Ann. of Appl. Math.
32:4(2016), 331-356

GLOBAL EXISTENCE AND LONG-TIME BEHAVIOR FOR THE STRONG SOLUTIONS IN H^2 TO THE 3D COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS*

Jincheng Gao, Boling Guo

(*Institute of Applied Physics and Computational Math., 100088, Beijing, PR China*)

Xiaoyu Xi†

(*Graduate School of China Academy of Engineering Physics, 100088, Beijing, PR China*)

Abstract

In this paper, we investigate the global existence and long time behavior of strong solutions for compressible nematic liquid crystal flows in three-dimensional whole space. The global existence of strong solutions is obtained by the standard energy method under the condition that the initial data are close to the constant equilibrium state in H^2-framework. If the initial datas in L^1-norm are finite additionally, the optimal time decay rates of strong solutions are established. With the help of Fourier splitting method, one also establishes optimal time decay rates for the higher order spatial derivatives of director.

Keywords compressible nematic liquid crystal flows; global solution; Green function; long-time behavior

2010 Mathematics Subject Classification 35Q35; 35B40; 76A15

1 Introduction

In this paper, we investigate the motion of compressible nematic liquid crystal flows, which are governed by the following simplified version of the Ericksen-Leslie equations

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\mu+\nu)\nabla \mathrm{div} u + \nabla P(\rho) = -\gamma \nabla d \cdot \Delta d, \\ d_t + u\cdot\nabla d = \theta(\Delta d + |\nabla d|^2 d), \end{cases} \tag{1.1}$$

where ρ, u and d stand for the density, velocity and macroscopic average of the nematic liquid crystal orientation field respectively. The pressure $P(\rho)$ is a smooth

*Manuscript received August 6, 2016

†Corresponding author. E-mail: xixiaoyu1357@126.com

function in a neighborhood of 1 with $P'(1) = 1$. The constants μ and ν are shear viscosity and the bulk viscosity coefficients of the fluid respectively, that satisfy the physical assumptions

$$\mu > 0, \quad 2\mu + 3\nu \geq 0.$$

The positive constants γ and θ present the competition between the kinetic energy and the potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. For simplicity, we set the constants γ and θ to be 1. The symbol $\otimes$ denotes the Kronecker tensor product such that $u \otimes u = (u_i u_j)_{1 \leq i,j \leq 3}$. To complete system (1.1), the initial data are given by

$$(\rho, u, d)(x,t)|_{t=0} = (\rho_0(x), u_0(x), d_0(x)). \tag{1.2}$$

Furthermore, as the space variable tends to infinity, we assume

$$\lim_{|x| \to \infty} (\rho_0 - 1, u_0, d_0 - w_0)(x) = 0, \tag{1.3}$$

where w_0 is a fixed unit constant vector. The system is a coupling between the compressible Navier-Stokes equations and a transported heat flow of harmonic maps into S^2. Generally speaking, we can obtain any better results for system (1.1) than those for the compressible Navier-Stokes equations.

The hydrodynamic theory of liquid crystals in the nematic case has been established by Ericksen [1] and Leslie [2] during the period of 1958 through 1968. Since then, the mathematical theory is still progressing and the study of the full Ericksen-Leslie model presents relevant mathematical difficulties. The pioneering work comes from [3-6]. For example, Lin and Liu [5] obtained the global weak and smooth solutions for the Ginzburg-Landau approximation to relax the nonlinear constraint $d \in S^2$. They also discussed the uniqueness and some stability properties of the system. Later, the decay rates for this approximate system were given by Wu [7] in a bounded domain. On the other hand, Dai et al. [8], Dai and Schonbek [9] established the time decay rates for the Cauchy problem respectively. More precisely, Dai and Schonbek [9] obtained the global existence of solutions in the Sobolev space $H^N(\mathbb{R}^3) \times H^{N+1}(\mathbb{R}^3)$ $(N \geq 1)$ only requiring the smallness of $\|u_0\|_{H^1}^2 + \|d_0 - w_0\|_{H^2}^2$, where w_0 is a fixed unit constant vector. If the initial data in L^1-norm are finitely additionally, they also established the following time decay rates

$$\|\nabla^k u(t)\|_{L^2} + \|\nabla^k (d - w_0)(t)\|_{L^2} \leq C(1+t)^{-\frac{3+2k}{4}},$$

for $k = 0, 1, 2, \cdots, N$. Recently, Liu and Zhang [10], for the density-dependent model, obtained the global weak solutions in dimension three with the initial density $\rho_0 \in L^2$, which was improved by Jiang and Tan [11] for the case $\rho_0 \in L^\gamma$ $(\gamma > \frac{3}{2})$. Under the constraint $d \in S^2$, Wen and Ding [12] established the local existence of the strong solutions and obtained the global solutions under the assumptions of small energy and positive initial density, which was improved by Li [13] to be of vacuum. Later, Hong [14] and Lin, Lin and Wang [15] showed independently the global existence of weak solutions in two-dimensional space. Recently, Wang [16] established a global well-posedness theory for rough initial data provided that $\|u_0\|_{\mathrm{BMO}^{-1}} + [d_0]_{BMO} \le \varepsilon_0$ for some $\varepsilon_0 > 0$. Under this condition, Du and Wang [17] obtained arbitrary space-time regularity for the Koch and Tataru type solution (u, d). As a corollary, they also got the decay rates. For more results, the readers can refer to [18-22] and the references therein.

Considering the compressible nematic liquid crystal flows (1.1), Ding, Lin, Wang and Wen [23] gained both the existence and uniqueness of global strong solutions for one dimensional space. And this result about the classical solutions was improved by Ding, Wang and Wen [24] by generalizing the fluids to be of vacuum. For the case of multi-dimensional space, Jiang, Jiang and Wang [25] established the global existence of weak solutions for the initial-boundary problem with large initial energy and without any smallness condition on the initial density and velocity if some component of initial direction field is small. Recently, Lin, Lai and Wang [26] established the existence of global weak solutions in three-dimensional space, provided the initial orientational director field d_0 lies in the hemisphere S_2^+. Local existence of unique strong solutions was proved provided that the initial datas were sufficiently regular and satisfied a natural compatibility condition in a recent work [27]. Some blow-up criterions that were derived for the possible breakdown of such local strong solutions at finite time could be found in [28–30]. The local existence and uniqueness of classical solutions to (1.1) were established by Ma in [31]. On one hand, Hu and Wu [32] obtained the existence and uniqueness of global strong solutions in critical Besov spaces provided that the initial data were close to an equilibrium state $(1, 0, \hat{d})$ with a constant vector $\hat{d} \in S^2$; on the other hand, Gao et al. [30] attained the global small classical solution in Sobolev spaces H^m $(m \ge 3)$ and established decay rates for the compressible nematic liquid crystal flows (1.1). For more results, the readers can refer to [34] for some recent developments of analysis for hydrodynamic flow of nematic liquid crystal flows and references therein.

Recently, Wang and Tan [35] established the global existence of strong solutions and built the time decay rates for the compressible Navier-Stokes equations in H^2-framework (See Matsumura and Nishida [36] in H^3-framework). Precisely, if small initial perturbation belongs to H^2 and initial perturbation in L^1-norm is finite, they built optimal time decay rates as follows

$$\|(\rho - 1)(t)\|_{H^{2-k}} + \|u(t)\|_{H^{2-k}} \le C(1+t)^{-\frac{3+2k}{4}},$$

where $k = 0, 1$. This framework of time convergence rates for compressible flows has been applied to other compressible models, refer to [37-39].

In this paper, motivated by the work [35], we hope to establish the global existence and time decay rates of strong solutions for the compressible nematic liquid crystal flows under the H^2-framework. First, we construct the global existence of strong solutions by the standard energy method under the condition that the initial data are close to the constant equilibrium state $(1, 0, w_0)$ (w_0 is a fixed unit constant vector) in H^2-framework. Second, if the initial data in L^1-norm are finite additionally, the optimal time decay rates of strong solutions are established by the method of Green function. Precisely, we obtain the following time decay rates for all $t \geq 0$

$$\|(\rho - 1)(t)\|_{H^{2-k}} + \|u(t)\|_{H^{2-k}} + \|(d - w_0)(t)\|_{H^{3-k}} \leq C(1 + t)^{-\frac{3+2k}{4}},$$

where $k = 0, 1$. Although angular momentum equations $(1.1)_3$ are nonlinear parabolic equations, we hope to establish optimal time decay rates for higher order spatial derivatives of director under the condition of small initial perturbation. Motivated by Lemma 3.2, we move the nonlinear terms to the right hand side of $(1.1)_3$ and deal with the nonlinear terms as external force with the property on fast time decay rates. Then, the optimal time decay rates for higher order spatial derivatives of director are built with the help of Fourier splitting method by Schonbek [40]. Finally, we also study the decay rates for the time derivatives of velocity and the mixed space-time derivatives of density and director.

Notation In this paper, we use $H^s(\mathbb{R}^3)$ $(s \in \mathbb{R})$ to denote the usual Sobolev spaces with the norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$ $(1 \leq p \leq \infty)$ to denote the usual L^p spaces with the norm $\|\cdot\|_{L^p}$. The symbol ∇^l with an integer $l \geq 0$ stands for the usual any spatial derivatives of order l. When l is not a positive integer, ∇^l stands for Λ^l defined by $\Lambda^l f := \mathscr{F}^{-1}(|\xi|^l \mathscr{F} f)$, where $\mathscr{F}$ is the usual Fourier transform operator ($\mathscr{F}(f) := \hat{f}$) and $\mathscr{F}^{-1}$ is its inverse. We will employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$ independent of time t. $a \approx b$ means $a \lesssim b$ and $b \lesssim a$. For simplicity, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$ and $\int f \mathrm{d}x := \int_{\mathbb{R}^3} f \mathrm{d}x$.

Now, we establish the first result concerning the global existence of solutions for the compressible nematic liquid crystal flows (1.1)-(1.3).

Theorem 1.1 *Assume that the initial data* $(\rho_0 - 1, u_0, \nabla d_0) \in H^2$, $|d_0(x)| = 1$ *in* $\mathbb{R}^3$ *and there exists a small constant* $\delta_0 > 0$ *such that*

$$\|(\rho_0 - 1, u_0, \nabla d_0)\|_{H^2} \leq \delta_0, \tag{1.4}$$

then problem (1.1)-(1.3) *admits a unique global solution* (ρ, u, d) *satisfying for all* $t \geq 0$,

$$\|(\rho-1,u,\nabla d)\|_{H^2}^2+\int_0^t(\|\nabla\rho\|_{H^1}^2+\|(\nabla u,\nabla^2 d)\|_{H^2}^2)\mathrm{d}\tau\le C\|(\rho_0-1,u_0,\nabla d_0)\|_{H^2}^2. \tag{1.5}$$

After obtaining the global existence of strong solutions at hand, we investigate the long-time behavior for the density, velocity and direction field.

Theorem 1.2 *Under the assumptions in Theorem* 1.1*, suppose the initial data* $\|d_0-w_0\|_{L^2}$ *and* $\|(\rho_0-1,u_0,d_0-w_0)\|_{L^1}$ *are finite additionally, then the solution* (ρ,u,d) *obtained in Theorem* 1.1 *satisfies for all* $t\ge 0$,

$$\begin{aligned}&\|\nabla^k(\rho-1)(t)\|_{H^{2-k}}+\|\nabla^k u(t)\|_{H^{2-k}}\le C(1+t)^{-\frac{3+2k}{4}},\\&\|\nabla^l(d-w_0)(t)\|_{L^2}\le C(1+t)^{-\frac{3+2l}{4}},\end{aligned} \tag{1.6}$$

where $k=0,1,$ *and* $l=0,1,2,3.$

Remark 1.1 For any $2\le p\le 6$, by virtue of Theorem 1.2 and the Sobolev interpolation inequality, we also obtain the following time decay rates:

$$\begin{aligned}&\|(\rho-1)(t)\|_{L^p}+\|u(t)\|_{L^p}\le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},\\&\|\nabla^k(d-w_0)(t)\|_{L^p}\le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k}{2}},\end{aligned}$$

where $k=0,1,2$. Furthermore, in the same manner, we also have

$$\begin{aligned}&\|(\rho-1)(t)\|_{L^\infty}+\|u(t)\|_{L^\infty}\le C(1+t)^{-\frac{5}{4}},\\&\|\nabla^k(d-w_0)(t)\|_{L^\infty}\le C(1+t)^{-\frac{3+k}{2}},\end{aligned}$$

where $k=0,1$.

Remark 1.2 Under the assumption of finiteness of $\|d_0-w_0\|_{L^2}$ in Theorem 1.2, one can obtain the rate of director $d(x,t)$ converging to the constant equilibrium state w_0 in $L^\infty(\mathbb{R}^3)$-norm.

Finally, we also study the convergence rates for time derivatives of velocity and mixed space-time derivatives of density and director.

Theorem 1.3 *Under the assumptions in Theorem* 1.2*, the global solution* (ρ,u,d) *of problem* (1.1)-(1.3) *has the following time decay rates for all* $t\ge 0$,

$$\begin{aligned}&\|\rho_t\|_{H^1}+\|u_t\|_{L^2}\le C(1+t)^{-\frac{5}{4}},\\&\|\nabla^k d_t\|_{L^2}\le C(1+t)^{-\frac{7+2k}{4}},\end{aligned} \tag{1.7}$$

where $k=0,1.$

This paper is organized as follows. In Section 2, we establish some energy estimates that will play an important role for us to construct the global existence of strong solutions. Then, we close the estimates by the standard continuity argument and the global existence of strong solutions follows immediately. In Section 3, we build the time decay rates by taking the method of Green function and establish optimal time decay rates for the higher order spatial derivatives of director. Finally, we also study the decay rates for the time derivatives of velocity and the mixed space-time derivatives of density and director.

2 Proof of Theorem 1.1

In this section, we construct the global existence of strong solutions for the compressible nematic liquid crystal flows (1.1)-(1.3). By a classical argument (see [36]), the global existence of solutions are obtained by combining the local existence result with a priori estimates. Since the local existence and uniqueness of strong solutions were established by Huang et al. [27], the global solutions follow in a standard continuity argument after we establish (1.5) a priori.

2.1 Energy estimates

Denoting $\varrho = \rho - 1$ and $n = d - w_0$, we rewrite (1.1) in the perturbation form as

$$\begin{cases} \varrho_t + \operatorname{div} u = S_1, \\ u_t - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u + \nabla\varrho = S_2, \\ n_t - \Delta n = S_3. \end{cases} \tag{2.1}$$

Here S_i $(i = 1, 2, 3)$ are defined as

$$\begin{cases} S_1 = -\varrho \operatorname{div} u - u\cdot\nabla\varrho, \\ S_2 = -u\cdot\nabla u - h(\varrho)[\mu\Delta u + (\mu+\nu)\nabla\operatorname{div} u] - f(\varrho)\nabla\varrho - g(\varrho)\nabla n\cdot\Delta n, \\ S_3 = -u\cdot\nabla n + |\nabla n|^2(n + w_0), \end{cases} \tag{2.2}$$

where the three nonlinear functions of ϱ are defined by

$$h(\varrho) := \frac{\varrho}{\varrho+1}, \quad f(\varrho) := \frac{P'(\varrho+1)}{\varrho+1} - 1, \quad g(\varrho) := \frac{1}{\varrho+1}. \tag{2.3}$$

The associated initial condition is given by

$$(\varrho, u, n)(x,t)|_{t=0} = (\varrho_0, u_0, n_0)(x). \tag{2.4}$$

Assume there exists a small positive constant δ satisfying the following estimate

$$\|(\varrho, u, \nabla n)(t)\|_{H^2} \le \delta, \tag{2.5}$$

for all $t \in [0, T]$. By virtue of (2.5) and Sobolev inequality, it is easy to get

$$\frac{1}{2} \le \varrho + 1 \le \frac{3}{2}.$$

Hence, we immediately have

$$|h(\varrho)|, |f(\varrho)| \le C|\varrho| \quad \text{and} \quad |g^{(k-1)}(\varrho)|, |h^{(k)}(\varrho)|, |f^{(k)}(\varrho)| \le C \quad \text{for any } k \ge 1, \tag{2.6}$$

which can be used frequently to derive a priori estimates. The following analytic tool has been proved in Wang and Tan [41]. For simplicity, we only state the results here and omit the proof for brevity.

Lemma 2.1 *Let $2 \le p \le \infty$ and $0 \le m, \alpha \le l$; when $p = \infty$ we require further that $m \le \alpha + 1$ and $l \ge \alpha + 2$. Then we have that for any $f \in C_0^\infty(\mathbb{R}^3)$,*

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|_{L^2}^{1-\theta} \|\nabla^l f\|_{L^2}^{\theta},$$

where $0 \le \theta \le 1$ and α satisfy

$$\alpha + 3\left(\frac{1}{2} - \frac{1}{p}\right) = m(1-\theta) + l\theta.$$

Remark 2.1 If $\|f\|_{H^2} \le M$, then according to Lemma 2.1 we obtain

$$\|\nabla^\alpha f\|_{L^2} \lesssim \|f\|_{L^2}^{1-\frac{\alpha}{2}} \|\nabla^2 f\|_{L^2}^{\frac{\alpha}{2}} \lesssim M,$$

for any $\alpha \in [0, 2]$. Hence, under assumption (2.5), it is easy to obtain

$$\|(\nabla^\alpha \varrho, \nabla^\alpha u, \nabla^\alpha \nabla n)(t)\|_{L^2} \lesssim \delta,$$

for any $\alpha \in [0, 2]$.

First of all, we will derive the following energy estimates.

Lemma 2.2 *Under condition* (2.5)*, then for $k = 0, 1$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k(\varrho, u, \nabla n)\|_{L^2}^2 + C\|\nabla^{k+1}(u, \nabla n)\|_{L^2}^2 \lesssim \delta \|\nabla^{k+1}\varrho\|_{L^2}^2. \tag{2.7}$$

Proof Taking k-th spatial derivatives to $(2.1)_1$ and $(2.1)_2$ respectively, multiplying the resulting identities by $\nabla^k \varrho$ and $\nabla^k u$ respectively and integrating over $\mathbb{R}^3$ (by parts), it is easy to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|\nabla^k \varrho|^2 + \nabla^k u|^2)\mathrm{d}x + \int (\mu|\nabla^{k+1}u|^2 + (\mu+\nu)|\nabla^k \mathrm{div} u|^2)\mathrm{d}x \\ &= \int \nabla^k S_1 \nabla^k \varrho \mathrm{d}x + \int \nabla^k S_2 \nabla^k u \mathrm{d}x.\end{aligned} \tag{2.8}$$

Taking $(k+1)$-th spatial derivatives to $(2.1)_3$, multiplying the resulting identities $\nabla^{k+1} n$ and integrating over $\mathbb{R}^3$ (by parts), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^{k+1} n|^2 \mathrm{d}x + \int |\nabla^{k+2} n|^2 \mathrm{d}x = \int \nabla^{k+1} S_3 \nabla^{k+1} n \mathrm{d}x. \tag{2.9}$$

Ading (2.8) to (2.9), it follows immediately that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(|\nabla^k\varrho|^2+|\nabla^k u|^2+|\nabla^{k+1}n|^2)\mathrm{d}x+\int(\mu|\nabla^{k+1}u|^2+(\mu+\nu)|\nabla^k\mathrm{div}u|^2+|\nabla^{k+2}n|^2)\mathrm{d}x\\&=\int\nabla^k S_1\nabla^k\varrho\mathrm{d}x+\int\nabla^k S_2\nabla^k u\mathrm{d}x+\int\nabla^{k+1}S_3\nabla^{k+1}n\mathrm{d}x.\end{aligned}\tag{2.10}$$

For the case $k=0$, the differential identity (2.10) has the following form

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(|\varrho|^2+|u|^2+|\nabla n|^2)\mathrm{d}x+\int(\mu|\nabla u|^2+(\mu+\nu)|\mathrm{div}u|^2+|\nabla^2 n|^2)\mathrm{d}x\\&=\int S_1\cdot\varrho\mathrm{d}x+\int S_2\cdot u\mathrm{d}x-\int S_3\cdot\Delta n\mathrm{d}x=I_1+I_2+I_3.\end{aligned}\tag{2.11}$$

Applying the Hölder, Sobolev and Young inequalities, it is easy to obtain

$$\begin{aligned}I_1&\le\|\varrho\|_{L^3}\|\mathrm{div}u\|_{L^2}\|\varrho\|_{L^6}+\|\varrho\|_{L^3}\|\nabla\varrho\|_{L^2}\|u\|_{L^6}\\&\lesssim\|\varrho\|_{H^1}\|\nabla u\|_{L^2}\|\nabla\varrho\|_{L^2}+\|\varrho\|_{H^1}\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2}\\&\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned}\tag{2.12}$$

Integrating by parts and applying (2.6), Hölder, Sobolev and Young inequalities, it arrives at directly

$$\begin{aligned}&-\int h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)u\mathrm{d}x\\&\approx\int(h'(\varrho)\nabla\varrho\cdot u+h(\varrho)\nabla u)\nabla u\mathrm{d}x\\&\lesssim\|\nabla\varrho\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^3}+\|\varrho\|_{L^\infty}\|\nabla u\|_{L^2}^2\\&\lesssim(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2)\\&\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned}\tag{2.13}$$

Hence, with the help of (2.6), Hölder, Sobolev and Young inequalities, we deduce

$$\begin{aligned}I_2&\lesssim(\|u\|_{L^3}\|\nabla u\|_{L^2}+\|\varrho\|_{L^3}\|\nabla\varrho\|_{L^2}+\|\nabla n\|_{L^3}\|\Delta n\|_{L^2})\|u\|_{L^6}+\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2)\\&\lesssim(\|u\|_{H^1}\|\nabla u\|_{L^2}+\|\varrho\|_{H^1}\|\nabla\varrho\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^2 n\|_{L^2})\|\nabla u\|_{L^2}+\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2)\\&\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla^2 n\|_{L^2}^2).\end{aligned}\tag{2.14}$$

By virtue of $|d|=1$ (that is, $|n+w_0|=1$), it follows immediately from the Hölder and Sobolev inequalities that

$$\begin{aligned}I_3&\le(\|u\|_{L^3}\|\nabla n\|_{L^6}+\|\nabla n\|_{L^3}\|\nabla n\|_{L^6})\|\nabla^2 n\|_{L^2}\\&\lesssim(\|u\|_{H^1}+\|\nabla n\|_{H^1})\|\nabla^2 n\|_{L^2}^2\\&\lesssim\delta\|\nabla^2 n\|_{L^2}^2.\end{aligned}\tag{2.15}$$

Substituting (2.12), (2.14) and (2.15) into (2.11) completes the proof of (2.7) for the case of $k=0$. Now, we turn to give the proof of (2.7) for the case of $k=1$. Indeed, taking $k=1$ in (2.10) and integrating by part yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(|\nabla\varrho|^2+|\nabla u|^2+|\nabla^2 n|^2)\mathrm{d}x+\int(\mu|\nabla^2 u|^2+(\mu+\nu)|\nabla\mathrm{div}u|^2+|\nabla^3 n|^2)\mathrm{d}x\\&=-\int S_1\Delta\varrho\mathrm{d}x-\int S_2\Delta u\mathrm{d}x-\int\nabla S_3\nabla\Delta n\mathrm{d}x=\mathit{II}_1+\mathit{II}_2+\mathit{II}_3.\end{aligned}\tag{2.16}$$

Applying Hölder, Sobolev and Young inequalities, we obtain

$$\begin{aligned}\mathit{II}_1&\le(\|\varrho\|_{L^3}\|\mathrm{div}u\|_{L^6}+\|u\|_{L^3}\|\nabla\varrho\|_{L^6})\|\nabla^2\varrho\|_{L^2}\\&\lesssim(\|\varrho\|_{H^1}+\|u\|_{H^1})(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2)\\&\lesssim\delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2).\end{aligned}\tag{2.17}$$

Similarly, it is easy to deduce

$$\begin{aligned}\mathit{II}_2\le&(\|u\|_{L^3}\|\nabla u\|_{L^6}+\|\varrho\|_{L^\infty}\|\nabla^2 u\|_{L^2})\|\nabla^2 u\|_{L^2}\\&+(\|\varrho\|_{L^3}\|\nabla\varrho\|_{L^6}+\|\nabla n\|_{L^3}\|\Delta n\|_{L^6})\|\nabla^2 u\|_{L^2}\\\lesssim&(\|\varrho\|_{H^2}+\|u\|_{H^1}+\|\nabla n\|_{H^1})(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 n\|_{L^2}^2)\\\lesssim&\delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 n\|_{L^2}^2),\end{aligned}\tag{2.18}$$

and

$$\begin{aligned}\mathit{II}_3\le&(\|\nabla u\|_{L^6}\|\nabla n\|_{L^3}+\|u\|_{L^3}\|\nabla^2 n\|_{L^6})\|\nabla^3 n\|_{L^2}\\&+(\|\nabla n\|_{L^3}\|\nabla^2 n\|_{L^6}+\|\nabla n\|_{L^6}^3)\|\nabla^3 n\|_{L^2}\\\lesssim&(\|u\|_{H^1}+\|\nabla n\|_{H^1}+\|\nabla^{\frac{3}{2}}n\|_{L^2}^2)(\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 n\|_{L^2}^2)\\\lesssim&\delta(\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 n\|_{L^2}^2).\end{aligned}\tag{2.19}$$

Substituting (2.17)-(2.19) into (2.16), then we complete the proof of (2.7) for the case of $k=1$. The proof is completed.

Next, we derive the second type of energy estimates involving the higher order spatial derivatives of ϱ and u.

Lemma 2.3 *Under condition* (2.5)*, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2(\varrho,u,\nabla n)\|_{L^2}^2+C\|\nabla^3(u,\nabla n)\|_{L^2}^2\lesssim\delta\|\nabla^2\varrho\|_{L^2}^2.\tag{2.20}$$

Proof Taking 2-th spatial derivatives to $(2.1)_1$ and $(2.1)_2$ respectively, multiplying the resulting identities by $\nabla^2\varrho$ and $\nabla^2 u$ respectively and integrating over $\mathbb{R}^3$ (by parts), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(|\nabla^2\varrho|^2+|\nabla^2 u|^2)\mathrm{d}x+\int(\mu|\nabla^3 u|^2+(\mu+\nu)|\nabla^2\mathrm{div}u|^2)\mathrm{d}x\\&=\int\nabla^2 S_1\nabla^2\varrho\mathrm{d}x+\int\nabla^2 S_2\nabla^2 u\mathrm{d}x.\end{aligned}\tag{2.21}$$

Applying Hölder, Sobolev and Young inequalities, it is easy to obtain

$$\begin{aligned}
&-\int \nabla^2(\varrho \mathrm{div} u)\nabla^2\varrho \mathrm{d}x\\
=&-\int(\nabla^2\varrho \mathrm{div} u+2\nabla\varrho\nabla \mathrm{div} u+\varrho\nabla^2 \mathrm{div} u)\nabla^2\varrho \mathrm{d}x\\
\lesssim&(\|\nabla u\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}+\|\nabla\varrho\|_{L^3}\|\nabla^2 u\|_{L^6}+\|\varrho\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^2\varrho\|_{L^2}\\
\lesssim&(\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla^2\varrho\|_{L^2}+\|\nabla\varrho\|_{H^1}\|\nabla^3 u\|_{L^2}+\|\varrho\|_{H^2}\|\nabla^3 u\|_{L^2})\|\nabla^2\varrho\|_{L^2}\\
\lesssim&(\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^2\varrho\|_{L^2}^{\frac12}+\|\nabla\varrho\|_{H^1}+\|\varrho\|_{H^2})(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2)\\
\lesssim&\delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2).
\end{aligned}\tag{2.22}$$

Integrating by part and applying Hölder, Sobolev and Young inequalities, it arrives at

$$\begin{aligned}
-\int \nabla^2(u\cdot\nabla\varrho)\nabla^2\varrho \mathrm{d}x=&\int\Big[-(\nabla^2 u\nabla\varrho+2\nabla u\nabla^2\varrho)\nabla^2\varrho+\frac12|\nabla^2\varrho|^2 \mathrm{div} u\Big]\mathrm{d}x\\
\lesssim&(\|\nabla^2 u\|_{L^6}\|\nabla\varrho\|_{L^3}+\|\nabla u\|_{L^\infty}\|\nabla^2\varrho\|_{L^2})\|\nabla^2\varrho\|_{L^2}\\
\lesssim&\|\nabla\varrho\|_{H^1}\|\nabla^2\varrho\|_{L^2}\|\nabla^3 u\|_{L^2}+\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla^2\varrho\|_{L^2}^2\\
\lesssim&(\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^2\varrho\|_{L^2}^{\frac12}+\|\nabla\varrho\|_{H^1})(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2)\\
\lesssim&\delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2).
\end{aligned}\tag{2.23}$$

The combination of (2.22) and (2.23) gives rise to

$$\int \nabla^2 S_1\nabla^2\varrho \mathrm{d}x\lesssim\delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2).\tag{2.24}$$

Now, we turn to give the estimate for the second term on the right hand side of (2.21). First of all, by virtue of Hölder and Sobolev inequalities, we have

$$\begin{aligned}
\int \nabla(u\cdot\nabla u)\nabla\Delta u \mathrm{d}x=&\int(\nabla u\nabla u+u\nabla^2 u)\nabla\Delta u \mathrm{d}x\\
\le&\|\nabla u\|_{L^3}\|\nabla u\|_{L^6}\|\nabla^3 u\|_{L^2}+\|u\|_{L^3}\|\nabla^2 u\|_{L^6}\|\nabla^3 u\|_{L^2}\\
\lesssim&\|u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}+\|u\|_{H^1}\|\nabla^3 u\|_{L^2}^2\\
\lesssim&\delta\|\nabla^3 u\|_{L^2}^2.
\end{aligned}\tag{2.25}$$

In view of (2.6), Hölder and Sobolev inequalities, we have

$$\begin{aligned}
&\int \nabla(h(\varrho)(\mu\Delta+(\mu+\lambda)\nabla \mathrm{div} u))\nabla\Delta u dx\\
\lesssim&(\|\nabla\varrho\|_{L^3}\|\nabla^2 u\|_{L^6}+\|\varrho\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^3 u\|_{L^2}\\
\lesssim&(\|\nabla\varrho\|_{H^1}\|\nabla^3 u\|_{L^2}+\|\varrho\|_{H^2}\|\nabla^3 u\|_{L^2})\|\nabla^3 u\|_{L^2}\\
\lesssim&\delta\|\nabla^3 u\|_{L^2}^2
\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}\int \nabla(f(\varrho)\nabla\varrho)\nabla\Delta u \mathrm{d}x &\lesssim (\|\nabla\varrho\|_{L^4}^2 + \|\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2})\|\nabla^3 u\|_{L^2}\\ &\lesssim (\|\nabla^{\frac{3}{2}}\varrho\|_{L^2}\|\nabla^2\varrho\|_{L^2} + \|\varrho\|_{H^2}\|\nabla^2\varrho\|_{L^2})\|\nabla^3 u\|_{L^2}\\ &\lesssim (\|\nabla^{\frac{3}{2}}\varrho\|_{L^2} + \|\varrho\|_{H^2})\|\nabla^2\varrho\|_{L^2}\|\nabla^3 u\|_{L^2}\\ &\lesssim \delta(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2).\end{aligned} \tag{2.27}$$

Similarly, it is easy to deduce

$$\begin{aligned}&\int \nabla(g(\varrho)\nabla n\cdot\Delta n)\nabla\Delta u \mathrm{d}x\\ \lesssim\,& (\|\nabla n\|_{L^\infty}\|\nabla\varrho\|_{L^3}\|\Delta n\|_{L^6} + \|\nabla^2 n\|_{L^4}^2 + \|\nabla n\|_{L^3}\|\nabla^3 n\|_{L^6})\|\nabla^3 u\|_{L^2}\\ \lesssim\,& \|\nabla n\|_{H^2}\|\nabla^{\frac{5}{4}}\varrho\|_{L^2}^{\frac{2}{3}}\|\nabla^2\varrho\|_{L^2}^{\frac{1}{3}}\|\nabla n\|_{L^2}^{\frac{1}{3}}\|\nabla^4 n\|_{L^2}^{\frac{2}{3}}\|\nabla^3 u\|_{L^2}\\ &+ (\|\nabla^{\frac{3}{2}}n\|_{L^2}\|\nabla^4 n\|_{L^2} + \|\nabla n\|_{H^1}\|\nabla^4 n\|_{L^2})\|\nabla^3 u\|_{L^2}\\ \lesssim\,& (\|\nabla n\|_{H^2}\|\nabla^{\frac{5}{4}}\varrho\|_{L^2}^{\frac{2}{3}}\|\nabla n\|_{L^2}^{\frac{1}{3}} + \|\nabla^{\frac{3}{2}}n\|_{L^2} + \|\nabla n\|_{H^1})(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2)\\ \lesssim\,& \delta(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2).\end{aligned} \tag{2.28}$$

Combining (2.25)-(2.27) with (2.28), we deduce

$$\int \nabla^2 S_2\nabla^2 u \mathrm{d}x \lesssim \delta(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2). \tag{2.29}$$

Inserting (2.24) and (2.29) into (2.21), it arrives at immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(|\nabla^2\varrho|^2 + |\nabla^2 u|^2)\mathrm{d}x + \int|\nabla^3 u|^2\mathrm{d}x \lesssim \delta(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2). \tag{2.30}$$

Taking 3-th spatial derivatives to $(2.1)_3$, multiplying the resulting identities by $\nabla^3 n$ and integrating over $\mathbb{R}^3$ (by parts), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla^3 n|^2\mathrm{d}x + \int|\nabla^4 n|^2\mathrm{d}x = \int\nabla^3 S_3\cdot\nabla^3 n\mathrm{d}x. \tag{2.31}$$

The application of Hölder, Sobolev and Young inequalities, it is easy to deduce

$$\begin{aligned}\int\nabla^3 S_3\cdot\nabla^3 n\mathrm{d}x \lesssim\,& (\|\nabla^2 u\|_{L^6}\|\nabla n\|_{L^3} + \|\nabla u\|_{L^3}\|\nabla^2 n\|_{L^6} + \|u\|_{L^3}\|\nabla^3 n\|_{L^6}\\ &+ \|\nabla^2 n\|_{L^4}^2 + \|\nabla n\|_{L^3}\|\nabla^3 n\|_{L^6} + \|\nabla n\|_{L^6}^2\|\nabla^2 n\|_{L^6})\|\nabla^4 n\|_{L^2}\\ \lesssim\,& (\|\nabla n\|_{H^1}\|\nabla^3 u\|_{L^2} + \|\nabla^{\frac{3}{4}}u\|_{L^2}^{\frac{2}{3}}\|\nabla^2 u\|_{L^2}^{\frac{1}{3}}\|\nabla n\|_{L^2}^{\frac{1}{3}}\|\nabla^4 n\|_{L^2}^{\frac{2}{3}}\\ &+ \|u\|_{H^1}\|\nabla^4 n\|_{L^2} + \|\nabla^{\frac{3}{2}}n\|_{L^2}\|\nabla^4 n\|_{L^2} + \|\nabla n\|_{H^1}\|\nabla^4 n\|_{L^2}\\ &+ \|\nabla n\|_{L^2}^{\frac{4}{3}}\|\nabla^4 n\|_{L^6}^{\frac{2}{3}}\|\nabla^{\frac{5}{2}}n\|_{L^2}^{\frac{2}{3}}\|\nabla^4 n\|_{L^2}^{\frac{1}{3}})\|\nabla^4 n\|_{L^2}\\ \lesssim\,& \delta(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2).\end{aligned} \tag{2.32}$$

Substituting (2.32) into (2.31), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla^3 n|^2\mathrm{d}x+\int|\nabla^4 n|^2\mathrm{d}x\lesssim\delta\|\nabla^3 u\|_{L^2}^2. \tag{2.33}$$

The combination of (2.30) and (2.33) completes the proof of lemma.

Finally, we will use equations (2.1) to recover the dissipation estimate for ϱ.

Lemma 2.4 *Under condition* (2.5)*, then for* $k=0,1$*, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x+C\|\nabla^{k+1}\varrho\|_{L^2}^2\lesssim\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+3}n\|_{L^2}^2. \tag{2.34}$$

Proof Taking k-th spatial derivatives to the second equation of (2.1), multiplying by $\nabla^{k+1}\varrho$ and integrating over $\mathbb{R}^3$, then we obtain

$$\begin{aligned}&\int\nabla^k u_t\cdot\nabla^{k+1}\varrho\mathrm{d}x+\int|\nabla^{k+1}\varrho|^2\mathrm{d}x\\=&\int\nabla^k[\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u]\nabla^{k+1}\varrho\mathrm{d}x+\int\nabla^k S_2\nabla^{k+1}\varrho\mathrm{d}x.\end{aligned} \tag{2.35}$$

In order to deal with $\int\nabla^k u_t\cdot\nabla^{k+1}\varrho\mathrm{d}x$, following the idea in Guo and Wang [42], we turn the time derivatives of velocity to the density. Then, applying the mass equation $(2.1)_1$, we can transform time derivatives to the spatial derivatives, that is,

$$\begin{aligned}&\int\nabla^k u_t\cdot\nabla^{k+1}\varrho dxd\\=&\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x-\int\nabla^k u\cdot\nabla^{k+1}\varrho_t\mathrm{d}x\\=&\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x+\int\nabla^k u\cdot\nabla^{k+1}(\mathrm{div}u+\mathrm{div}(\varrho u))\mathrm{d}x\\=&\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x-\int\nabla^k\mathrm{div}u\cdot\nabla^k(\mathrm{div}u+\mathrm{div}(\varrho u))\mathrm{d}x\\=&\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^k u\cdot\nabla^{k+1}\varrho dx-\int|\nabla^k\mathrm{div}u|^2\mathrm{d}x-\int\nabla^k\mathrm{div}u\cdot\nabla^k\mathrm{div}(\varrho u)\mathrm{d}x.\end{aligned} \tag{2.36}$$

Substituting (2.36) into (2.35), it is easy to deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x+\int|\nabla^{k+1}\varrho|^2\mathrm{d}x\\=&\int|\nabla^k\mathrm{div}u|^2\mathrm{d}x+\int\nabla^k\mathrm{div}u\cdot\nabla^k\mathrm{div}(\varrho u)\mathrm{d}x+\int\nabla^k S_2\nabla^{k+1}\varrho\mathrm{d}x\\&+\int\nabla^k[\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u]\nabla^{k+1}\varrho\mathrm{d}x.\end{aligned} \tag{2.37}$$

For the case $k=0$, applying Hölder, Sobolev and Young inequalities, we obtain

$$\begin{aligned}\int \mathrm{div}u\cdot\mathrm{div}(\varrho u)\mathrm{d}x &\lesssim \|\varrho\|_{L^\infty}\|\nabla u\|_{L^2}^2+\|u\|_{L^3}\|\mathrm{div}u\|_{L^6}\|\nabla\varrho\|_{L^2}\\ &\lesssim (\|\varrho\|_{H^2}+\|u\|_{H^1})(\|\nabla\varrho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2)\\ &\lesssim \delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2).\end{aligned}\tag{2.38}$$

By virtue of Hölder inequality and (2.5), it is easy to deduce

$$\begin{aligned}\int S_2\ \nabla\varrho\mathrm{d}x &\lesssim (\|u\|_{L^3}\|\nabla u\|_{L^6}+\|\varrho\|_{L^\infty}\|\nabla^2u\|_{L^2})\|\nabla\varrho\|_{L^2}\\ &\quad+(\|\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^2}+\|\nabla n\|_{L^3}\|\Delta n\|_{L^6})\|\nabla\varrho\|_{L^2}\\ &\lesssim \delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2+\|\nabla^3n\|_{L^2}^2)\end{aligned}\tag{2.39}$$

and

$$\int[\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u]\nabla\varrho\mathrm{d}x\le\frac{1}{2}\|\nabla\varrho\|_{L^2}^2+\frac{1}{2}\|\nabla^2u\|_{L^2}^2.\tag{2.40}$$

The combination of (2.38), (2.39) and (2.40) complete the proof of (2.34) for the case of $k=0$. As for the case $k=1$, applying Hölder, Sobolev and Young inequalities, we deduce

$$\begin{aligned}\int \nabla\mathrm{div}u\cdot\nabla\mathrm{div}(\varrho u)\mathrm{d}x &\lesssim (\|\nabla\varrho\|_{L^3}\|\mathrm{div}u\|_{L^6}+\|\varrho\|_{L^\infty}\|\nabla\mathrm{div}u\|_{L^2})\|\nabla^2u\|_{L^2}\\ &\quad+(\|\nabla\varrho\|_{L^3}\|\nabla u\|_{L^6}+\|u\|_{L^\infty}\|\nabla^2\varrho\|_{L^2})\|\nabla^2u\|_{L^2}\\ &\lesssim \delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2).\end{aligned}\tag{2.41}$$

With the help of Hölder inequality and Lemma 2.3, it arrives at

$$\int\nabla S_2\ \nabla^2\varrho\mathrm{d}x\lesssim\delta(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3u\|_{L^2}^2+\|\nabla^4n\|_{L^2}^2),\tag{2.42}$$

and

$$\int\nabla[\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u]\nabla^2\varrho\mathrm{d}x\le\frac{1}{2}\|\nabla^2\varrho\|_{L^2}^2+\frac{1}{2}\|\nabla^3u\|_{L^2}^2.\tag{2.43}$$

The combination of (2.41), (2.42) and (2.43) gives rise to the proof of (2.34) for the case of $k=1$. The proof is completed.

2.2 Global existence of solutions

In this subsection, we shall combine the energy estimates derived in the previous section to prove the global existence of strong solutions in Theorem 1.1. Summing up (2.7) from $k=l$ $(l=0,1)$ to $k=1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^l(\varrho,u,\nabla n)\|_{H^{1-l}}^2+C\|\nabla^l(\nabla u,\nabla^2n)\|_{H^{1-l}}^2\lesssim\delta\|\nabla^{l+1}\varrho\|_{H^{1-l}}^2,$$

which, together with (2.20), arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^l(\varrho,u,\nabla n)\|_{H^{2-l}}^2+C\|\nabla^{l+1}(u,\nabla n)\|_{H^{2-l}}^2\le\delta C_1\|\nabla^{l+1}\varrho\|_{H^{1-l}}^2.\tag{2.44}$$

On the other hand, summing (2.34) from $k = l$ ($l = 0, 1$) to $k = 1$, we obtain immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\le k\le 1}\int \nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x + C_2\|\nabla^{l+1}\varrho\|^2_{H^{1-l}} \le C_3\big(\|\nabla^{l+1}u\|^2_{H^{2-l}} + \|\nabla^{l+3}n\|^2_{H^{1-l}}\big). \tag{2.45}$$

Multiplying (2.45) by $2\delta C_1/C_2$ and adding the resulting inequality to (2.44), it arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l^2(t) + C_3\big(\|\nabla^{l+1}\varrho\|^2_{H^{1-l}} + \|\nabla^{l+1}(u,\nabla n)\|^2_{H^{2-l}}\big) \le 0, \tag{2.46}$$

where $\mathcal{E}_l^2(t)$ is defined as

$$\mathcal{E}_l^2(t) = \|\nabla^l(\varrho, u, \nabla n)\|^2_{H^{2-l}} + \frac{2\delta C_1}{C_2}\sum_{l\le k\le 1}\int \nabla^k u\cdot\nabla^{k+1}\varrho\mathrm{d}x.$$

By virtue of the smallness of δ, it is easy to obtain

$$C_4^{-1}\|\nabla^l(\varrho, u, \nabla n)\|^2_{H^{2-l}} \le \mathcal{E}_l^2(t) \le C_4\|\nabla^l(\varrho, u, \nabla n)\|^2_{H^{2-l}}. \tag{2.47}$$

Choosing $l = 0$ in (2.46) and integrating over $[0, t]$ yield

$$\|(\varrho, u, \nabla n)(t)\|_{H^2} \le C\|(\varrho_0, u_0, \nabla n_0)\|_{H^2}. \tag{2.48}$$

Since $\|(\varrho, u, \nabla n)(t)\|_{H^2}$ is a continuous function with respect to time (see [27]), there exists a small and positive constant T_0 such that

$$\max_{0\le t\le T_0}\|(\varrho, u, \nabla n)(t)\|_{H^2} \le 2\|(\varrho_0, u_0, \nabla n_0)\|_{H^2}. \tag{2.49}$$

Choosing

$$\|(\varrho_0, u_0, \nabla n_0)\|_{H^2} \le \min\left\{\frac{\delta}{2}, \frac{\delta}{2C}\right\},$$

this, together with (2.49), gives directly

$$\max_{0\le t\le T_0}\|(\varrho, u, \nabla n)(t)\|_{H^2} \le \delta.$$

Then, applying estimate (2.48), it is easy to deduce

$$\max_{0\le t\le T_0}\|(\varrho, u, \nabla n)(t)\|_{H^2} \le \frac{\delta}{2}.$$

Thus, problem (2.1)-(2.4) with the initial data $(\varrho, u, \nabla n)(x, T_0)$ admits a unique solution on $[T_0, 2T_0]\times\mathbb{R}^3$ satisfying the estimate

$$\max_{T_0\le t\le 2T_0}\|(\varrho, u, \nabla n)(t)\|_{H^2} \le 2\|(\varrho, u, \nabla n)(T_0)\|_{H^2} \le \delta,$$

which, together with (2.48), yields directly

$$\max_{0\le t\le 2T_0}\|(\varrho,u,\nabla n)(t)\|_{H^2}\le\frac{\delta}{2}.$$

Thus, we can continue the same process for $0\le t\le nT_0$ $(n=1,2,\cdots)$ and finally get a global solution on $[0,\infty)\times\mathbb{R}^3$. The uniqueness of global strong solutions follows immediately from the uniqueness of local existence of solutions. Choosing $l=0$ in (2.46), integrating over $[0,t]$ and applying the equivalent relation of (2.47), we obtain

$$\|(\varrho,u,\nabla n)\|_{H^2}^2+\int_0^t(\|\nabla\varrho\|_{H^1}^2+\|(\nabla u,\nabla^2 n)\|_{H^2}^2)\mathrm{d}\tau\le C\|(\varrho_0,u_0,\nabla n_0)\|_{H^2}^2,$$

which completes the proof of Theorem 1.1.

3 Proof of Theorems 1.2 and 1.3

In this section, we will establish the time decay rates for the compressible nematic liquid crystal flows (1.1)-(1.3). First of all, the decay rates are built by the method of the Green function. Secondly, motivated by Lemma 3.2, we enhance the time decay rates for the higher order derivatives of director. Finally, we also establish the convergence rates for the time derivatives of density, velocity and director.

3.1 Decay rates for the nonlinear systems

First of all, let us to consider the following linearized systems

$$\begin{cases}\varrho_t+\mathrm{div}u=0,\\ u_t-\mu\Delta u-(\mu+\nu)\nabla\mathrm{div}u+\nabla\varrho=0,\\ n_t-\Delta n=0,\end{cases}\tag{3.1}$$

with the initial data

$$(\varrho,u,n)|_{t=0}=(\varrho_0,u_0,n_0).\tag{3.2}$$

Obviously, the solution (ϱ,u,n) for the linear problem (3.1)-(3.2) can be expressed as

$$(\varrho,u,n)^{tr}=G(t)*(\varrho_0,u_0,n_0)^{tr},\quad t\ge 0.\tag{3.3}$$

Here $G(t):=G(x,t)$ is the Green matrix for system (3.1) and the exact expression of the Fourier transform $\widehat{G}(\xi,t)$ of Green function $G(x,t)$ as

$$\widehat{G}(\xi,t)=\begin{pmatrix}\dfrac{\lambda_+\mathrm{e}^{\lambda_-t}-\lambda_-\mathrm{e}^{\lambda_+t}}{\lambda_+-\lambda_-} & \dfrac{-i\xi^t(\mathrm{e}^{\lambda_+t}-\mathrm{e}^{\lambda_-t})}{\lambda_+-\lambda_-} & 0\\ \dfrac{-i\xi(\mathrm{e}^{\lambda_+t}-\mathrm{e}^{\lambda_-t})}{\lambda_+-\lambda_-} & \dfrac{\lambda_+\mathrm{e}^{\lambda_+t}-\lambda_-\mathrm{e}^{\lambda_-t}}{\lambda_+-\lambda_-}\dfrac{\xi\xi^t}{|\xi|^2}+\mathrm{e}^{\lambda_0t}\left(I_{3\times3}-\dfrac{\xi\xi^t}{|\xi|^2}\right) & 0\\ 0 & 0 & \mathrm{e}^{\lambda_1t}I_{3\times3}\end{pmatrix},$$

where

$$\lambda_0 = -\mu|\xi|^2, \quad \lambda_1 = -|\xi|^2,$$
$$\lambda_+ = -\Big(\mu + \frac{1}{2}\nu\Big)|\xi|^2 + i\sqrt{|\xi|^2 - \Big(\mu + \frac{1}{2}\nu\Big)^2|\xi|^4},$$
$$\lambda_- = -\Big(\mu + \frac{1}{2}\nu\Big)|\xi|^2 - i\sqrt{|\xi|^2 - \Big(\mu + \frac{1}{2}\nu\Big)^2|\xi|^4}.$$

Since systems (3.1) is an independent coupling of the classical linearized Navier-Stokes equation and heat equation, the representation of Green function $\widehat{G}(\xi,t)$ is easy to be verified. Furthermore, we have the following decay rates for systems (3.1)-(3.2), refer to [33, 43].

Proposition 3.1 *Assume that* (ϱ, u, n) *is a solution of the linearized compressible nematic liquid crystal system* (3.1)-(3.2) *with the initial data* $(\varrho_0, u_0, n_0) \in L^1 \cap H^2$*, then*

$$\|\nabla^k \varrho\|_{L^2}^2 \le C\big(\|(\varrho_0, u_0)\|_{L^1}^2 + \|\nabla^k(\varrho_0, u_0)\|_{L^2}^2\big)(1+t)^{-\frac{3}{2}-k},$$
$$\|\nabla^k u\|_{L^2}^2 \le C\big(\|(\varrho_0, u_0)\|_{L^1}^2 + \|\nabla^k(\varrho_0, u_0)\|_{L^2}^2\big)(1+t)^{-\frac{3}{2}-k},$$
$$\|\nabla^k n\|_{L^2}^2 \le C\big(\|n_0\|_{L^1}^2 + \|\nabla^k n_0\|_{L^2}^2\big)(1+t)^{-\frac{3}{2}-k},$$

for $0 \le k \le 2$.

In the sequel, we want to verify some simplified inequalities that play an important role to derive the time decay rates for the compressible nematic liquid crystal flows (2.1)-(2.4). More precisely, we have

$$\begin{aligned} &\|(S_1, S_2, S_3)\|_{L^1} \lesssim \delta(\|\nabla\varrho\|_{L^2} + \|\nabla u\|_{H^1} + \|\nabla n\|_{H^1}), \\ &\|(S_1, S_2, S_3)\|_{L^2} \lesssim \delta(\|\nabla\varrho\|_{L^2} + \|\nabla u\|_{H^1} + \|\nabla n\|_{H^1}), \\ &\|\nabla(S_1, S_2, S_3)\|_{L^2} \lesssim \delta(\|\nabla^2\varrho\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla^2 n\|_{H^1}) + \|\nabla\varrho\|_{H^1}\|\nabla^3 u\|_{L^2}. \end{aligned} \tag{3.4}$$

Next, we establish decay rates for the compressible nematic liquid crystal flows (2.1)-(2.4).

Lemma 3.1 *Under the assumptions in Theorem* 1.2*, the global solution* (ϱ, u, n) *of problem* (2.1)-(2.4) *satisfies*

$$\|\nabla^k \varrho(t)\|_{H^{2-k}}^2 + \|\nabla^k u(t)\|_{H^{2-k}}^2 + \|\nabla^k n(t)\|_{H^{3-k}}^2 \le C(1+t)^{-\frac{3}{2}-k} \tag{3.5}$$

for $k = 0, 1$.

Proof First of all, taking $k = 0$ in (2.9), which together with inequality (2.15), we obtain the following inequality immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla n|^2 \mathrm{d}x + \int |\nabla^2 n|^2 \mathrm{d}x \le 0. \tag{3.6}$$

Taking $l=1$ specially in (2.46), it arrives at directly

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^2(t)+C_3\left(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2 u\|_{H^1}^2+\|\nabla^3 n\|_{H^1}^2\right)\leq 0,$$

which, together with (3.6), yields directly

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1^2(t)+C_4\left(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2 u\|_{H^1}^2+\|\nabla^2 n\|_{H^2}^2\right)\leq 0, \tag{3.7}$$

where $\mathcal{F}_1^2(t)$ is defined as

$$\mathcal{F}_1^2(t)=\|\nabla(\varrho,u)\|_{H^1}^2+\|\nabla n\|_{H^2}^2+\frac{2C_1\delta}{C_2}\int\nabla u\cdot\nabla^2\varrho\mathrm{d}x.$$

With the help of Young inequality, it is easy to deduce

$$C_5^{-1}\big(\,\|\nabla(\varrho,u)\|_{H^1}^2+\|\nabla n\|_{H^2}^2\,\big)\leq\mathcal{F}_1^2(t)\leq C_5\big(\,\|\nabla(\varrho,u)\|_{H^1}^2+\|\nabla n\|_{H^2}^2\,\big). \tag{3.8}$$

Ading both hand sides of (3.7) by $\|\nabla(\varrho,u,n)\|_{L^2}^2$ and applying the equivalent relation (3.8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1^2(t)+C\mathcal{F}_1^2(t)\leq\|\nabla(\varrho,u,n)\|_{L^2}^2. \tag{3.9}$$

In view of the Gronwall inequality, it follows immediately

$$\mathcal{F}_1^2(t)\leq\mathcal{F}_1^2(0)\mathrm{e}^{-Ct}+\int_0^t\mathrm{e}^{-C(t-\tau)}\|\nabla(\varrho,u,n)\|_{L^2}^2\mathrm{d}\tau. \tag{3.10}$$

In order to derive the time decay rates for $\mathcal{F}_1^2(t)$, we need to control the term $\|\nabla(\varrho,u,n)\|_{L^2}^2$. In fact, by Duhamel principle, we can represent the solution for system (2.1)-(2.4) as

$$(\varrho,u,n)^{tr}(t)=G(t)*(\varrho_0,u_0,n_0)^{tr}+\int_0^t G(t-s)*(S_1,S_2,S_3)^{tr}(s)\mathrm{d}s. \tag{3.11}$$

Denoting $F(t)=\sup\limits_{0\leq\tau\leq t}(1+\tau)^{\frac{5}{2}}(\|\nabla\varrho(\tau)\|_{H^1}^2+\|\nabla u(\tau)\|_{H^1}^2+\|\nabla n(\tau)\|_{H^2}^2)$, by virtue of (3.4), (3.11) and Proposition 3.1, we have

$$\begin{aligned}\|\nabla(\varrho,u,n)\|_{L^2}&\leq C(1+t)^{-\frac{5}{4}}+C\int_0^t(\|(S_1,S_2,S_3)\|_{L^1}+\|\nabla(S_1,S_2,S_3)\|_{L^2})\,(1+t-\tau)^{-\frac{5}{4}}\mathrm{d}\tau\\&\leq C(1+t)^{-\frac{5}{4}}+C\int_0^t\delta\,(\|\nabla\varrho\|_{H^1}+\|\nabla u\|_{H^1}+\|\nabla n\|_{H^2})\,(1+t-\tau)^{-\frac{5}{4}}\mathrm{d}\tau\\&\quad+C\int_0^t\|\nabla\varrho\|_{H^1}\|\nabla^3 u\|_{L^2}(1+t-\tau)^{-\frac{5}{4}}\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\leq C(1+t)^{-\frac{5}{4}} + C\delta\sqrt{F(t)}\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}\mathrm{d}\tau \\
&\quad + C\sqrt{F(t)}\left[\int (1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{5}{2}}\mathrm{d}\tau\right]^{\frac{1}{2}}\left[\int_0^t \|\nabla^3 u(\tau)\|_{L^2}^2\mathrm{d}\tau\right]^{\frac{1}{2}} \\
&\leq C(1+t)^{-\frac{5}{4}} + C\delta\sqrt{F(t)}(1+t)^{-\frac{5}{4}} \\
&\leq (1+t)^{-\frac{5}{4}}(1+\delta\sqrt{F(t)}),
\end{aligned}
$$

where we have used the fact

$$
\begin{aligned}
&\int_0^t (1+t-\tau)^{-r}(1+\tau)^{-r}\mathrm{d}\tau \\
=&\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t (1+t-\tau)^{-r}(1+\tau)^{-r}\mathrm{d}\tau \\
\leq&\Big(1+\frac{t}{2}\Big)^{-r}\int_0^{\frac{t}{2}}(1+\tau)^{-r}\mathrm{d}\tau + \Big(1+\frac{t}{2}\Big)^{-r}\int_{\frac{t}{2}}^t (1+t-\tau)^{-r}\mathrm{d}\tau \\
\leq&(1+t)^{-r},
\end{aligned}
$$

for $r=\frac{5}{2}$ and $r=\frac{5}{4}$ respectively. Thus, we have the estimate

$$
\|\nabla(\varrho,u,n)\|_{L^2}^2 \leq C(1+t)^{-\frac{5}{2}}(1+\delta F(t)). \tag{3.12}
$$

Inserting (3.12) into (3.10), it follows immediately

$$
\begin{aligned}
\mathcal{F}_1^2(t) &\leq \mathcal{F}_1^2(0)\mathrm{e}^{-Ct} + C\int_0^t \mathrm{e}^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}}(1+\delta F(\tau))\mathrm{d}\tau \\
&\leq \mathcal{F}_1^2(0)\mathrm{e}^{-Ct} + C(1+\delta F(t))\int_0^t \mathrm{e}^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}}\mathrm{d}\tau \\
&\leq \mathcal{F}_1^2(0)\mathrm{e}^{-Ct} + C(1+\delta F(t))(1+t)^{-\frac{5}{2}} \\
&\leq C(1+\delta F(t))(1+t)^{-\frac{5}{2}},
\end{aligned} \tag{3.13}
$$

where we have used the fact

$$
\begin{aligned}
\int_0^t \mathrm{e}^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}}\mathrm{d}\tau &= \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \mathrm{e}^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}}\mathrm{d}\tau \\
&\leq \mathrm{e}^{-\frac{c}{2}t}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{5}{2}}\mathrm{d}\tau + \Big(1+\frac{t}{2}\Big)^{-\frac{5}{2}}\int_{\frac{t}{2}}^t \mathrm{e}^{-C(t-\tau)}\mathrm{d}\tau \\
&\leq C\,(1+t)^{-\frac{5}{2}}.
\end{aligned}
$$

Hence, by virtue of the definition of $F(t)$ and (3.13), it follows immediately

$$
F(t) \leq C(1+\delta F(t)),
$$

which, in view of the smallness of δ, gives

$$F(t) \le C.$$

Therefore, we have the following time decay rates

$$\|\nabla \varrho(t)\|_{H^1}^2 + \|\nabla u(t)\|_{H^1}^2 + \|\nabla n(t)\|_{H^2}^2 \le C(1+t)^{-\frac{5}{2}}. \tag{3.14}$$

On the other hand, by (3.4), (3.11), (3.14) and Proposition 3.1, it is easy to deduce

$$\begin{aligned}
\|(\varrho, u, n)\|_{L^2}^2 &\le C(1+t)^{-\frac{3}{2}} + C\int_0^t \left(\|(S_1, S_2, S_3)\|_{L^1}^2 + \|(S_1, S_2, S_3)\|_{L^2}^2\right)(1+t-\tau)^{-\frac{3}{2}}\mathrm{d}\tau \\
&\le C(1+t)^{-\frac{3}{2}} + C\int_0^t \delta\left(\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla n\|_{H^1}^2\right)(1+t-\tau)^{-\frac{3}{2}}\mathrm{d}\tau \\
&\le C(1+t)^{-\frac{3}{2}} + C\int_0^t (1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau \\
&\le C(1+t)^{-\frac{3}{2}},
\end{aligned}$$

where we have used the fact

$$\int_0^t (1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau \le C(1+t)^{-\frac{3}{2}}.$$

Hence, we have the following decay rates

$$\|(\varrho, u, n)(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}}. \tag{3.15}$$

Therefore, the combination of (3.14) and (3.15) completes the proof of the lemma.

3.2 Optimal decay rates for the higher order derivatives of director

In this subsection, we will enhance the time decay rates for the higher order spatial derivatives of direction field. This improvement is motivated by the following lemma.

Lemma 3.2 *For some smooth function $F(x,t)$, suppose the smooth function $v(x,t)$ is a solution of heat equation*

$$v_t(x,t) - \Delta v(x,t) = F(x,t), \tag{3.16}$$

for $(x,t) \in \mathbb{R}^3 \times R^+$ with the smooth initial data $v(x,0) = v_0(x)$. If the function $F(x,t)$ and the solution $v(x,t)$ have the time decay rates

$$\|\nabla^k v(t)\|_{L^2}^2 \le C(1+t)^{-(\frac{3}{2}+k)}, \quad \|\nabla^k F(t)\|_{L^2}^2 \le C(1+t)^{-\alpha}, \tag{3.17}$$

where $\alpha \ge k + \frac{7}{2}$. Then, we have the following time decay rate for the $(k+1)$-th order of spatial derivatives

$$\|\nabla^{k+1} v(t)\|_{L^2}^2 \le C(1+t)^{-(k+\frac{5}{2})}.$$

Proof Taking $(k+1)$-th spatial derivatives on both hand sides of (3.16), multiplying by $\nabla^{k+1}v$ and integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla^{k+1}v|^2\mathrm{d}x+\int|\nabla^{k+2}v|^2\mathrm{d}x\le\frac{1}{2}\int|\nabla^k F|^2\mathrm{d}x+\frac{1}{2}\int|\nabla^{k+2}v|^2\mathrm{d}x,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla^{k+1}v|^2\mathrm{d}x+\int|\nabla^{k+2}v|^2\mathrm{d}x\le\int|\nabla^k F|^2\mathrm{d}x. \tag{3.18}$$

For some constant R defined below, denoting the time sphere (see [40])

$$S_0=\Big\{\ \xi\in\mathbb{R}^3\big|\ |\xi|\le\Big(\frac{R}{1+t}\Big)^{\frac{1}{2}}\Big\},$$

it follows immediately

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla^{k+2}v|^2\mathrm{d}x&\ge\int_{\mathbb{R}^3/S_0}|\xi|^{2(k+2)}|\widehat{v}|^2\mathrm{d}\xi\\&\ge\frac{R}{1+t}\int_{\mathbb{R}^3/S_0}|\xi|^{2(k+1)}|\widehat{v}|^2\mathrm{d}\xi\\&\ge\frac{R}{1+t}\int_{\mathbb{R}^3}|\xi|^{2(k+1)}|\widehat{v}|^2\mathrm{d}\xi-\Big(\frac{R}{1+t}\Big)^2\int_{S_0}|\xi|^{2k}|\widehat{v}|^2\mathrm{d}\xi,\end{aligned}$$

or equivalently

$$\int_{\mathbb{R}^3}|\nabla^{k+2}v|^2\mathrm{d}x\ge\frac{R}{1+t}\int_{\mathbb{R}^3}|\nabla^{k+1}v|^2\mathrm{d}x-\Big(\frac{R}{1+t}\Big)^2\int_{\mathbb{R}^3}|\nabla^k v|^2\mathrm{d}x. \tag{3.19}$$

Choosing $R=k+3$ and combining inequalities (3.18), (3.19) and the time decay rates (3.17), it arrives at directly

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla^{k+1}v|^2\mathrm{d}x+\frac{k+3}{1+t}\int|\nabla^{k+1}v|^2\mathrm{d}x&\le\Big(\frac{k+3}{1+t}\Big)^2\int|\nabla^k v|^2\mathrm{d}x+\int|\nabla^k F|^2\mathrm{d}x\\&\le C(1+t)^{-(k+\frac{7}{2})}.\end{aligned} \tag{3.20}$$

Multiplying (3.20) by $(1+t)^{k+3}$ and integrating over $[0,t]$, we have

$$\|\nabla^{k+1}v(t)\|_{L^2}^2\le(1+t)^{-k-3}\big[\|\nabla^{k+1}v_0\|_{L^2}^2+C(1+t)^{\frac{1}{2}}\big],$$

which implies the time decay rates

$$\|\nabla^{k+1}v(t)\|_{L^2}^2\le C(1+t)^{-(k+\frac{5}{2})}.$$

Therefore, we complete the proof of the lemma.

Motivated by Lemma 3.2, we will improve the time decay rates for the second and third order derivatives of director.

Lemma 3.3 *Under the assumptions in Theorem* 1.2, *the global solution* (ϱ, u, n) *for problem* (2.1)-(2.4) *satisfies*

$$\|\nabla^k n(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}-k} \tag{3.21}$$

where $k = 2, 3$.

Proof Taking $k = 1$ in (2.9), it follows immediately

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^2 n|^2 \mathrm{d}x + \int |\nabla^3 n|^2 \mathrm{d}x = \int \nabla[u\cdot\nabla n - |\nabla n|^2(n+w_0)]\nabla^3 n \mathrm{d}x. \tag{3.22}$$

In view of (2.19), we have

$$-\int \nabla[|\nabla n|^2(n+w_0)]\nabla^3 n \mathrm{d}x \lesssim \delta\|\nabla^3 n\|_{L^2}^2. \tag{3.23}$$

By virtue of (3.5), Hölder, Sobolev and Young inequalities, it arrives at

$$\begin{aligned}\int \nabla(u\cdot\nabla n)\nabla^3 n\mathrm{d}x &\lesssim \|\nabla u\|_{L^3}\|\nabla n\|_{L^6}\|\nabla^3 n\|_{L^2} + \|u\|_{L^3}\|\nabla^2 n\|_{L^6}\|\nabla^3 n\|_{L^2}\\ &\lesssim \|\nabla u\|_{H^1}^2\|\nabla^2 n\|_{L^2}^2 + (\varepsilon+\delta)\|\nabla^3 n\|_{L^2}^2\\ &\lesssim (1+t)^{-\frac{5}{2}}(1+t)^{-\frac{5}{2}} + (\varepsilon+\delta)\|\nabla^3 n\|_{L^2}^2\\ &\lesssim (1+t)^{-5} + (\varepsilon+\delta)\|\nabla^3 n\|_{L^2}^2.\end{aligned} \tag{3.24}$$

Inserting (3.23) and (3.24) into (3.22) and applying the smallness of ε and δ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^2 n|^2\mathrm{d}x + \int |\nabla^3 n|^2 \mathrm{d}x \le C(1+t)^{-5}. \tag{3.25}$$

On the other hand, from inequality (2.31), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^3 n|^2 \mathrm{d}x + \int |\nabla^4 n|^2 \mathrm{d}x = \int \nabla^2[u\cdot\nabla n - |\nabla n|^2(n+w_0)]\nabla^4 n \mathrm{d}x. \tag{3.26}$$

By virtue of Hölder, Sobolev and Young inequalities, we obtain

$$\begin{aligned}&\int \nabla^2(u\cdot\nabla n)\nabla^4 n\mathrm{d}x\\ \lesssim\ &\|\nabla^2 u\|_{L^2}\|\nabla n\|_{L^\infty}\|\nabla^4 n\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla^2 n\|_{L^6}\|\nabla^4 n\|_{L^2} + \|u\|_{L^3}\|\nabla^3 n\|_{L^6}\|\nabla^4 n\|_{L^2}\\ \lesssim\ &\|\nabla^2 u\|_{L^2}^2\|\nabla^2 n\|_{L^2}\|\nabla^3 n\|_{L^2} + \|\nabla u\|_{H^1}^2\|\nabla^3 n\|_{L^2}^2 + (\varepsilon+\delta)\|\nabla^4 n\|_{L^2}^2\\ \lesssim\ &\|\nabla u\|_{H^1}^2\|\nabla^2 n\|_{H^1}^2 + (\varepsilon+\delta)\|\nabla^4 n\|_{L^2}^2.\end{aligned} \tag{3.27}$$

Following from the idea of inequality (2.32), we have

$$-\int \nabla^2[|\nabla n|^2(n+w_0)]\nabla^4 n\mathrm{d}x \lesssim \delta\|\nabla^4 n\|_{L^2}^2. \tag{3.28}$$

Inserting (3.27) and (3.28) into (3.26) and applying the smallness of ε and δ, it arrives at immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^3 n|^2\mathrm{d}x + \int |\nabla^4 n|^2\mathrm{d}x \lesssim \|\nabla u\|_{H^1}^2\|\nabla^2 n\|_{H^1}^2. \tag{3.29}$$

Combining (3.25) and (3.29) and applying the time decay rates (3.5), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (|\nabla^2 n|^2 + |\nabla^3 n|^2)\mathrm{d}x + \int (|\nabla^3 n|^2 + |\nabla^4 n|^2)\mathrm{d}x \le C(1+t)^{-5}. \tag{3.30}$$

Similar to the analysis of inequality (3.19), it follows immediately

$$\int |\nabla^3 n|^2\mathrm{d}x \ge \frac{4}{1+t}\int |\nabla^2 n|^2\mathrm{d}x - \Big(\frac{4}{1+t}\Big)^2\int |\nabla n|^2\mathrm{d}x, \tag{3.31}$$

and

$$\int |\nabla^4 n|^2\mathrm{d}x \ge \frac{5}{1+t}\int |\nabla^3 n|^2\mathrm{d}x - \Big(\frac{5}{1+t}\Big)^2\int |\nabla^2 n|^2\mathrm{d}x. \tag{3.32}$$

The combination of (3.30), (3.31) and (3.32) yields directly

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int (|\nabla^2 n|^2 + |\nabla^3 n|^2)\mathrm{d}x + \frac{4}{1+t}\int (|\nabla^2 n|^2 + |\nabla^3 n|^2)\mathrm{d}x\\
\le& \frac{25}{(1+t)^2}\int (|\nabla n|^2 + |\nabla^2 n|^2)\mathrm{d}x + C(1+t)^{-5}\\
\le& C(1+t)^{-\frac{9}{2}},
\end{aligned} \tag{3.33}$$

where have used the convergence rates (3.5). Multiplying (3.33) by $(1+t)^4$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^4(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2)\right] \le C(1+t)^{-\frac{1}{2}}. \tag{3.34}$$

Integrating (3.34) over $[0,t]$, we have the following decay rate

$$\|\nabla^2 n(t)\|_{L^2}^2 + \|\nabla^3 n(t)\|_{L^2}^2 \le C(1+t)^{-\frac{7}{2}}. \tag{3.35}$$

On the other hand, applying the convergence rates (3.5), (3.35) and inequality (3.29), it arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^3 n|^2\mathrm{d}x + \int |\nabla^4 n|^2\mathrm{d}x \lesssim (1+t)^{-\frac{5}{2}}(1+t)^{-\frac{7}{2}} \lesssim (1+t)^{-6},$$

which, together with (3.32) and (3.35), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla^3 n|^2\mathrm{d}x+\frac{5}{1+t}\int|\nabla^3 n|^2\mathrm{d}x\le C(1+t)^{-\frac{11}{2}}. \tag{3.36}$$

Multiplying (3.36) by $(1+t)^5$ and integrating over $[0,t]$, it follows immediately

$$\|\nabla^3 n(t)\|_{L^2}^2\le C(1+t)^{-\frac{9}{2}}.$$

Therefore, we complete the proof of the lemma.

Proof of Theorem 1.2 With the help of Lemmas 3.1 and 3.3, we complete the proof of Theorem 1.2.

Remark 3.1 In order to obtain the rate of $d(x,t)$ converging to w_0, we suppose the finiteness of $\|d_0-w_0\|_{L^2}$ in Theorem 1.2 additionally. Then, the density and velocity (ρ,u) enjoy the same decay rate with the director field $d(x,t)-w_0$. However, (ρ,u) will have the same decay rate with $\nabla(d(x,t)-w_0)$ without the assumption of finiteness of $\|d_0-w_0\|_{L^2}$.

3.3 Decay rates for the mixed space-time derivatives of density and velocity

In this subsection, we will establish the decay rates for the time derivatives of velocity and the mixed space-time derivatives of density and director.

Lemma 3.4 *Under the assumptions in Theorem* 1.2, *the global solution* (ϱ,u,n) *of problem* (2.1)-(2.4) *satisfies*

$$\begin{aligned}&\|\varrho_t(t)\|_{H^1}+\|u_t(t)\|_{L^2}\le C(1+t)^{-\frac{5}{4}},\\&\|\nabla^k n_t(t)\|_{L^2}\le C(1+t)^{-\frac{7+2k}{4}},\end{aligned}$$

for $k=0,1$.

Proof By virtue of equation $(2.1)_1$ and the convergence rates (1.6), we have

$$\begin{aligned}\|\varrho_t\|_{L^2}&=\|\mathrm{div}u+\varrho\mathrm{div}u+u\cdot\nabla\varrho\|_{L^2}\\&\le\|\mathrm{div}u\|_{L^2}+\|\varrho\|_{L^\infty}\|\mathrm{div}u\|_{L^2}+\|\nabla\varrho\|_{L^3}\|u\|_{L^6}\\&\le C(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Similarly, it follows immediately that

$$\begin{aligned}\|\nabla\varrho_t\|_{L^2}&=\|\nabla\mathrm{div}u+\nabla\varrho\mathrm{div}u+\varrho\nabla\mathrm{div}u+\nabla u\cdot\nabla\varrho+u\cdot\nabla^2\varrho\|_{L^2}\\&\lesssim\|\nabla\mathrm{div}u\|_{L^2}+\|\nabla\varrho\|_{L^3}\|\nabla u\|_{L^6}+\|u\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}\\&\lesssim\|\nabla^2 u\|_{L^2}+\|\nabla^2\varrho\|_{L^2}\\&\le C(1+t)^{-\frac{5}{4}},\end{aligned}$$

and

$$\begin{aligned}\|u_t\|_{L^2} \lesssim & \|\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u\|_{L^2}+\|\nabla\varrho\|_{L^2}+\|u\|_{L^3}\|\nabla u\|_{L^6}+\|h(\varrho)\|_{L^\infty}\|\mu\Delta u\\ & +(\mu+\nu)\nabla\mathrm{div}u\|_{L^2}+\|f(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^2}+\|g(\varrho)\|_{L^\infty}\|\nabla n\|_{L^\infty}\|\nabla^2 n\|_{L^2}\\ \lesssim & \|\nabla^2 u\|_{L^2}+\|\nabla\varrho\|_{L^2}+\|\nabla^2 n\|_{L^2}\\ \lesssim & (1+t)^{-\frac{5}{4}}+(1+t)^{-\frac{7}{4}}\\ \le & C(1+t)^{-\frac{5}{4}}.\end{aligned}$$

By virtue of $(2.1)_3$, (1.6), Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}\|n_t\|_{L^2} &= \|-u\cdot\nabla n+\Delta n+|\nabla n|^2(n+w_0)\|_{L^2}\\ &\lesssim \|u\|_{L^3}\|\nabla n\|_{L^6}+\|\Delta n\|_{L^2}+\|\nabla n\|_{L^3}\|\nabla n\|_{L^6}\\ &\lesssim \|u\|_{H^1}\|\nabla^2 n\|_{L^2}+\|\nabla^2 n\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^2 n\|_{L^2}\\ &\le C(1+t)^{-\frac{7}{4}}.\end{aligned}$$

In the same manner, it arrives at directly

$$\begin{aligned}\|\nabla n_t\|_{L^2} &= \|\nabla(-u\cdot\nabla n+\Delta n+|\nabla n|^2(n+w_0))\|_{L^2}\\ &\lesssim \|\nabla u\|_{L^3}\|\nabla n\|_{L^6}+\|u\|_{L^3}\|\nabla^2 n\|_{L^6}+\|\nabla\Delta n\|_{L^2}+\|\nabla n\|_{L^3}\|\nabla^2 n\|_{L^6}+\|\nabla n\|_{L^6}^3\\ &\lesssim \|\nabla u\|_{H^1}\|\nabla^2 n\|_{L^2}+\|\nabla^3 n\|_{L^2}+\|\nabla^2 n\|_{L^2}^3\\ &\lesssim (1+t)^{-\frac{5}{4}}(1+t)^{-\frac{7}{4}}+(1+t)^{-\frac{9}{4}}+(1+t)^{-\frac{21}{4}}\\ &\le C(1+t)^{-\frac{9}{4}}.\end{aligned}$$

Therefore, we complete the proof of the lemma.

Proof of Theorem 1.3 With the help of Lemma 3.4, we complete the proof of Theorem 1.3.

References

[1] J.L. Ericksen, Hydrostatic theory of liquid crystals, *Arch. Rational Mech. Anal.*, **9**(1962),371-378.

[2] F.M. Leslie, Some constitutive equations for liquid crystals, *Arch. Rational Mech. Anal.*, **28**(1968),265-283.

[3] R. Hardt, D. Kinderlehrer, F.H. Lin, Existence and partial regularity of static liquid configurations, *Commun. Math. Phys.*, **105**(1986),547-570.

[4] F.H. Lin, Nonlinear theory of defects in nematic liquid crystals: Phsse transition and flow phenomena, *Commun. Pure Appl. Math.*, **42**(1989),789-814.

[5] F.H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, *Commun. Pure Appl. Math.*, **48**(1995),501-537.

[6] F.H. Lin, C. Liu, Partial regularity of the dynamic system modeling the flow of liquid crystals, *Discrete Contin. Dyn. Syst.*, **2**(1996),1-22.

[7] H. Wu, Long-time behavior for nonlinear hydrodynamic system modeling the namatic liquid crystal flows, *Discrete Contin. Dyn. Syst.*, **26**(2010),379-396.

[8] M.M. Dai, J. Qing, M.E. Schonbek, Asymptotic behavior of solutions to the liquid crystals systems in $\mathbb{R}^3$, *Comm. Partial Differential Equations*, **37**(2012),2138-2164.

[9] M.M. Dai, M.E. Schonbek, Asymptotic behavior of solutions to the liquid crystal system in $H^m(\mathbb{R}^3)$, *SIAM J. Math. Anal.*, **46**(2014),3131-3150.

[10] X. Liu, Z. Zhang, Existence of the flow of liquid crystals system, *Chinese Ann. Math.*, **30A**(2009),1-20.

[11] F. Jiang, Z. Tan, Global weak solution to the flow of liquid crystals system, *Math. Methods Appl. Sci.*, **32**(2009),2243-2266.

[12] H. Y. Wen, S.J. Ding, Solution of incomressible hydrodynamic flow of liquid crystals, *Nonlinear Anal. Real World Appl.*, **12**(2011),1510-1531.

[13] J.K. Li, Global strong solutions to incompressible nematic liquid crystal flow, arXiv:1211.5864v1.

[14] M.C. Hong, Global existence of solutions of the simplified Ericksen-Leslie system in dimension two, *Calc. Var. Partial Differential Equations*, **40**(2011),15-36.

[15] F.H. Lin, J.Y. Lin, C.Y. Wang, Liquid crystal flows in two dimensions, *Arch. Ration. Mech. Anal.*, **197**(2010),297-336.

[16] C.Y. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, *Arch. Ration. Mech. Anal.*, **200**(2011),1-19.

[17] Y. Du, K.Y. Wang, Space-time regularity of the Koch and Tataru solutions to the liquid crystal equations, *SIAM J. Math. Anal.*, **45**(2013),3838-3853.

[18] Y. Du, K.Y. Wang, Regularity of the solutions to the liquid crystal equations with small rough data, *J. Differential Equations*, **256**(2014),65-81.

[19] J.K. Li, Global strong and weak solutions to nematic liquid crystal flow in two dimensions, *Nonlinear Anal.*, **99**(2014),80-94.

[20] X.L. Li, D.H. Wang, Global solution to the incompressible flow of liquid crystals, *J. Differential Equations*, **252**(2012),745-767.

[21] X.L. Li, D.H. Wang, Global strong solution to the density-dependent incompressible flow of liquid crystal, arXiv:1202.1011v1.

[22] Y.H. Hao, X.G. Liu, The existence and blow-up criterion of liquid crystals system in critical Besov space, arXiv:1305.1395v2.

[23] S.J. Ding, J.L. Lin, C.Y. Wang, H.Y. Wen, Compressible hydrodynamic flow of liquid crystals in 1-D, *Discrete Contin. Dyn. Syst.*, **32**(2012),539-563.

[24] S.J. Ding, C.Y. Wang, H.Y. Wen, Weak solution to comprssible hydrodynamic flow of liquid crystals in dimension one, *Discrete Contin. Dyn. Syst.*, **15**(2011),357-371.

[25] F. Jiang, S. Jiang, D.H. Wang, On multi-dimensional compressible flows of nematic liquid crystals with large initial energy in a bounded domain, *J. Funct. Anal.*, **265**(2013),3369-3397.

[26] J.Y. Lin, B.S. Lai, C.Y. Wang, Global finite energy weak solutions to the compressible nematic liquid crystal flow in dimension three, arXiv:1408.4149.

[27] T. Huang, C.Y. Wang, H.Y. Wen, Strong solutions of the compressible nematic liquid crystal, *J. Differential Equations*, **252**(2012),2222-2265.

[28] T. Huang, C.Y. Wang, H.Y. Wen, Blow up criterion for compressible nematic liquid crystal flows in dimension three, *Arch. Rational Mech. Anal.*, **204**(2012),285-311.

[29] X.D. Huang, Y. Wang, A Serrin criterion for compressible nematic liquid crystal flows, *Math. Meth. Appl. Sci.*, **36**(2013),1363-1375.

[30] J.C. Gao, Q. Tao, Z.A. Yao, A blowup criterion for the compressible nematic liquid crystal flows in dimension two, *J. Math. Anal. Appl.*, **415**(2014),33-52.

[31] S.X. Ma, Classical solutions for the compressible liquid crystal flows with nonnegative initial densities, *J. Math. Anal. Appl.*, **397**(2013),595-618.

[32] X.P. Hu, H. Wu, Global solution to the three-dimensional compressible flow of liquid crystals, *SIAM J. Math. Anal.*, **45**(2013),2678-2699.

[33] J.C. Gao, Q. Tao, Z.A. Yao, Long-time behavior of solution for the compressible nematic liquid crystal flows in $\mathbb{R}^3$, *J. Differential Equations*, **261**(2016),2334-2383.

[34] F.H. Lin, C.Y. Wang, Recent developments of analysis for hydrodynamic flow of nematic liquid crystals, arXiv:1408.4138.

[35] Y.J. Wang, Z. Tan, Global existence and optimal decay rate for the strong solutions in H^2 to the compressible Navier-Stokes equations, *Appl. Math. Lett.*, **24**(2011),1778-1784.

[36] A. Matsumura, T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.*, **20**(1980),67-104.

[37] X.P. Hu, G.C. Wu, Global existence and optimal decay rates for three-dimensional compressible viscoelastic flows, *SIAM J. Math. Anal.*, **45**(2013),2815-2833.

[38] W.J. Wang, W.K. Wang, Decay rates of the compressible Navier-Stokes-Korteweg equations with potential forces, *Discrete Contin. Dyn. Syst.*, **35**(2015).513-536.

[39] W.J. Wang, Large time behavior of solutions to the compressible Navier-Stokes equations with potential force, *J. Math. Anal. Appl.*, **423**(2015),1448-1468.

[40] M.E. Schonbek, L^2 decay for weak solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.*, **88**(1985),209-222.

[41] Z. Tan, Y. Wang, Global solution and large-time behavior of the 3D compressible Euler equations with damping, *J. Differential Equations*, **254**(2013),1686-1704.

[42] Y. Guo, Y.J. Wang, Decay of dissipative equations and negative Sobolev spaces, *Comm. Partial Differential Equations*, **37**(2012),2165-2208.

[43] R.J. Duan, H.X. Liu, S.J. Ukai, T. Yang, Optimal $L^p - L^q$ convergence rates for the compressible Navier-Stokes equations with potential force, *J. Differential Equations*, **238**(2007),220-233.

(*edited by Mengxin He*)