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THE SYMMETRIC POSITIVE SOLUTIONS OF $2n$-ORDER BOUNDARY VALUE PROBLEMS ON TIME SCALES[*†]

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Abstract

In this paper, we are concerned with the symmetric positive solutions of a $2n$-order boundary value problems on time scales. By using induction principle, the symmetric form of the Green's function is established. In order to construct a necessary and sufficient condition for the existence result, the method of iterative technique will be used. As an application, an example is given to illustrate our main result.

Keywords symmetric positive solutions; boundary value problems; induction principle; time scales; iterative technique

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1 Introduction

The theory of measure chains (time scales) was first introduced by Stefan Hilger in his Ph.D. thesis (see [1]) in 1988. Although it is a new research area of mathematics, it has already caused a lot of applications, e.g., insect population models, neural networks, heat transfer and epidemic models (see [2,3]). Some of these models can be found in [4-6]. Such as in [5], Q.K. Song and Z.J. Zhao discussed the problem on the global exponential stability of complex-valued neural networks with both leakage delay and time-varying delays on time scales. By constructing appropriate Lyapunov-Krasovskii functionals and using matrix inequality technique, a delay-dependent condition assuring the global exponential stability for the considered neural networks was established.

In the past few years, more and more scholars concentrated on a positive solution of boundary value problems for differential equations on time scales (see [7-12]). In

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[13,14], by using some fixed point theorems, the existences of pseudo-symmetric solutions of dynamic equations on time scales were obtained. In [15,16], the fourth order integral boundary value problems on time scales for an increasing homeomorphism and homomorphism were discussed. Recently, the conditions for the existence of symmetric positive solutions of boundary value problems were constructed in [17,18]. By applying an iterative technique, the existence and uniqueness of symmetric positive solutions of the $2n$-order nonlinear singular boundary value problems of differential equation

$$\begin{cases}(-1)^n u^{(2n)}(t) = f(t, u(t)), \quad t \in (0,1),\\ u^{(2k)}(0) = u^{(2k)}(1) = 0, \quad k = 0,1,2,\cdots,n-1,\end{cases}$$

were obtained.

In this paper, we are concerned with the existence of symmetric positive solutions of the following $2n$-order boundary value problems (BVP) on time scales

$$\begin{cases}(-1)^n u^{\Delta^{2n}}(t) = f(\sigma(t), u^{\sigma}(t)), \quad t \in [0,\sigma(1)],\\ u^{\Delta^{2i}}(0) = u^{\Delta^{2i}}(\sigma(1)) = 0, \quad 0 \le i \le n-1,\end{cases} \tag{1}$$

where $f : [0,\sigma(1)] \times [0,\infty) \to [0,\infty)$ is continuous and $f(t,u)$ may be singular at $u = 0$, $t = 0$ (and/or $t = \sigma(1)$). If a function $u : [0,\sigma(1)] \to \mathbb{R}$ is continuous and satisfies $u(t) = u(\sigma(1) - t)$ for $t \in [0,\sigma(1)]$, then we say that $u(t)$ is symmetric on $[0,\sigma(1)]$. By a symmetric positive solution of BVP (1), we mean a symmetric function $u \in C^{2n}[0,\sigma(1)]$ such that $(-1)^i u^{\Delta^{2i}}(t) > 0$ for $t \in (0,\sigma(1))$ and $i = 0,1,\cdots,n-1$, and $u(t)$ satisfies BVP (1). We assume that $\sigma(1)$ and 0 are all right dense. Throughout this paper we let $\mathbb{T}$ be any time scale (nonempty closed subset of $\mathbb{R}$) and $[a,b]$ be a subset of $\mathbb{T}$ such that $[a,b] = \{t \in \mathbb{T} : a \le t \le b\}$. $\mathbb{T}$ satisfies

$$\sigma(a - \sigma(b)) = \sigma(a) - \sigma(b), \tag{2}$$

and it is easy to see that $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$ satisfies (2). And thus $\widetilde{\mathbb{T}} = \{\sigma(t) | t \in \mathbb{T}\} = \mathbb{T}$.

2 Preliminary

Before discussing the problems of this paper, we introduce some basic materials for time scales which are useful in proving our main results. These preliminaries can be found in [17-20].

Lemma 2.1[20] (Substitution) *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f\circ\nu^{-1})(s)\widetilde{\Delta}s.$$

Throughout this paper, similar to [18], we assume that:

(A1) For $(t,u)\in[0,\sigma(1)]\times[0,\infty)$, $f(t,u)$ is symmetric in t, that is, f satisfies

$$f(\sigma(1)-t,u)=f(t,u), \quad \text{for } t\in(0,\sigma(1)). \tag{3}$$

(A2) For $(t,u)\in[0,\sigma(1)]\times[0,\infty)$, f is non-decreasing with respect to u and there exists a constant $\lambda\in(0,1)$, such that if $\sigma\in(0,1]$, then

$$\sigma^{\lambda}f(t,u)\le f(t,\sigma u). \tag{4}$$

It is easy to see that (4) implies that if $\sigma\in[1,\infty)$, then

$$f(t,\sigma u)\le\sigma^{\lambda}f(t,u). \tag{5}$$

For convenience, in this paper we let

$$e(t)=\frac{t}{\sigma(1)}(\sigma(1)-t), \quad \text{for } t\in[0,\sigma(1)]. \tag{6}$$

Lemma 2.2 *Let $v\in C[0,\sigma(1)]$, then the following BVP*

$$\begin{cases}(-1)^n u^{\Delta^{2n}}(t)=v(t), & t\in[0,\sigma(1)],\\ u^{\Delta^{2i}}(0)=u^{\Delta^{2i}}(\sigma(1))=0, & 0\le i\le n-1\end{cases} \tag{7}$$

has a unique solution

$$u(t)=\int_0^{\sigma(1)} G_n(t,s)v(s)\Delta s, \tag{8}$$

where $G_n(t,s)$ is defined in $[0,\sigma(1)]\times[0,\sigma(1)]$, and it follows from [19] *that*

$$G_i(t,s)=\int_0^{\sigma(1)} G(t,\tau)G_{i-1}(\tau,s)\Delta\tau, \quad 2\le i\le n, \tag{9}$$

$$G_1(t,s)=G(t,s)=\frac{1}{\sigma(1)}\begin{cases}t(\sigma(1)-\sigma(s)), & t\le s,\\ \sigma(s)(\sigma(1)-t), & t>\sigma(s),\end{cases} \tag{10}$$

which satisfies

$$G(t,s)>0, \quad (t,s)\in(0,\sigma(1))\times(0,1).$$

It is easy to see

$$\begin{aligned} e(s)e(t)\le G(t,s)\le G(t,t)&=\frac{t}{\sigma(1)}(\sigma(1)-\sigma(t))\\ &\le\frac{t}{\sigma(1)}(\sigma(1)-t)=e(t), \quad (t,s)\in[0,\sigma(1)]\times[0,1]. \end{aligned} \tag{11}$$

Lemma 2.3 *For any $t,s\in[0,\sigma(1)]$, from (2) we have*

$$G_n(\sigma(1)-t,\sigma(1)-\sigma(s))=G_n(t,s),\quad n\geq 1. \tag{12}$$

Proof For any $t,s\in[0,\sigma(1)]$, when $n=1$, it is easy to see $G_1(\sigma(1)-t,\sigma(1)-\sigma(s))=G_1(t,s)$.

Assume that

$$\begin{aligned}
&G_n(\sigma(1)-t,\sigma(1)-\sigma(s))\\
=&\int_0^{\sigma(1)}G(\sigma(1)-t,\tau)G_{n-1}(\tau,\sigma(1)-\sigma(s))\Delta\tau\\
=&\int_0^{\sigma(1)}G(\sigma(1)-t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)G_{n-2}(\tau,\sigma(1)-\sigma(s))\Delta\tau\Delta\tau\\
=&\int_0^{\sigma(1)}G(\sigma(1)-t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)\cdots\int_0^{\sigma(1)}G(\tau,\tau)G_1(\tau,\sigma(1)-\sigma(s))\Delta^{n-1}\tau\\
=&\int_0^{\sigma(1)}G(t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)\cdots\int_0^{\sigma(1)}G(\tau,\tau)G_1(\tau,s)\Delta^{n-1}\tau\\
=&G_n(t,s).
\end{aligned}\tag{13}$$

We consider $G_{n+1}(\sigma(1)-t,\sigma(1)-\sigma(s))$.

From (13), we obtain

$$\begin{aligned}
&G_{n+1}(\sigma(1)-t,\sigma(1)-\sigma(s))\\
=&\int_0^{\sigma(1)}G(\sigma(1)-t,\tau)G_n(\tau,\sigma(1)-\sigma(s))\Delta\tau\\
=&\int_0^{\sigma(1)}G(\sigma(1)-t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)\cdots\int_0^{\sigma(1)}G(\tau,\tau)G_1(\tau,\sigma(1)-\sigma(s))\Delta^n\tau\\
=&\int_0^{\sigma(1)}G(\sigma(1)-t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)\cdots\int_0^{\sigma(1)}G(\tau,\tau)G_1(\tau,\sigma(1)-\sigma(s))\Delta^{n-1}\tau\int_0^{\sigma(1)}G(\tau,\tau)\Delta\tau\\
=&\int_0^{\sigma(1)}G(t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)\cdots\int_0^{\sigma(1)}G(\tau,\tau)G_1(\tau,s)\Delta^{n-1}\tau\int_0^{\sigma(1)}G(\tau,\tau)\Delta\tau\\
=&\int_0^{\sigma(1)}G(t,\tau)\int_0^{\sigma(1)}G(\tau,\tau)\cdots\int_0^{\sigma(1)}G(\tau,\tau)G_1(\tau,s)\Delta^n\tau\\
=&\int_0^{\sigma(1)}G(t,\tau)G_n(\tau,s)\Delta\tau=G_{n+1}(t,s).
\end{aligned}$$

Therefore, the proof of Lemma 2.3 is complete.

Remark 2.1 For any $t,s\in[0,1]$, we have

$$G_n(1-t,1-s)=G_n(t,s),\quad n\geq 1$$

in a real space $\mathbb{R}$, with the Green's function

$$G_1(t,s) = G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Let $\mathbb{E}$ be the Banach space $C^{(2n)}[0,\sigma(1)]$, and define

$$\begin{aligned} \mathbb{P} = \{u \in \mathbb{E} : & \ u(0) = u(\sigma(1)) = 0, u(t) > 0 \text{ for } t \in (0,\sigma(1)), u(t) = u(\sigma(1)-t) \\ & \text{and there exist constants } l_u, L_u \text{ with } 0 < l_u < 1 < L_u \text{ such that} \\ & l_u e(t) \le u(t) \le L_u e(t) \text{ for } t \in [0,\sigma(1)]\}. \end{aligned} \tag{14}$$

Lemma 2.4 *Assume that* $u \in C^2[0,\sigma(1)]$ *with* $u(t) \ge 0$ *and* $u^{\Delta\Delta}((t)) \le 0$ *for* $t \in [0,\sigma(1)]$*, then*

$$u(t) \ge \max_{s\in[0,\sigma(1)]} u(s)e(t), \quad t \in [0,\sigma(1)].$$

Proof The proof is similar to that of Lemma 2.3 in [18].

Lemma 2.5 *If* $u(t)$ *is a symmetric solution of BVP* (1)*, then there exist constants* c_1*,* c_2 *with* $0 < c_1 < 1 < c_2$ *such that*

$$c_1 e(t) \le u(t) \le c_2 e(t), \quad t \in [0,\sigma(1)]. \tag{15}$$

Proof For $c_1 e(t) \le u(t)$, since $u(t)$ is a symmetric positive solution of (1), we obtain $u(t) > 0$ and $u^{\Delta\Delta}((t)) \le 0$ for $t \in [0,\sigma(1)]$. Choose a positive number $c_1 < \min\big\{1, \max\limits_{s\in[0,\sigma(1)]} u(s)\big\}$, then Lemma 2.4 implies $c_1 e(t) \le u(t)$, $t \in [0,\sigma(1)]$.

For $u(t) \le c_2 e(t)$, again from the fact that $u(t) \ge 0$ and $u^{\Delta\Delta}((t)) \le 0$ for $t \in [0,\sigma(1)]$, we obtain

$$u(t) \le u^{\Delta}(0)t, \quad t \in [0,\sigma(1)],$$

and

$$u(t) \le -u^{\Delta}(\sigma(1))(\sigma(1)-t), \quad t \in [0,\sigma(1)].$$

When $t \in [0,\sigma(1)/2]$, we have $\sigma(1) \le 2(\sigma(1)-t)$, then

$$u(t) \le u^{\Delta}(0)t \le \frac{2u^{\Delta}(0)t}{\sigma(1)}(\sigma(1)-t) = 2u^{\Delta}(0)e(t), \quad t \in [0,\sigma(1)/2].$$

When $t \in [\sigma(1)/2,\sigma(1)]$, we have $\sigma(1) \le 2t$, then

$$u(t) \le -\frac{2u^{\Delta}(\sigma(1))t}{\sigma(1)}(\sigma(1)-t) = -2u^{\Delta}(\sigma(1))e(t), \quad t \in [\sigma(1)/2,\sigma(1)].$$

The symmetry of $u(t)$ implies that $u^{\Delta}(0) = -u^{\Delta}(\sigma(1))$. Choose a number $c_2 > \max\big\{1, 2u^{\Delta}(0)\big\}$. Then $u(t) \le c_2 e(t)$, $t \in [0,\sigma(1)]$.

Clearly, (15) holds, and this completes the proof of Lemma 2.5.

Lemma 2.6[20] *Let* $[a,b] \in \mathbb{T}$*, and* f *be right-dense continuous. If* $[a,b]$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum\limits_{t\in[a,b)} (\sigma(t)-t)f(t), & if\ a<b,\\ 0, & if\ a=b,\\ -\sum\limits_{t\in[b,a)} (\sigma(t)-t)f(t), & if\ a>b. \end{cases} \tag{16}$$

3 The Existence Result

Theorem 3.1 *Assume* (A1) *and* (A2) *hold. Then BVP* (1) *has at least one symmetric positive solution if and only if*

$$0 < \int_0^{\sigma(1)} f(\sigma(t), e^{\sigma}(t))\Delta t < \infty. \tag{17}$$

Proof Necessity Assume first that $u(t)$ is a symmetric positive solution of (1). We will show that (17) holds. Let c_1 and c_2 be given as in Lemma 2.5 for this $u(t)$. By Lemma 2.5, $u(t)$ satisfies (15).

From (1), for $t \in (0, \sigma(1))$, when n is odd, $u^{\Delta^{2n}}((t)) \le 0$ and when n is even, $u^{\Delta^{2n}}((t)) \ge 0$. Then

$$u^{\Delta^{2n-1}}(\sigma(1)) \le u^{\Delta^{2n-1}}(0), \quad \text{when } n \text{ is odd}, \tag{18}$$

$$u^{\Delta^{2n-1}}(\sigma(1)) \ge u^{\Delta^{2n-1}}(0), \quad \text{when } n \text{ is even}. \tag{19}$$

By (4),(5),(15),(18) and (19),

$$\begin{aligned} \int_0^{\sigma(1)} f(\sigma(t), e^{\sigma}(t))\Delta t &\le \int_0^{\sigma(1)} f(\sigma(t), c_1^{-1}u^{\sigma}(t))\Delta t \le c_1^{-\lambda} \int_0^{\sigma(1)} f(\sigma(t), u^{\sigma}(t))\Delta t \\ &= (-1)^n c_1^{-\lambda}(u^{\Delta^{2n-1}}(\sigma(1)) - u^{\Delta^{2n-1}}(0)) < \infty, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \int_0^{\sigma(1)} f(\sigma(t), e^{\sigma}(t))\Delta t &\ge \int_0^{\sigma(1)} f(\sigma(t), c_2^{-1}u^{\sigma}(t))\Delta t \ge c_2^{-\lambda} \int_0^{\sigma(1)} f(\sigma(t), u^{\sigma}(t))\Delta t \\ &= (-1)^n c_2^{-\lambda}\big(u^{\Delta^{2n-1}}(\sigma(1)) - u^{\Delta^{2n-1}}(0)\big) > 0. \end{aligned} \tag{21}$$

Now, (17) follows from (20) and (21).

Sufficiency Now assume that (17) holds. We will show that BVP (1) has at least one symmetric positive solution.

Define an operator $T : \mathbb{E} \to \mathbb{E}$ by

$$Tu(t) = \int_0^{\sigma(1)} G_n(t,s) f(\sigma(s), u^{\sigma}(s))\Delta s, \tag{22}$$

where $G_n(t,s)$ is defined by (9) and (10). It is clear that u is a solution if and only if u is a fixed point of T.

Claim 1 The operator $T:\mathbb{P}\to\mathbb{P}$ is completely continuous and non-decreasing.

In fact, for $u\in\mathbb{P}$, it is obvious that $Tu\in\mathbb{E}$, $Tu(0)=Tu(\sigma(1))=0$, $Tu(t)>0$, for $t\in(0,\sigma(1))$. From (22), we have

$$Tu(\sigma(1)-t)=\int_0^{\sigma(1)}G_n(\sigma(1)-t,s)f(\sigma(s),u^\sigma(s))\Delta s.$$

Assuming $s=\sigma(1)-\sigma(\xi)$, since $\widetilde{\Delta}(\xi)=\Delta\xi$, from (3),(12)and Lemma 2.1, we can obtain

$$\begin{aligned}
&Tu(\sigma(1)-t)\\
=&\int_{\sigma(1)}^{\sigma^{-1}(0)}G_n(\sigma(1)-t,\sigma(1)-\sigma(\xi))f(\sigma(\sigma(1)-\sigma(\xi)),u^\sigma(\sigma(1)-\sigma(\xi)))\Delta(\sigma(1)-\sigma(\xi))\\
=&\int_{\sigma^{-1}(0)}^{\sigma(1)}G_n(\sigma(1)-t,\sigma(1)-\sigma(\xi))f(\sigma(\sigma(1)-\sigma(\xi)),u^\sigma(\sigma(1)-\sigma(\xi)))\widetilde{\Delta}(\xi)\\
=&\int_{\sigma^{-1}(0)}^{\sigma(1)}G_n(\sigma(1)-t,\sigma(1)-\sigma(\xi))f(\sigma(1)-\sigma(\xi),u^\sigma(\sigma(1)-\sigma(\xi)))\widetilde{\Delta}(\xi)\\
=&\int_{\sigma^{-1}(0)}^{\sigma(1)}G_n(\sigma(1)-t,\sigma(1)-\sigma(\xi))f(\sigma(1)-\sigma(\xi),u(\sigma(1)-\sigma(\xi)))\widetilde{\Delta}(\xi)\\
=&\int_0^{\sigma(1)}G_n(t,\xi)f(\sigma(\xi),u^\sigma(\xi))\Delta\xi=Tu(t).
\end{aligned}$$

Thus, for any $u\in\mathbb{P}$, from (4),(5),(10),(11) and (17), we obtain that for $t\in[0,\sigma(1)]$,

$$\begin{aligned}
Tu(t)&=\int_0^{\sigma(1)}G_n(t,s)f(\sigma(s),u^\sigma(s))\Delta s\\
&=\int_0^{\sigma(1)}\int_0^{\sigma(1)}G(t,\tau)G_{n-1}(\tau,s)\Delta\tau f(\sigma(s),u^\sigma(s))\Delta s\\
&=\int_0^{\sigma(1)}G(t,\tau)\int_0^{\sigma(1)}G_{n-1}(\tau,s)f(\sigma(s),u^\sigma(s))\Delta s\Delta\tau\\
&\le\int_0^{\sigma(1)}e(t)\int_0^{\sigma(1)}G_{n-1}(\tau,s)f(\sigma(s),u^\sigma(s))\Delta s\Delta\tau\\
&\le\int_0^{\sigma(1)}e(t)\int_0^{\sigma(1)}G_{n-1}(\tau,s)f(\sigma(s),L_ue^\sigma(s))\Delta s\Delta\tau\\
&\le L_u^\lambda\int_0^{\sigma(1)}\int_0^{\sigma(1)}G_{n-1}(\tau,s)f(\sigma(s),e^\sigma(s))\Delta s\Delta\tau e(t)\\
&\le L_{Tu}e(t),
\end{aligned}\tag{23}$$

and

$$\begin{aligned} Tu(t) &= \int_0^{\sigma(1)} G(t,\tau)\int_0^{\sigma(1)} G_{n-1}(\tau,s)f(\sigma(s),u^\sigma(s))\Delta s\Delta\tau \\ &\geq \int_0^{\sigma(1)} e(t)e(\tau)\int_0^{\sigma(1)} G_{n-1}(\tau,s)f(\sigma(s),u^\sigma(s))\Delta s\Delta\tau \\ &\geq \int_0^{\sigma(1)} e(t)e(\tau)\int_0^{\sigma(1)} G_{n-1}(\tau,s)f(\sigma(s),l_u e^\sigma(s))\Delta s\Delta\tau \\ &\geq l_u^\lambda\int_0^{\sigma(1)} e(\tau)\int_0^{\sigma(1)} G_{n-1}(\tau,s)f(\sigma(s),e^\sigma(s))\Delta s\Delta\tau e(t) \\ &\geq l_{Tu}e(t), \end{aligned} \tag{24}$$

where L_{Tu} and l_{Tu} are positive constants satisfying

$$L_{Tu} > \max\left\{1, L_u^\lambda\int_0^{\sigma(1)}\int_0^{\sigma(1)} G_{n-1}(\tau,s)f(\sigma(s),e^\sigma(s))\Delta s\Delta\tau\right\},$$

$$l_{Tu} < \min\left\{1, l_u^\lambda\int_0^{\sigma(1)} e(\tau)\int_0^{\sigma(1)} G_{n-1}(\tau,s)f(\sigma(s),e^\sigma(s))\Delta s\Delta\tau\right\}.$$

Thus, it follows from (23) and (24) that there exist constants L_{Tu} and l_{Tu} with $0 < l_{Tu} < 1 < L_{Tu}$ such that

$$l_{Tu}e(t) \leq Tu(t) \leq L_{Tu}e(t), \quad t \in [0,\sigma(1)]. \tag{25}$$

Therefore, $Tu(t) \in \mathbb{P}$, that is, $T : \mathbb{P} \to \mathbb{P}$. A standard argument can be used to show that $T : \mathbb{P} \to \mathbb{P}$ is completely continuous.

From (A2), it is easy to see that T is non-decreasing with respect to u. Hence, Claim 1 holds.

Claim 2 Let δ and γ be fixed numbers satisfying

$$0 < \delta \leq l_{Te}^{\frac{1}{1-\lambda}} \quad \text{and} \quad \lambda \geq L_{Te}^{\frac{1}{1-\lambda}}, \tag{26}$$

and assume

$$u_0 = \delta e(t), \quad v_0 = \gamma e(t), \tag{27}$$

$$u_n = Tu_{n-1} \quad \text{and} \quad v_n = Tv_{n-1}, \quad \text{for } n = 1, 2, \cdots. \tag{28}$$

Then,

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \tag{29}$$

and there exists a $u^* \in \mathbb{P}$ such that

$$u_n(t) \to u^*(t), \quad v_n(t) \to u^*(t), \quad \text{uniformly on } [0,\sigma(1)]. \tag{30}$$

In fact, $0 > l_{Te} < 1 < L_{Te}$ since $Te \in \mathbb{P}$. So $0 < \delta < 1 < \gamma$. From (27), we have $u_0, v_0 \in \mathbb{P}$ and $u_0 \le v_0$.

On the other hand,

$$\begin{aligned} u_1(t) = Tu_0(t) &= \int_0^{\sigma(1)} G_n(t,s) f(\sigma(s), \delta e^\sigma(s)) \Delta s \\ &\ge \delta^\lambda \int_0^{\sigma(1)} G_n(t,s) f(\sigma(s), e^\sigma(s)) \Delta s \\ &= \delta^\lambda Te \ge \delta^\lambda l_{Te} e(t) \ge \delta^\lambda \delta^{1-\lambda} e(t) = u_0(t), \\ v_1(t) = Tv_0(t) &= \int_0^{\sigma(1)} G_n(t,s) f(\sigma(s), \gamma e^\sigma(s)) \Delta s \\ &\le \gamma^\lambda \int_0^{\sigma(1)} G_n(t,s) f(\sigma(s), e^\sigma(s)) \Delta s \\ &= \gamma^\lambda Te \le \gamma^\lambda L_{Te} e(t) \le \gamma^\lambda \gamma^{1-\lambda} e(t) = v_0(t). \end{aligned}$$

Since $u_0 \le v_0$ and T is nondecreasing, by induction, (27) holds.

Let $c_0 = \frac{\delta}{\gamma}$, then $0 < c_0 < 1$. It follows from

$$T(cu) \ge c^\lambda Tu, \quad \text{if } 0 < c < 1,\ u \in \mathbb{P},$$

that for any natural number n,

$$u_n = Tu_{n-1} = T^n u_0 = T^n(\delta e(t)) = T^n(c_0 \gamma e(t)) \ge c_0^{\lambda^n} T^n(\gamma e(t)) = c_0^{\lambda^n} v_n.$$

Thus, for each natural numbers n and p^*, we have

$$0 \le u_{n+p^*} - u_n \le v_n - u_n \le (1 - c_0^{\lambda^n}) v_n \le c_0^{\lambda^n} \gamma e(t),$$

which implies that there exists a $u^* \in \mathbb{P}$ such that (30) holds, and Claim 2 holds.

Let $n \to \infty$ in (28), we obtain $u^*(t) = Tu^*(t)$, which is a symmetric positive solution of BVP (1). Thus, the proof of Theorem 3.1 is complete.

4 Example

Example 4.1 Consider

$$\begin{cases} (-1)^n u^{\Delta^{2n}}(t) = (\sigma(t))^\alpha (\sigma(1) - \sigma(t))^\alpha u^\beta(\sigma(t)), & t \in [0, \sigma(1)], \\ u^{\Delta^{2i}}(0) = u^{\Delta^{2i}}(\sigma(1)) = 0, \quad 0 \le i \le n-1, \end{cases} \tag{31}$$

where $\alpha \in \mathbb{R}$, $0 < \beta < 1$. Let $f(\sigma(t), u^\sigma(t)) = (\sigma(t))^\alpha (\sigma(1) - \sigma(t))^\alpha u^\beta(\sigma(t))$, $(t, u) \in [0, \sigma(1)] \times [0, \infty)$, then, for $\alpha > -\beta - 1$, there is at least one symmetric positive solution of BVP (31).

Note that the function f satisfies assumptions (A1) and (A2). In fact, for $(t,u) \in [0,\sigma(1)] \times [0,\infty)$, $f(\sigma(1)-\sigma(t),u^\sigma(t)) = f(\sigma(t),u^\sigma(t))$, f is non-decreasing with respect to u and if $\sigma \in (0,1]$, there exists a constant λ with $0<\beta\leq\lambda<1$, such that $f(\sigma(t),\sigma u^\sigma(t)) \geq \sigma^\lambda f(\sigma(t),u^\sigma(t))$.

Thus, from Theorem 3.1, there is at least one symmetric positive solution of BVP (31) if and only if

$$0<\int_0^{\sigma(1)}(\sigma(t))^\alpha(\sigma(1)-\sigma(t))^\alpha\Big(\frac{\sigma(t)}{\sigma(1)}(\sigma(1)-\sigma(t))\Big)^\beta\Delta t<\infty. \tag{32}$$

In fact, the integration

$$\int_0^{\sigma(1)}\Big(\frac{1}{\sigma(1)}\Big)^\beta(\sigma(t))^{\alpha+\beta}(\sigma(1)-\sigma(t))^{\alpha+\beta}\Delta t$$

converges if and only if $\alpha>-\beta-1$, then (32) holds. That is, for $\alpha>-\beta-1$, there is at least one symmetric positive solution of BVP (31).

5 Discussion

If $\mathbb{T}^*=\{0\}\cup\{\frac{1}{n}\}\cup\{1-\frac{1}{n}\}\cup\{1\}$, from (16), we can obtain the following result.

Theorem 5.1 *Assume that* (A1) *and* (A2) *hold. Then BVP* (1) *has at least one symmetric positive solution if and only if*

$$0<\sum_{t\in[0,\frac{1}{2})\cap\mathbb{T}^*}\frac{t^2}{(1-2t)(1-t)}f(\sigma(t),e^\sigma(t))+\sum_{t\in[\frac{1}{2},1)\cap\mathbb{T}^*}\frac{(1-t)^2}{(3-2t)(2-t)}f(\sigma(t),e^\sigma(t))<\infty.$$

In particular, letting

$$f(\sigma(t),u^\sigma(t))=\sigma(t)(\sigma(1)-\sigma(t)), \quad (t,u)\in\mathbb{T}^*\times[0,\infty),$$

it is easy to see that f satisfies assumptions (A1) and (A2), and there is

$$0<\sum_{n=2}^{\infty}\Big(\frac{1}{(n-1)^3}+\frac{n}{(n+2)(n+1)^3}\Big)<\infty.$$

Thus, from Theorem 5.1, BVP (1) has at least one symmetric positive solution.

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