac{2}{3}B\psi^3=\frac{2}{3}|B|(z_1-\psi)(\psi-z_2)(\psi-z_3)$. Then we obtain additional travelling wave solutions of (1.1) as follows:

$$\begin{cases}u(x,t)=m\mathrm{e}^{i(px+lt)}\Big[z_1-(z_1-z_2)\mathrm{sn}^2\Big(\frac{\sqrt{6|B|(z_1-z_3)}}{6}(kx-ct),\sqrt{\frac{z_1-z_2}{z_1-z_3}}\Big)\Big],\\ v(x,t)=z_1-(z_1-z_2)\mathrm{sn}^2\Big(\frac{\sqrt{6|B|(z_1-z_3)}}{6}(kx-ct),\sqrt{\frac{z_1-z_2}{z_1-z_3}}\Big).\end{cases}\tag{4.2}$$

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