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WELL-POSEDNESS AND SPACE-TIME REGULARITY OF SOLUTIONS TO THE LIQUID CRYSTAL EQUATIONS IN CRITICAL SPACE*†

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Abstract

In this paper, we consider a hydrodynamic flow of nematic liquid crystal system. We prove the local well-posedness for the system in the critical Lebesgue space, and study the space-time regularity of the local solution.

Keywords space-time regularity; liquid crystal system; critical Sobolev space

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1 Introduction

In this paper, we consider the following hydrodynamic flow of nematic liquid crystal system:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d), \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ (u(x,t), d(x,t))|_{t=0} = (u_0(x), d_0(x)), \quad |d_0(x)| = 1, \end{cases} \tag{1.1}$$

which was proposed by Lin and Liu [25,26], as a simplified system of Ericksen-Leslie model. Here u is the velocity of the flow, $d(\cdot,t) : \mathbb{R}^n \to \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, is the unit vector field to depict the macroscopic molecular orientation of nematic liquid crystal material, P is pressure. We denote by $\nabla d \otimes \nabla d$ the 3×3-matrix whose (i,j)-entry is $\nabla_i d \cdot \nabla_j d$ and $1 \le i, j \le 3$.

The hydrodynamic theory of liquid crystal flow due to Ericksen and Leslie was developed in 1960's [5,6,21,22]. The model (1.1) is a simplified system of Ericksen-Leslie model, and it is a macroscopic continuum description of the time evolution of

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material under the influence of both the flow field $u(x,t)$ and the macroscopic description of the microscopic orientation configuration $d(x,t)$ of rod-like liquid crystal.

Many efforts on rigorous mathematical analysis of system (1.1) have been made, see [23,25-27,29] etc. Since the liquid crystal system (1.1) is a coupling system between the incompressible Navier-Stokes equations and the heat flow of harmonic maps, we shall first recall some results of Navier-Stokes equations as follows.

For the incompressible Navier-Stokes equations, in [19], Leray proved that for any finite square-integrable initial data there exists a (possibly not unique) global-in-time weak solution. Moreover, for two space dimensions case, [20] proved the uniqueness of the weak solution. Although the problems of uniqueness and regularity for $n \geq 3$ of Leray-Hopf weak solutions are still open, since the seminal work of Leray, there is an extensive literature on conditional results under various criteria. The most well-known condition is so-called Ladyzhenskaya-Prodi-Serrin condition, that is for some $T>0$, $u \in L^p(0,T;L^q(\mathbb{R}^n))$, where the pair (p,q) satisfies

$$\frac{2}{p}+\frac{n}{q} \leq 1, \quad q \in (n,+\infty]. \tag{1.2}$$

Under condition (1.2), the uniqueness of Leray-Hopf weak solutions was proved by Prodi [33] and Serrin [34], and the smoothness was obtained by Ladyzhenskaya [15]. The borderline case $(p,q)=(\infty,n)$ is much more subtle.

Subsequently, [8] proved the well-posedness for the Navier-Stokes equations in a scaling invariant space $\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$. The scaling invariant in the context of the Navier-Stokes equations is defined as: if a pair of functions $(u(x,t),P(x,t))$ solves the incompressible Navier-Stokes equations, then

$$(u_\lambda,P_\lambda)(x,t)=(\lambda u(\lambda x,\lambda^2 t),\lambda^2 P(\lambda x,\lambda^2 t)) \tag{1.3}$$

is also the solution of the incompressible Navier-Stokes equations with initial data $(u_\lambda(x,0),P_\lambda(x,0))=(\lambda u_0(\lambda x),\lambda^2 P_0(\lambda x))$. The spaces which are invariant under such a scaling are also called critical spaces. Examples of critical spaces for the Navier-Stokes in n dimensions are:

$$\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n) \subset L^n(\mathbb{R}^n) \subset \dot{B}^{-1+\frac{n}{p},p}_{p|p<\infty}(\mathbb{R}^n) \subset BMO^{-1}(\mathbb{R}^n) \subset B^{-1,\infty}_{\infty}(\mathbb{R}^n). \tag{1.4}$$

The study of the Navier-Stokes equations in critical spaces was initiated by Fujita-Kato [8,13], and continued by many authors, see [1,7,10,14,32] etc.

In 2003, Escauriaza, Seregin, and Sverak [7] obtained many perfect results, such as the backward uniqueness of the parabolic system and the regularity results for weak Leary-Hopf solutions u satisfying the additional condition $u \in \mathbb{L}^\infty(0,T;\mathbb{L}^3(\mathbb{R}^3))$, as well as the local well-posedness in the critical Lebesgue space, which verified the borderline case of (1.2) for $n=3$. The results of [7] is the borderline case for the

Ladyzhenskaya-Prodi-Serrin condition (1.2), which implied that the bound of weak solution in $\mathbb{L}^\infty(0,T;\mathbb{L}^3(\mathbb{R}^3))$ plays a crucial role to the uniqueness of the weak solutions to the Navier-Stokes equations. And for the borderline case of (1.2) with $n \geq 4$, the results were established by Du and Dong [3].

Subsequently, the space-time regularity for those local solutions in critical Lebesgue space of the Navier-Stokes equations was presented by [2] and [10]. Similarly the space-time analyticity results of the Navier-Stokes equations in other critical spaces, please see [9, 11, 16, 30, 31] etc.

Now, we turn to the liquid crystal system of (1.1). Recently, Lin, Lin, and Wang [24] studied the Dirichlet initial boundary value problem of (1.1), and proved the results that for any initial data $(u_0, d_0) \in L^2(\mathbb{R}^2) \times H^1(\Omega, S^2)$, there exists a global Leray-Hopf weak solution (u, d) that is smooth away from at most finitely many singularity times. Under the initial data $(u_0, d_0) \in BMO^{-1} \times BMO$, the local and global well-posedness were studied by Wang [35]. Very recently, in [12], the authors were established some Serrin type (not in borderline case, see (1.2)) and Beal-Kato-Majida type regularity criterion for the weak solution to (1.1) in $\mathbb{R}^3$. In [28], Lin and Wang proved the borderline case for the Serrin type criterion which is more intrinsic and difficult. For classical solutions to the Cauchy problem in the two-dimensional incompressible liquid crystal equation and the heat flows of harmonic maps equation, under a natural geometric angle condition, in [17], Lei, Li, and Zhang proved the global smooth solutions to a class of large initial data in energy space. After that, the existence of a pair of exact strong solutions to the 2D incompressible liquid crystal equations with finite energy was constructed by Dong and Lei [4].

Define

$$(u_\lambda, P_\lambda, d_\lambda)(x,t) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t), d(\lambda x, \lambda^2 t)), \tag{1.5}$$

then we can establish the critical space for the liquid crystal equations (1.1) as (1.4). There are similar results for the liquid crystal system (1.1) in the so-called critical spaces. For example in [29] and [35] the well-posedness to system (1.1) in critical Sobolev space $\mathbb{L}^n \times \dot{\mathbb{W}}^{1,n}$ and in $BMO \times BMO^{-1}$ were studied respectively.

Similar to the results of [7] of Navier-Stokes equations, the bound of the weak solutions (u, d) in the space $L^\infty([0,T); \mathbb{L}^n(\mathbb{R}^n)) \times L^\infty([0,T); \dot{\mathbb{W}}^{1,n}(\mathbb{R}^n))$ will be crucial to determine the uniqueness of the weak solutions to the liquid crystal equations (1.1), see [28]. In this paper, we shall present the well-posedness of the solutions to system (1.1) in critical Lebesgue spaces $L^\infty([0,T); \mathbb{L}^n(\mathbb{R}^n)) \times L^\infty([0,T); \dot{\mathbb{W}}^{1,n}(\mathbb{R}^n))$. Furthermore, the space-time regularity of the solutions are also presented.

There are several ingredients in this paper. Firstly, we shall prove the local well-posedness for system (1.1) in the critical Sobolev spaces. This part have extended the corresponding results of Navier-Stokes equations to the liquid crystal system. Subsequently, we shall study the space-time regularity of the local solutions, which implies not only the smoothness of the local solution, but also the decay rate about time t. To prove our results, we need to verify the space-time regularity of the local solution in the time interval $[0, T_0]$. By the standard method, it is easy to prove our results in $[0, T_1]$ with $T_1 \ll T_0$, and then some iteration method to verity our results always hold on $[T_1, T_0]$.

Our results are stated as follows.

Theorem 1.1(Local well-posedness) *Suppose that* (u_0, d_0) *is a pair of initial data of* (1.1) *with* $(u_0, d_0) \in L^n(\mathbb{R}^n) \times \dot{W}^{1,n}(\mathbb{R}^n)$*, then there exists a constant* $T > 0$*, which depends on* (u_0, d_0)*, such that system* (1.1) *admits a pair of unique solution* (u, d) *with the following properties:*

$$u, \nabla d \in C([0, T]; L^2(\mathbb{R}^n)) \cap L^2([0, T]; \dot{H}^1(\mathbb{R}^n)), \tag{1.6}$$

$$u, \nabla d \in C([0, T]; L^n(\mathbb{R}^n)) \cap L^{n+2}(0, T, \mathbb{R}^n) \cap L^{n+1}(0, T, \mathbb{R}^n). \tag{1.7}$$

Moreover, if the initial data satisfying $\|u_0\|_{L^n(\mathbb{R}^n)} + \|\nabla d_0\|_{L^n(\mathbb{R}^n)}$ *is small enough, then we can take* $T = +\infty$.

Remark 1.1 In [28], when $u \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap C([0, T); L^n(\mathbb{R}^n))$ and $d \in L^2(0, T; \dot{H}^1(\mathbb{R}^n)) \cap C([0, T); \dot{W}^{1,n}(\mathbb{R}^n))$, the Leray-Hopf type weak solution is unique on $\mathbb{R}^n \times [0, T]$, moreover, the local solution is smooth on $[0, T] \times \mathbb{R}^n$.

Theorem 1.2 *Let* (u, d) *be the local solution presented in Theorem* 1.1 *on* $[0, T]$*, then for any positive integers* k *and* m*, by letting* $(p, q) \in [2, \infty] \times [n, \infty]$ *with* $\frac{2}{p} + \frac{n}{q} = 1$*, we have*

$$t^{m+\frac{k}{2}} \partial_t^m \nabla^k u, t^{m+\frac{k}{2}} \partial_t^m \nabla^{k+1} d \in L^p(0, T, L^q(\mathbb{R}^n)). \tag{1.8}$$

This paper is organized as follows: In Section 2, we shall present some well-known results for the Leray Projector operator and some estimates for linear stokes system. In Section 3, the local well-posedness of (1.1) in the critical Lebesgue spaces is proved. The space-time regularity properties of the local solutions are proved in Section 4. Throughout this paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Preliminaries

At the beginning, we recall some properties for the Leary projection operator $\mathbb{P}$ to divergence free vector fields, which is defined by its matrix valued Fourier multiplier $\hat{\mathbb{P}}(\xi) = \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}$. For any multi-indices α, this symbol satisfies Mihlin-Hormander condition $\sup\limits_{|\xi|\neq 0}|\xi|^{\alpha}|\partial_{\xi}^{\alpha}\hat{\mathbb{P}}(\xi)|\leq C$. Furthermore, we have the following pointwise bound (see [18] Proposition 11.1).

Lemma 2.1 *Denote* $\mathrm{e}^{t\Delta}$ *as the heat operator,* n *as the space dimension and* $\widetilde{\mathbb{P}}(x,t)$ *as the kernel of* $\nabla^{k+1}\mathbb{P}\mathrm{e}^{t\Delta}$ *respectively, then there holds*

$$\widetilde{\mathbb{P}}(x,t)\leq C(k)\frac{1}{(\sqrt{t}+|x|)^{n+k+1}}, \tag{2.1}$$

where $C(k)$ *is a constant depending only on* k.

Lemma 2.2 *Let* $\mathbb{K}(x,t)=\frac{1}{\sqrt{t}^n}\mathrm{e}^{-\frac{|x|^2}{4t}}$*, then there exists a polynomial* $J^{k+2m}(\frac{|x|}{\sqrt{t}})$ *with degree* $k+2m$*, such that*

$$\partial_t^m\nabla^k\mathbb{K}(x,t)=\frac{1}{t^{m+\frac{k}{2}}}\mathbb{K}(x,t)J^{k+2m}\Big(\frac{|x|}{\sqrt{t}}\Big). \tag{2.2}$$

Proof It can be proved by induction.

We also need the following properties of the solution to the heat equation:

Lemma 2.3 *Denote* ϕ *to be the solution to the linear heat equation* $\partial_t\phi-\Delta\phi=0$ *with the initial data* $\phi|_{t=0}=\phi_0$*. Then for* $n\geq 2$ *there hold:*

(1) *For* $s\geq s_1\geq 1$*, denote* $\frac{1}{l}=\frac{n}{2}(\frac{1}{s_1}-\frac{1}{s})$,

$$\|\phi(\cdot,t)\|_{L^s(\mathbb{R}^n)}\leq C(s,l)t^{-1/l}\|\phi_0\|_{L^{s_1}(\mathbb{R}^n)}, \tag{2.3}$$

moreover, if $s>s_1\geq\max\{2,\frac{s(n-2)}{n}\}$ *then*

$$\|\phi(\cdot,t)\|_{L^l([0,T],L^s(\mathbb{R}^n))}\leq C(s,l)\|\phi_0\|_{L^{s_1}}. \tag{2.4}$$

Particularly, for the case $n=s_1\geq 2$, $s=l=n+2$*, we have*

$$\|\phi\|_{L^{n+2}(0,T,\mathbb{R}^n)}\leq\|\phi_0\|_{L^n(\mathbb{R}^n)}. \tag{2.5}$$

(2) *When the initial data* $\phi_0\in L^n(\mathbb{R}^n)$*, for any positive integers* M *and* K *we have*

$$\sum_{m=0}^{M}\sum_{k=0}^{K}\lim_{t\to 0}t^{m+\frac{k+1}{2}}\|\partial_t^m\nabla^k\phi\|_{L^\infty(\mathbb{R}^n)}=0. \tag{2.6}$$

(3) *For any positive integers* m,n *and* $p\in[n+2,+\infty]$, $q\in[n,n+2]$ *satisfying the condition* $\frac{2}{p}+\frac{n}{q}=1$*, we have*

$$\|t^{m+\frac{k}{2}}\partial_t^m\nabla^k\phi\|_{L^p(0,\infty;L^q(\mathbb{R}^n))}\leq C(m,k,n)\|\phi_0\|_{L^n(\mathbb{R}^n)}, \tag{2.7}$$

where $C(m,k,n)$ *is a constant depending on* m,k *and* n.

Proof The case (3) was proved by Dong-Du [2]. For the case (1), when $n=3$, it was proved by Lemma 7.1 of [7]. In fact (2.3) can easily be proved by using Young inequality. For the case $n = s_1 = 3$, $s = l = 5$, (2.4) is proved by [7] (see Lemma 7.1). We shall prove (2.4) for arbitrary dimensions $n \geq 2$ for completeness.

Multiplying $|\phi|^{s_1-2}\phi$ to the heat equation and integrating on $\mathbb{R}^n$, we have

$$\sup_{0\leq t\leq T}\|\phi(\cdot,t)\|^{s_1}_{L^{s_1}(\mathbb{R}^n)} + \int_0^t\int_{\mathbb{R}^n}[\nabla(|\phi|^{s_1/2})]^2\mathrm{d}x\mathrm{d}t \lesssim \|\phi_0\|^{s_1}_{L^{s_1}(\mathbb{R}^n)}. \tag{2.8}$$

Let $g = |\phi|^{\frac{s_1}{2}}$, then by Hölder inequality and (2.8), we get

$$\begin{aligned}\|\phi(\cdot,t)\|^l_{L^l([0,T],L^s(\mathbb{R}^n))} &= \int_0^T\left(\int_{\mathbb{R}^n}|g|^{\frac{2s}{s_1}}\mathrm{d}x\right)^{\frac{l}{s}}\mathrm{d}t\\ &\lesssim \int_0^T\left(\int_{\mathbb{R}^n}|g|^2\mathrm{d}x\right)^{\frac{s_1n-s(n-2)}{n(s-s_1)}}\left(\int_{\mathbb{R}^n}|g|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{n}}\mathrm{d}t \qquad (2.9)\\ &\lesssim \sup_{0\leq t\leq T}\|\phi(\cdot,t)\|^{\frac{s_1^2n-ss_1(n-2)}{n(s-s_1)}}_{L^{s_1}(\mathbb{R}^n)}\|\nabla g\|^2_{L^2(0,T,\mathbb{R}^n)} \lesssim \|\phi_0\|^l_{L^{s_1}(\mathbb{R}^n)}.\end{aligned}$$

Now, we are going to prove (2.6). Let ω_ϵ be the smoother kernel and

$$\phi_{0\epsilon} = \omega_\epsilon * \phi_0, \tag{2.10}$$

then

$$\phi_\epsilon = \mathrm{e}^{t\Delta}\phi_{0\epsilon} = \omega_\epsilon * \mathrm{e}^{t\Delta}u_0 = \omega_\epsilon * \phi. \tag{2.11}$$

For any positive integers m and k, by Lemma 2.2, we have

$$\begin{aligned}t^{m+\frac{k+1}{2}}\|\partial_t^m\nabla^k\phi\|_{L^\infty(\mathbb{R}^n)} &\leq t^{m+\frac{k+1}{2}}(\|\partial_t^m\nabla^k\phi_\epsilon\|_{L^\infty(\mathbb{R}^n)}+\|\partial_t^m\nabla^k(\phi_\epsilon-\phi)\|_{L^\infty(\mathbb{R}^n)})\\ &\leq t^{m+\frac{k+1}{2}}(\|\nabla^{k+2m}\phi_\epsilon\|_{L^\infty(\mathbb{R}^n)}+\|\nabla^{k+2m}(\phi_\epsilon-\phi)\|_{L^\infty(\mathbb{R}^n)})\\ &\leq t^{\frac{1}{2}}\|\phi_{0\epsilon}(x,t)\|_{L^\infty(\mathbb{R}^n)}+\|\phi_{0\epsilon}(x,t)-\phi_0\|_{L^n(\mathbb{R}^n)}. \qquad (2.12)\end{aligned}$$

Recalling that $\phi_0 \in L^n(\mathbb{R}^n)$ and (2.10), let $\epsilon \to 0$ and $t \to 0$, we get

$$t^{m+\frac{k+1}{2}}\|\partial_t^m\nabla^k\phi\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{t\to 0} 0. \tag{2.13}$$

The proof is completed.

Remark 2.1 Particularly, for given positive constants m and k, we can prove the following estimate just similar to (2.3):

$$t^{m+\frac{k+1}{2}}\|\partial_t^m\nabla^k\phi\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\phi_0\|_{L^n(\mathbb{R}^n)}. \tag{2.14}$$

We also recall some results for the following linear Stokes system:

$$\begin{cases}u_t - \Delta u = \mathrm{div}f, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\\ \mathrm{div}u = 0.\end{cases} \tag{2.15}$$

For given initial data $u_0 \in L^n(\mathbb{R}^n)$, we have:

Proposition 2.1 *For any $T>0$, suppose that $f \in L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)\cap L^2(0,T;\mathbb{R}^n)$ and the initial data $u_0 \in L^n(\mathbb{R}^n)$, then for the linear equation* (2.15)*, there exists a uniform constant C_0, such that the solution u satisfies:*

$$u \in C([0,T];L^2(\mathbb{R}^n)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^n)); \tag{2.16}$$

$$u \in C([0,T];L^n(\mathbb{R}^n)) \cap L^{n+2}(0,T;\mathbb{R}^n) \cap L^{n+1}(0,T;\mathbb{R}^n); \tag{2.17}$$

$$\|u(\cdot,t)\|_{L^n(\mathbb{R}^n)} + \|u\|_{L^{n+2}(0,T;\mathbb{R}^n)} \le C_0(\|f\|_{L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)} + \|u_0\|_{L^n(\mathbb{R}^n)}); \tag{2.18}$$

$$\|u\|_{L^{n+1}(0,T;\mathbb{R}^n)} \le C_0(\|f\|_{L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)} + \|u_0\|_{L^n(\mathbb{R}^n)}). \tag{2.19}$$

Proof This Proposition comes from [7], where the authors proved it for the case $n=3$. For completeness, we shall give a brief proof of (2.18)-(2.19) for the general case $n \ge 2$.

Write $g = |u|^{n/2}$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |u|^{n+2}\mathrm{d}x\right)^{\frac{n-2}{2(n+2)}}\\
=&\left(\int_{\mathbb{R}^n} |g|^{\frac{2(n+2)}{n}}\mathrm{d}x\right)^{\frac{n-2}{2(n+2)}} \lesssim \left(\int_{\mathbb{R}^n} |g|^2\mathrm{d}x\right)^{\frac{n-2}{n(n+2)}} \left(\int_{\mathbb{R}^n} |g|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{(n-2)^2}{2n(n+2)}}\\
\lesssim&\left(\int_{\mathbb{R}^n} |u|^n\mathrm{d}x\right)^{\frac{n-2}{n(n+2)}} \left(\int_{\mathbb{R}^n} |g|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{(n-2)^2}{2n(n+2)}} \lesssim \|u\|_{L^n(\mathbb{R}^n)}^{\frac{n-2}{n+2}} \|\nabla g\|_{L^2(\mathbb{R}^n)}^{\frac{n-2}{n+2}}.
\end{aligned} \tag{2.20}$$

By multiplying $|u|^{n-2}u$ to (2.15) and integrating by parts, we have

$$\begin{aligned}
&\partial_t \|u\|_{L^n(\mathbb{R}^n)}^n + \int_{\mathbb{R}^n} |u|^{n-2}|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^n} |\nabla g|^2\mathrm{d}x\\
\lesssim&\left(\int_{\mathbb{R}^n} |f|^2|u|^{n-2}\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^n} |u|^{n-2}|\nabla u|^2\mathrm{d}x\right)^{1/2}\\
\lesssim&\|f\|_{L^{\frac{n+2}{2}}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} |u|^{n+2}\mathrm{d}x\right)^{\frac{n-2}{2(n+2)}}\left(\int_{\mathbb{R}^n} |u|^{n-2}|\nabla u|^2\mathrm{d}x\right)^{1/2}\\
\lesssim&\|f\|^2_{L^{\frac{n+2}{2}}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} |u|^{n+2}\mathrm{d}x\right)^{\frac{n-2}{n+2}}\\
\lesssim&\|f\|^2_{L^{\frac{n+2}{2}}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} |u|^{n}\mathrm{d}x\right)^{\frac{2(n-2)}{n(n+2)}}\left(\int_{\mathbb{R}^n} |g|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{(n-2)^2}{n(n+2)}}\\
\lesssim&\|f\|^2_{L^{\frac{n+2}{2}}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} |u|^{n}\mathrm{d}x\right)^{\frac{2(n-2)}{n(n+2)}}\left(\int_{\mathbb{R}^n} |\nabla g|^2\mathrm{d}x\right)^{\frac{n-2}{n+2}}\\
\lesssim&\|f\|^{\frac{n+2}{2}}_{L^{\frac{n+2}{2}}(\mathbb{R}^n)}\|u\|^{\frac{n-2}{2}}_{L^n(\mathbb{R}^n)}.
\end{aligned} \tag{2.21}$$

Then by Gronwall inequality, we get

$$\|u\|_{L^\infty(0,T;L^n(\mathbb{R}^n))} \lesssim \|f\|_{L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)} + \|u_0\|_{L^n(\mathbb{R}^n)}, \tag{2.22}$$

and

$$\|\nabla g\|_{L^2(0,T;\mathbb{R}^n)} \lesssim \|f\|^{n/2}_{L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)} + \|u_0\|^{n/2}_{L^n(\mathbb{R}^n)}. \tag{2.23}$$

By (2.22) and (2.23), we verify (2.18) as follows:

$$\|u\|_{L^{n+2}(0,T;\mathbb{R}^n)} \lesssim \|u\|^{\frac{2}{n+2}}_{L^n(\mathbb{R}^n)} \|\nabla g\|^{\frac{2}{n+2}}_{L^2(0,T;\mathbb{R}^n)} \lesssim \|f\|_{L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)} + \|u_0\|_{L^n(\mathbb{R}^n)}. \tag{2.24}$$

Similarly, (2.19) can be proved as

$$\begin{aligned}
\|u\|_{L^{n+1}(0,T;\mathbb{R}^n)} &= \left(\int_0^t\!\!\int_{\mathbb{R}^n} |u|^{n+1}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{n+1}} \\
&\lesssim \left[\int_0^t \left(\int_{\mathbb{R}^n} g^2 \mathrm{d}x\right)^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} g^{\frac{2n}{n-1}} \mathrm{d}x\right)^{\frac{n-1}{n}} \mathrm{d}t\right]^{\frac{1}{n+1}} \\
&\lesssim \|u\|^{\frac{1}{n+1}}_{L^n(\mathbb{R}^n)} \left(\int_0^t\!\!\int_{\mathbb{R}^n} |\nabla g|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{n+1}} \\
&\lesssim \|f\|_{L^{\frac{n+2}{2}}(0,T;\mathbb{R}^n)} + \|u_0\|_{L^n(\mathbb{R}^n)}.
\end{aligned} \tag{2.25}$$

The proof is complete.

Lemma 2.4 *For any constant $k_0 > 0$ and $d_0(x) \in \mathbb{S}^2$, there exists a constant $C(k_0)$ such that*

$$\mathrm{dist}(\mathrm{e}^{t\Delta} d_0(x), \mathbb{S}^2) \le k_0 + (C^n + 1)^{1/n} \|\nabla d_0\|_{L^n(\mathbb{R}^n)}, \tag{2.26}$$

where $\mathrm{dist}(\cdot,\cdot)$ *is the distance.*

Proof This Lemma follows directly from Lemma 2.1 of [35].

3 Local Existence

We prove Theorem 1.1 by using the fixed point argument. Given any $T > 0$, we write

$$\begin{aligned}
\kappa(T) = {} & \|\mathrm{e}^{t\Delta} u_0\|_{L^{n+1}(0,T;\mathbb{R}^n)} + \|\mathrm{e}^{t\Delta} u_0\|_{L^{n+2}(0,T;\mathbb{R}^n)} \\
& + \|\mathrm{e}^{t\Delta} \nabla d_0\|_{L^{n+1}(0,T;\mathbb{R}^n)} + \|\mathrm{e}^{t\Delta} \nabla d_0\|_{L^{n+2}(0,T;\mathbb{R}^n)} < +\infty.
\end{aligned} \tag{3.1}$$

At the beginning, we set a suitable space as follows (for more details of the suitable space see [18]):

Definition 3.1 For the functions $(f(x,t), g(x,t))$ defined on $\mathbb{R}^n \times [0,T]$ $(0 < T \le \infty)$, we say that $(f(x,t), g(x,t)) \in E^T$ if there hold:

$$\lim_{t\to 0} \sqrt{t}(\|f\|_{L^\infty(\mathbb{R}^n)} + \|\nabla g\|_{L^\infty(\mathbb{R}^n)}) = 0, \tag{3.2}$$

and

$$|||(f,g)|||_{E^T} \triangleq \|g\|_{L^\infty(\mathbb{R}^n)} + \|(f,g)\|_{E^T} < \infty, \tag{3.3}$$

where

$$\begin{aligned}\|(f,g)\|_{E^T} \triangleq \sup_{0\le t\le T} \sqrt{t}\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^{n+1}(0,T;\mathbb{R}^n)} + \|f\|_{L^{n+2}(0,T;\mathbb{R}^n)} \\ + \sup_{0\le t\le T} \sqrt{t}\|\nabla g\|_{L^\infty(\mathbb{R}^n)} + \|\nabla g\|_{L^{n+1}(0,T;\mathbb{R}^n)} + \|\nabla g\|_{L^{n+2}(0,T;\mathbb{R}^n)}.\end{aligned} \tag{3.4}$$

Furthermore, we say that $(f(x,t), g(x,t)) \in E^T_{\kappa(T)}$ if $(f(x,t), g(x,t)) \in E^T$ and

$$\|(f,g)\|_{E^T} \le 2\kappa(T). \tag{3.5}$$

It is easy to check that both E^T and $E^T_{\kappa(T)}$ are non-empty Banach spaces.

Let $u^{(1)}$ and $d^{(1)}$ be solutions to the following equations respectively,

$$\begin{cases} u^{(1)}_t - \Delta u^{(1)} = 0, \\ u^{(1)}(x,t)|_{t=0} = u_0. \end{cases} \tag{3.6}$$

and

$$\begin{cases} d^{(1)}_t - \Delta d^{(1)} = 0, \\ d^{(1)}(x,t)|_{t=0} = d_0. \end{cases} \tag{3.7}$$

We set a map $\mathbb{T}(\widetilde{u}, \widetilde{d}) = (\mathbb{T}_1(\widetilde{u}, \widetilde{d}), \mathbb{T}_2(\widetilde{u}, \widetilde{d}))$ as

$$\begin{cases} u^{(2)} = \mathbb{T}_1(\widetilde{u}, \widetilde{d}) = -\displaystyle\int_0^t S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u} + \nabla\widetilde{d}\otimes\nabla\widetilde{d}](\cdot,\tau)\mathrm{d}\tau, \\ d^{(2)} = \mathbb{T}_2(\widetilde{u}, \widetilde{d}) = \displaystyle\int_0^t S(t-\tau)\big(|\nabla\widetilde{d}|^2\widetilde{d} - \widetilde{u}\cdot\nabla\widetilde{d}\big)\mathrm{d}\tau, \\ u^{(2)}|_{t=0} = 0, \quad d^{(2)}|_{t=0} = 0. \end{cases} \tag{3.8}$$

Here and hereafter, we denote $S(t)$ as the heat operator and $\mathbb{P}$ is the Leary projection operator.

By Lemma 2.3, to prove Theorem 1.1, it is sufficient to estimate $(u^{(2)}, d^{(2)})$.

Proposition 3.1 *There exists a constant t_1, when $0 < t \le t_1$ we have*

$$\mathbb{T}(\widetilde{u}, \widetilde{d}) : E^t_{\kappa(t)} \to E^t_{\kappa(t)}. \tag{3.9}$$

Proof We need to prove

$$\|d^{(2)}\|_{L^\infty(\mathbb{R}^n)} < \infty, \tag{3.10}$$

and

$$\lim_{t\to 0} \sqrt{t}(\|u^{(2)}\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d^{(2)}\|_{L^\infty(\mathbb{R}^n)}) = 0, \tag{3.11}$$

as well as

$$\|(u^{(2)}, d^{(2)})\|_{E^{t_1}} \leq 2\kappa(t). \tag{3.12}$$

We shall prove (3.10)-(3.12) term by term.

$$\|d^{(2)}\|_{L^\infty} \leq \Big\| \int_0^t S(t-\tau)\big(|\nabla \widetilde{d}|^2 \widetilde{d} - \widetilde{u}\cdot\nabla\widetilde{d}\big)\mathrm{d}\tau \Big\|_{L^\infty(\mathbb{R}^n)}. \tag{3.13}$$

When $0<\tau<t/2$, by Hölder inequality we have

$$\begin{aligned}
&\Big\| \int_0^{t/2} S(t-\tau)\big(|\nabla \widetilde{d}|^2 \widetilde{d} - \widetilde{u}\cdot\nabla\widetilde{d}\big)\mathrm{d}\tau \Big\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim{}& \Big(\int_0^{t/2}\int_{\mathbb{R}^n} \frac{\mathrm{e}^{-\frac{(n+2)|y-\widetilde{y}|^2}{2n(t-\tau)}}}{\sqrt{t-\tau}^{n+2}}\mathrm{d}\widetilde{y}\mathrm{d}\tau\Big)^{\frac{n}{n+2}} (\|\widetilde{u}\|^2_{L^{n+2}(0,T;\mathbb{R}^n)} + \|\nabla\widetilde{d}\|^2_{L^{n+2}(0,T;\mathbb{R}^n)})\\
\lesssim{}& \kappa^2(t) < \infty.
\end{aligned} \tag{3.14}$$

For the case $t/2<\tau<t$, we get

$$\begin{aligned}
&\Big\| \int_{t/2}^{t} S(t-\tau)\big(|\nabla \widetilde{d}|^2 \widetilde{d} - \widetilde{u}\cdot\nabla\widetilde{d}\big)\mathrm{d}\tau \Big\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim{}& \Big\| \int_{t/2}^{t} \frac{1}{\tau}\int_{\mathbb{R}^n} \frac{\mathrm{e}^{-\frac{|y-\widetilde{y}|^2}{2(t-\tau)}}}{\sqrt{t-\tau}^{n}}\mathrm{d}\widetilde{y}(t\|u\|^2_{L^\infty(\mathbb{R}^n)} + t\|\nabla d\|^2_{L^\infty(\mathbb{R}^n)})\mathrm{d}\tau \Big\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim{}& \kappa^2(t) < \infty.
\end{aligned} \tag{3.15}$$

Then (3.10) comes from (3.13)-(3.15).

To verify (3.11), we begin with the term $u^{(2)}$ and we have

$$\begin{aligned}
t^{\frac12}\|u^{(2)}\|_{L^\infty(\mathbb{R}^n)} &= t^{\frac12}\Big\| \int_0^t S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u} + \nabla\widetilde{d}\otimes\nabla\widetilde{d}]\mathrm{d}\tau\Big\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim t^{\frac12}\Big\| \Big(\int_0^{t/2} + \int_{t/2}^t\Big) S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u} + \nabla\widetilde{d}\otimes\nabla\widetilde{d}]\mathrm{d}\tau\Big\|_{L^\infty(\mathbb{R}^n)}\\
&\triangleq I_{11} + I_{12}.
\end{aligned} \tag{3.16}$$

By Lemma 2.1 we have

$$\begin{aligned}
I_{11} &= t^{\frac12}\Big\| \int_0^{t/2} S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u} + \nabla\widetilde{d}\otimes\nabla\widetilde{d}]\mathrm{d}\tau\Big\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim \Big\| \int_0^{t/2}\int_{\mathbb{R}^n} \frac{1}{\sqrt{t-\tau}^n}\frac{1}{(1+\frac{|y-\widetilde{y}|}{\sqrt{t-\tau}})^{n+1}}(|\widetilde{u}|^2 + |\nabla\widetilde{d}|^2)\mathrm{d}\widetilde{y}\mathrm{d}\tau\Big\|_{L^\infty(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \Big(\int_0^{t/2}\int_{\mathbb{R}^n}\frac{1}{\sqrt{t-\tau}^{n+2}}\frac{1}{(1+\frac{|y-\widetilde{y}|}{\sqrt{t-\tau}})^{\frac{(n+1)(n+2)}{n}}}\mathrm{d}\widetilde{y}\mathrm{d}\tau\Big)^{\frac{n}{n+2}}\\
&\quad\cdot(\|\widetilde{u}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)})\\
&\lesssim \|\widetilde{u}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}. \qquad (3.17)
\end{aligned}$$

When $\frac{t}{2}\le\tau\le t$, by Lemma 2.1 we get

$$\begin{aligned}
&|S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]|\\
\lesssim&\Big|\int_{\mathbb{R}^n}\frac{1}{(\sqrt{t-\tau}+|y-\widetilde{y}|)^{n+1}}\mathrm{d}\widetilde{y}\Big|\|\big(\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}\big)\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim&\frac{1}{\sqrt{t-\tau}}\|\big(\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}\big)\|_{L^\infty(\mathbb{R}^n)}, \qquad (3.18)
\end{aligned}$$

therefore we have

$$\begin{aligned}
I_{12}&\lesssim t^{\frac12}\int_{t/2}^t\frac{1}{\sqrt{t-\tau}\cdot\tau}\mathrm{d}\tau\big(\sqrt{t}\|\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}+\sqrt{t}\|\nabla\widetilde{d}\|_{L^\infty(\mathbb{R}^n)}\big)^2\\
&\lesssim\big(\sqrt{t}\|\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}+\sqrt{t}\|\nabla\widetilde{d}\|_{L^\infty(\mathbb{R}^n)}\big)^2. \qquad (3.19)
\end{aligned}$$

Recalling that $(\widetilde{u},\widetilde{d})\in E^t$, from (3.16)-(3.19), we get

$$\lim_{t\to0}\sqrt{t}\|u^{(2)}\|_{L^\infty(\mathbb{R}^n)}=0. \qquad (3.20)$$

Furthermore, for a uniform constant C_1, from (3.16)-(3.19) we have

$$\sup_{0\le t\le T}\sqrt{t}\|u^{(2)}\|_{L^\infty(\mathbb{R}^n)}\le C_1\kappa^2(t). \qquad (3.21)$$

Similarly to the process of (3.16)-(3.21), we can get

$$\begin{aligned}
&\sqrt{t}\|\nabla d^{(2)}\|_{L^\infty(\mathbb{R}^n)}\\
=&\sqrt{t}\Big\|\Big(\int_0^{t/2}+\int_{t/2}^t\Big)S(t-\tau)\big(|\nabla\widetilde{d}|^2\widetilde{d}-\widetilde{u}\cdot\nabla\widetilde{d}\big)\mathrm{d}\tau\Big\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim&\Big(\int_0^{t/2}\int_{\mathbb{R}^n}\frac{1}{\sqrt{t-\tau}^{n+2}}\frac{1}{(1+\frac{|y-\widetilde{y}|}{\sqrt{t-\tau}})^{\frac{(n+1)(n+2)}{n}}}\mathrm{d}\widetilde{y}\mathrm{d}\tau\Big)^{\frac{n}{n+2}}\\
&\cdot(\|\widetilde{u}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}\|\widetilde{d}\|_{L^\infty(\mathbb{R}^n)})\\
&+t^{\frac12}\int_{t/2}^t\frac{1}{\sqrt{t-\tau}\cdot\tau}\mathrm{d}\tau\big(\sqrt{t}\|\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}+\sqrt{t}\|\nabla\widetilde{d}\|_{L^\infty(\mathbb{R}^n)}\|\widetilde{d}\|_{L^\infty(\mathbb{R}^n)}\big)^2\\
\lesssim&\|\widetilde{u}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,\frac{t}{2};\mathbb{R}^n)}+t\|\widetilde{u}\|^2_{L^\infty(\mathbb{R}^n)}+t\|\nabla\widetilde{d}\|^2_{L^\infty(\mathbb{R}^n)}\xrightarrow{t\to0}0. \qquad (3.22)
\end{aligned}$$

For a uniform constant C_2, from (3.22) we have

$$\sup_{0\le t\le T}\sqrt{t}\|\nabla d^{(2)}\|_{L^\infty(\mathbb{R}^n)}\le C_2\kappa^2(t). \tag{3.23}$$

Then (3.11) follows from (3.20) and (3.22).

By Proposition 2.1, we have

$$\begin{aligned}&\|u^{(2)}\|_{L^{n+1}(0,T^*;\mathbb{R}^n)}+\|u^{(2)}\|_{L^{n+2}(0,T^*;\mathbb{R}^n)}\\ \lesssim\ &\|\widetilde{u}\|^2_{L^{n+2}(0,T;\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,T;\mathbb{R}^n)}\le C_3\kappa^2(t),\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}&\|\nabla d^{(2)}\|_{L^{n+1}(0,T^*;\mathbb{R}^n)}+\|\nabla d^{(2)}\|_{L^{n+2}(0,T^*;\mathbb{R}^n)}\\ \lesssim\ &\|\widetilde{u}\|^2_{L^{n+2}(0,T;\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,T;\mathbb{R}^n)}\le C_4\kappa^2(t),\end{aligned} \tag{3.25}$$

where C_3 and C_4 are uniform constants.

Claim There exists a constant t_1, for $0<t<t_1$ there holds

$$(C_1+C_2+C_3+C_4)\kappa(t)\le 1. \tag{3.26}$$

Then from (3.21), (3.23) and (3.24)-(3.25), (3.12) follows immediately.

In the following, we shall verify Claim (3.26). Denote

$$u_0^\epsilon=\omega_\epsilon * u_0,\quad d_0^\epsilon=\omega_\epsilon * d_0, \tag{3.27}$$

where ω_ϵ is the usual smoother kernel.

Write $u_\epsilon^{(1)}=\mathrm{e}^{t\Delta}u_0^\epsilon$ and $d_\epsilon^{(1)}=\mathrm{e}^{t\Delta}d_0^\epsilon$, by Lemma 2.3 and Proposition 2.1, we get

$$\begin{aligned}\kappa(t)\lesssim\ &\|\mathrm{e}^{t\Delta}u_0^\epsilon\|_{L^{n+1}(0,T;\mathbb{R}^n)}+\|\mathrm{e}^{t\Delta}u_0^\epsilon\|_{L^{n+2}(0,T;\mathbb{R}^n)}+\|\mathrm{e}^{t\Delta}\nabla d_0^\epsilon\|_{L^{n+1}(0,T;\mathbb{R}^n)}\\ &+\|\mathrm{e}^{t\Delta}\nabla d_0^\epsilon\|_{L^{n+2}(0,T^*;\mathbb{R}^n)}+\|u_0^\epsilon-u_0\|_{L^n(\mathbb{R}^n)}+\|\nabla d_0^\epsilon-\nabla d_0\|_{L^n(\mathbb{R}^n)}\\ \lesssim\ &t^{\frac{1}{2(n+1)}}(\|u_0^\epsilon\|_{L^n(\mathbb{R}^n)}+\|u_0^\epsilon\|_{L^{n+1}(\mathbb{R}^n)}+\|\nabla d_0^\epsilon\|_{L^n(\mathbb{R}^n)}+\|\nabla d_0^\epsilon\|_{L^{n+1}(\mathbb{R}^n)})\\ &+(\|u_0^\epsilon-u_0\|_{L^n(\mathbb{R}^n)}+\|\nabla d_0^\epsilon-\nabla d_0\|_{L^n(\mathbb{R}^n)})\\ \le\ &C_5t^{\frac{1}{2(n+1)}}+C_6(\|u_0^\epsilon-u_0\|_{L^n(\mathbb{R}^n)}+\|\nabla d_0^\epsilon-\nabla d_0\|_{L^n(\mathbb{R}^n)}).\end{aligned} \tag{3.28}$$

Taking t_1 small enough, such that for $0<t\le t_1$, we have

$$(C_1+C_2+C_3+C_4)C_5t_1^{\frac{1}{2(n+1)}}\le\frac{1}{3}. \tag{3.29}$$

Recalling that ω_ϵ is the smoother kernel, we can choose the parameter ϵ such that

$$(C_1+C_2+C_3+C_4)C_6(\|u_0^\epsilon-u_0\|_{L^n(\mathbb{R}^n)}+\|\nabla d_0^\epsilon-\nabla d_0\|_{L^n(\mathbb{R}^n)})\le\frac{1}{3}. \tag{3.30}$$

Then (3.26) follows from (3.28)-(3.30). The proof is completed.

Proposition 3.2 *There exists a constant $t_2 > 0$, when $0 < t \leq t_2$, such that*

$$\mathbb{T}(\widetilde{u}, \widetilde{d}) : E^t_{\kappa(t)} \to E^t_{\kappa(t)} \tag{3.31}$$

is a contraction map. More precisely, let $(\widetilde{u}, \widetilde{d}), (\overline{u}, \overline{d}) \in \mathbb{E}^t$ with $(\widetilde{u}, \widetilde{d})|_{t=0} = (\overline{u}, \overline{d})|_{t=0} = (u_0, d_0)$, then there exists a constant $t_2 > 0$, such that for $0 < t \leq t_2$ there holds:

$$|||\big(\mathbb{T}_1(\widetilde{u}, \widetilde{d}) - \mathbb{T}_1(\overline{u}, \overline{d}), \mathbb{T}_2(\widetilde{u}, \widetilde{d}) - \mathbb{T}_2(\overline{u}, \overline{d})\big)|||_{E^t} \leq \frac{1}{2}|||(\widetilde{u} - \overline{u}, \widetilde{d} - \overline{d})|||_{E^t}. \tag{3.32}$$

Proof For simplicity, we write $u^* = \widetilde{u} - \overline{u}$ and $d^* = \widetilde{d} - \overline{d}$. Recalling (3.8), we obtain

$$\begin{aligned}|\mathbb{T}_1(\widetilde{u}, \widetilde{d}) - \mathbb{T}_1(\overline{u}, \overline{d})| &= \left| \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot \big(\widetilde{u} \otimes \widetilde{u} + \nabla\widetilde{d} \otimes \nabla\widetilde{d} - \overline{u} \otimes \overline{u} - \nabla\overline{d} \otimes \nabla\overline{d}\big)\mathrm{d}\tau \right| \\ &= \left| \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot \big(\widetilde{u} \otimes u^* + \nabla\widetilde{d} \otimes \nabla d^* + u^* \otimes \overline{u} + \nabla d^* \otimes \nabla\overline{d}\big)\mathrm{d}\tau \right|,\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}|\mathbb{T}_2(\widetilde{u}, \widetilde{d}) - \mathbb{T}_2(\overline{u}, \overline{d})| &= \left| \int_0^t S(t-\tau)\big(|\nabla\widetilde{d}|^2\widetilde{d} - \widetilde{u} \cdot \nabla\widetilde{d} - |\nabla\overline{d}|^2\overline{d} + \overline{u} \cdot \nabla\overline{d}\big)\mathrm{d}\tau \right| \\ &= \left| \int_0^t S(t-\tau)\big(|\nabla\widetilde{d}|^2 d^* - \widetilde{u} \cdot \nabla d^* - u^* \cdot \nabla\overline{d} + (\nabla\widetilde{d} + \nabla\overline{d}) : \nabla d^*\overline{d}\big)\mathrm{d}\tau \right|.\end{aligned} \tag{3.34}$$

Repeating the proof as in Proposition 3.1, we have

$$\begin{aligned}&\|\big(\mathbb{T}_1(\widetilde{u}, \widetilde{d}) - \mathbb{T}_1(\overline{u}, \overline{d}), \mathbb{T}_2(\widetilde{u}, \widetilde{d}) - \mathbb{T}_2(\overline{u}, \overline{d})\big)\|_{E^t} \\ &\lesssim \|d^*\|_{L^\infty(\mathbb{R}^n)}\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,t;\mathbb{R}^n)} + (\|(\overline{u}, \overline{d})\|_{E^t} + \|(\widetilde{u}, \widetilde{d})\|_{E^t})\|(u^*, d^*)\|_{E^t} \\ &\leq C_7\big(\|d^*\|_{L^\infty(\mathbb{R}^n)} + \|(u^*, d^*)\|_{E^t}\big) \cdot \kappa(t),\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}&\|\mathbb{T}_2(\widetilde{u}, \widetilde{d}) - \mathbb{T}_2(\overline{u}, \overline{d})\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \|d^*\|_{L^\infty(\mathbb{R}^n)}\|\nabla\widetilde{d}\|^2_{L^{n+2}(0,t;\mathbb{R}^n)} + (\|(\overline{u}, \overline{d})\|_{E^t} + \|(\widetilde{u}, \widetilde{d})\|_{E^t})\|(u^*, d^*)\|_{E^t} \\ &\leq C_8\big(\|d^*\|_{L^\infty(\mathbb{R}^n)} + \|(u^*, d^*)\|_{E^t}\big) \cdot \kappa(t).\end{aligned} \tag{3.36}$$

As the proof in (3.28)-(3.30), we can take $t_2 > 0$, such that for $0 < t \leq t_2$, we have

$$(C_7 + C_8)\kappa(t) \leq \frac{1}{2}. \tag{3.37}$$

Then (3.32) follows from (3.35)-(3.37). The proof is completed.

Proof of Theorem 1.1 By taking

$$T=\min\{t_1,t_2\}, \tag{3.38}$$

combing Lemma 2.3, Proposition 3.1 and Proposition 3.2, there exists a pair of unique solution (u,d) satisfying (1.6)-(1.7) in the time interval $[0,T]$. To finish the proof of Theorem 1.1, we still need to verify that $|d(x,t)|=1$.

Following the line of [35], by Lemma 2.4 and (3.13)-(3.15), we have

$$\operatorname{dist}(d,\mathbb{S}^2)\le\operatorname{dist}(d,d^{(1)})+\operatorname{dist}(d^{(1)},\mathbb{S}^2)\lesssim\kappa^2(t)+\delta+(K^n+1)^{1/n}\|\nabla d_0\|_{L^n(\mathbb{R}^n)}, \tag{3.39}$$

where we used the fact that $\operatorname{dist}(d,d^{(1)})\lesssim\|d^{(2)}\|_{L^\infty(\mathbb{R}^n)}$. Recalling that $\nabla d_0\in L^n(\mathbb{R}^n)$, for any $0\le t\le T$, we have

$$\operatorname{dist}(d,\mathbb{S}^2)\le C(T)<\infty. \tag{3.40}$$

Furthermore, we take the vector $\Pi(d)\in\mathbb{S}^2$ with

$$\operatorname{dist}(d,\Pi(d))=\min\{d,\mathbb{S}^2\}. \tag{3.41}$$

Write $Q(d)=d-\Pi(d)$ and $\rho(d)=\frac{1}{2}|Q(d)|^2$, we have

$$\begin{aligned}\rho_t+u\cdot\nabla\rho-\Delta\rho=&(d-\Pi(d))\cdot\nabla_dQ(d)\cdot(d_t+u\cdot\nabla d-\Delta d)\\&-|\nabla Q(d)|^2+Q(d)\cdot\nabla^2Q(d)=-|\nabla Q(d)|^2\le 0,\end{aligned} \tag{3.42}$$

where we used $\nabla Q,\nabla^2Q\in$ the tangent plate of $\mathbb{S}^2$ and

$$d-\Pi(d)\perp\text{ the tangent plate of }\mathbb{S}^2. \tag{3.43}$$

Meanwhile, $d|_{t=0}=d_0=\Pi(d_0)\in\mathbb{S}^2$ implies that $\rho(d)|_{t=0}=0$. Due to the maximum principle, we conclude $\rho=0$ from (3.42), which implies that $d\in\mathbb{S}^2$.

Then we finish the proof of Theorem 1.1.

4 Space-time Regularity of the Local Solution

In the following, we shall prove Theorem 1.2. For any positive integers M,K and $(p,q)\in[2+n,\infty]\times[n,n+2]$ satisfying

$$\frac{2}{p}+\frac{n}{q}=1, \tag{4.1}$$

it is sufficient to prove that the local solution (u,d) of (1.1) satisfies

$$\sum_{m=0}^{M}\sum_{k=0}^{K}\left(\|t^{m+\frac{k}{2}}\partial_t^m\nabla^ku\|_{L^p(0,T;L^q(\mathbb{R}^n))}+\|t^{m+\frac{k}{2}}\partial_t^m\nabla^{k+1}d\|_{L^p(0,T;L^q(\mathbb{R}^n))}\right)<+\infty. \tag{4.2}$$

To prove (4.2), it is sufficient to verify the special case $m=0$. Since when $m\ge1$, by using the linear heat equation $\partial_t\Phi-\Delta\Phi=F$, we have

$$\partial_t^m \Phi = \Delta^m \Phi + \sum_{l=0}^{m-1} \Delta^{m-1-l} \partial_t^l F, \tag{4.3}$$

with a small modification of the following proof, and the general case $m \geq 1$ can be proved by induction.

We write

$$\begin{aligned}\|(f,g)\|_{E_K^{[0,T]}} \triangleq \sum_{k=0}^{K} \Big(&\sup_{0\leq t\leq T} t^{\frac{k+1}{2}} (\|\nabla^k f\|_{L^\infty(\mathbb{R}^n)} + \|\nabla^{k+1} g\|_{L^\infty(\mathbb{R}^n)}) \\ &+ \|t^{\frac{k}{2}} \nabla^k f\|_{L^p(0,T;L^q(\mathbb{R}^n))} + \|t^{\frac{k}{2}} \nabla^{k+1} g\|_{L^p(0,T;L^q(\mathbb{R}^n))}\Big).\end{aligned} \tag{4.4}$$

Therefore, we verify (4.2), it is sufficient to prove

$$\|(u,d)\|_{E_K^{[0,T]}} < \infty. \tag{4.5}$$

To prove (4.5), we firstly give the following proposition.

Proposition 4.1 *Let (u,d) be a local solution on $t \in [0,T]$ to (1.1) with the initial data u_0 and $\nabla d_0 \in L^n(\mathbb{R}^n)$, for any positive integers M, K and $(p,q) \in [2+n,\infty] \times [n,n+2]$ satisfying (4.1), then there exists a constant $0<\delta<T$, such that*

$$\begin{aligned}&\sum_{k=0}^{K} \Big(\sup_{0<t<\delta} t^{\frac{k+1}{2}} (\|\nabla^k u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla^{k+1} d\|_{L^\infty(\mathbb{R}^n)}) \\ &+ \|t^{\frac{k}{2}} \nabla^k u\|_{L^p(0,\delta;L^q(\mathbb{R}^n))} + \|t^{\frac{k}{2}} \nabla^{k+1} d\|_{L^p(0,\delta;L^q(\mathbb{R}^n))}\Big) < +\infty.\end{aligned} \tag{4.6}$$

Proof We prove this proposition by fixed point argument.

Recalling Lemma 2.3, it is sufficient to estimate $(u^{(2)}, d^{(2)})$ with $(u^{(2)}, d^{(2)})$ satisfying

$$\begin{cases} u^{(2)} = -\displaystyle\int_0^t S(t-\tau)\mathbb{P}\nabla\cdot[u\otimes u + \nabla d\otimes\nabla d](\cdot,\tau)\mathrm{d}\tau, \\ d^{(2)} = \displaystyle\int_0^t S(t-\tau)\big(|\nabla d|^2 d - u\cdot\nabla d\big)\mathrm{d}\tau. \end{cases} \tag{4.7}$$

And we define the map $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2)$ as in (3.8).

Write

$$\theta(t) \triangleq \|\mathrm{e}^{t\Delta} u_0\|_{L^{n+2}(0,t;\mathbb{R}^n)} + \|\mathrm{e}^{t\Delta}\nabla d_0\|_{L^{n+2}(0,t;\mathbb{R}^n)}, \tag{4.8}$$

we define the following space

$$E_K^{[0,t]} = \{(f,g) |\, |||(f,g)|||_{E_K^{[0,t]}} \triangleq \|g\|_{L^\infty(\mathbb{R}^n)} + \|(f,g)\|_{E_K^{[0,t]}} < \infty\}, \tag{4.9}$$

with

$$\|(f,g)\|_{E_K^{[0,t]}} \le 2\theta(t). \tag{4.10}$$

It is obvious that $E_K^{[0,t]}$ is a non-empty Banach spaces.

We shall prove the map $\mathbb{T}=(\mathbb{T}_1,\mathbb{T}_2)$ is a contracting map in the space $E_K^{[0.t]}$ when t is small enough.

Firstly, the estimates $\|d\|_{L^\infty(\mathbb{R}^n)}<\infty$ were verified in (3.13)-(3.15). It is sufficient to proceed the proof in two steps.

Step 1 There exists a constant $\delta_0>0$ such that $\mathbb{T}: E_K^{[0,\delta_0]} \to E_K^{[0,\delta_0]}$.

We begin the estimates with the term $u^{(2)}$. Taking the positive integers $k\le K$, we have

$$t^{\frac{k+1}{2}}\|\nabla^k u^{(2)}\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{\frac{k+1}{2}}\left\|\nabla^k\int_0^t S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\mathrm{d}\tau\right\|_{L^\infty(\mathbb{R}^n)}, \tag{4.11}$$

By Remark 2.1, when $0<\tau<\frac{t}{2}$, we have

$$\begin{aligned}
&t^{\frac{k+1}{2}}\left\|\nabla^k\int_0^{t/2} S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\mathrm{d}\tau\right\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim&\left\|\int_0^{t/2}\int_{\mathbb{R}^n}\frac{1}{\sqrt{t-\tau}^n}\frac{|\widetilde{u}|^2+|\nabla\widetilde{d}|^2}{(1+\frac{|y-\widetilde{y}|}{\sqrt{t-\tau}})^{n+k+1}}\mathrm{d}\widetilde{y}\mathrm{d}\tau\right\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim&\left(\int_0^{t/2}\frac{1}{\sqrt{t-\tau}^{\frac{2n}{q}\frac{p}{p-2}}}\mathrm{d}\tau\right)^{\frac{p-2}{p}}\left(\|\widetilde{u}\|^2_{L^p(0,\frac{t}{2};L^q(\mathbb{R}^n))}+\|\nabla\widetilde{d}\|^2_{L^p(0,\frac{t}{2};L^q(\mathbb{R}^n))}\right)\\
\lesssim&\,\theta^2(t).
\end{aligned}\tag{4.12}$$

When $\frac{t}{2}\le\tau\le t$, we have

$$\begin{aligned}
&|\nabla^k S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]|\\
\lesssim&\left|\int_{\mathbb{R}^n}\frac{1}{(\sqrt{t-\tau}+|y-\widetilde{y}|)^{n+1}}\mathrm{d}\widetilde{y}\right|\|\nabla^k\big(\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}\big)\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim&\frac{\sum\limits_{l=0}^{k}(\|\nabla^{k-l}\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}\|\nabla^l\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}+\|\nabla^{k-l+1}\widetilde{d}\|_{L^\infty(\mathbb{R}^n)}\|\nabla^{l+1}\widetilde{d}\|_{L^\infty(\mathbb{R}^n)})}{\sqrt{t-\tau}},
\end{aligned}\tag{4.13}$$

therefore we have

$$\begin{aligned}
&t^{\frac{k+1}{2}}\left\|\int_{t/2}^t\nabla^k S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\mathrm{d}\tau\right\|_{L^\infty(\mathbb{R}^n)}\\
\lesssim&\int_{t/2}^t\frac{1}{\sqrt{t-\tau}\cdot\sqrt{\tau}}\mathrm{d}\tau\,\theta^2(t)\lesssim\theta^2(t).
\end{aligned}\tag{4.14}$$

Similarly to the process of (4.11)-(4.14), we get

$$t^{\frac{k+1}{2}}\|\nabla^{k+1}d^{(2)}\|_{L^\infty(\mathbb{R}^n)}\lesssim\theta^2(t). \tag{4.15}$$

Next, we give an estimate for $\|t^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^\infty(0,t;L^n(\mathbb{R}^n))}$. Due to Minkovski inequality, we have

$$\begin{aligned}\|t^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^\infty(0,t;L^n(\mathbb{R}^n))}&\le t^{\frac{k}{2}}\left(\int_0^{t/2}+\int_{t/2}^t\right)\|\nabla^k S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\|_{L^n(\mathbb{R}^n)}\mathrm{d}\tau\\&\triangleq II_1+II_2.\end{aligned} \tag{4.16}$$

From Lemma 2.2 and Young inequality

$$\begin{aligned}II_1&=t^{\frac{k}{2}}\int_0^{t/2}\|\nabla^k S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\|_{L^n(\mathbb{R}^n)}\mathrm{d}\tau\\&\lesssim t^{\frac{k}{2}}\int_0^{t/2}\left\|\frac{1}{\sqrt{t-\tau}^{n+k+1}}\frac{1}{(1+\frac{|y|}{\sqrt{t-\tau}})^{n+k+1}}*(|\widetilde{u}|^2+|\nabla\widetilde{d}|^2)\right\|_{L^n(\mathbb{R}^n)}\mathrm{d}\tau\\&\lesssim\int_0^{t/2}\left\|\frac{1}{\sqrt{t-\tau}^{n+1}}\frac{1}{(1+\frac{|y|}{\sqrt{t-\tau}})^{n+k+1}}\right\|_{L^{\frac{nq}{(n+1)q-2n}}(\mathbb{R}^n)}(\|\widetilde{u}\|^2_{L^q(\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^q(\mathbb{R}^n)})\mathrm{d}\tau\\&\lesssim\left(\int_0^{t/2}\frac{1}{\sqrt{t-\tau}^{\frac{2n}{q}\frac{p}{p-2}}}\mathrm{d}\tau\right)^{\frac{p-2}{p}}(\|\widetilde{u}\|^2_{L^p(0,\frac{t}{2};L^q(\mathbb{R}^n))}+\|\nabla\widetilde{d}\|^2_{L^p(0,\frac{t}{2};L^q(\mathbb{R}^n))})\\&\lesssim\theta^2(t).\end{aligned} \tag{4.17}$$

Similarly to (4.13), we have

$$\begin{aligned}II_2&=t^{\frac{k}{2}}\int_{t/2}^t\|\nabla^k S(t-\tau)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\|_{L^n(\mathbb{R}^n)}\mathrm{d}\tau\\&\lesssim t^{\frac{k+2}{2}}\sum_{l=0}^k(\|\nabla^{k-l}\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}\|\nabla^l\widetilde{u}\|_{L^\infty(\mathbb{R}^n)}+\|\nabla^{k-l+1}\widetilde{d}\|_{L^\infty(\mathbb{R}^n)}\|\nabla^{l+1}\widetilde{d}\|_{L^\infty(\mathbb{R}^n)})\\&\quad\cdot\int_{t/2}^t\frac{1}{\sqrt{t-\tau}\sqrt{\tau}}\mathrm{d}\tau\lesssim\theta^2(t).\end{aligned} \tag{4.18}$$

From (4.16)-(4.18), we have

$$\|t^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^\infty(0,t;L^n(\mathbb{R}^n))}\lesssim\theta^2(t). \tag{4.19}$$

Similarly to (4.16)-(4.18), we also have

$$\|t^{\frac{k}{2}}\nabla^{k+1}d^{(2)}\|_{L^\infty(0,t;L^n(\mathbb{R}^n))}\lesssim\theta^2(t). \tag{4.20}$$

Now, we are going to estimate $\|\tau^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^p(0,t;L^q(\mathbb{R}^n))}$.

$$\begin{aligned}&\|\tau^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^q(\mathbb{R}^n)}\\&=\tau^{\frac{k}{2}}\left(\int_0^{\tau/2}+\int_{\tau/2}^\tau\right)\|\nabla^k S(\tau-s)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\|_{L^q(\mathbb{R}^n)}\mathrm{d}s.\end{aligned} \tag{4.21}$$

From Lemma 2.2 and Young inequality

$$
\begin{aligned}
&\tau^{\frac{k}{2}}\int_0^{\tau/2}\|\nabla^k S(\tau-s)\mathbb{P}\nabla\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\|_{L^q(\mathbb{R}^n)}\mathrm{d}s\\
\lesssim\ &\tau^{\frac{k}{2}}\int_0^{\tau/2}\left\|\frac{1}{\sqrt{\tau-s}^{n+k+1}}\frac{1}{(1+\frac{|y|}{\sqrt{\tau-s}})^{n+k+1}}*(|\widetilde{u}|^2+|\nabla\widetilde{d}|^2)\right\|_{L^q(\mathbb{R}^n)}\mathrm{d}s\\
\lesssim\ &\int_0^{\tau/2}\left\|\frac{1}{\sqrt{\tau-s}^{n+1}}\frac{1}{(1+\frac{|y|}{\sqrt{\tau-s}})^{n+k+1}}\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)}(\|\widetilde{u}\|^2_{L^q(\mathbb{R}^n)}+\|\nabla\widetilde{d}\|^2_{L^q(\mathbb{R}^n)})\mathrm{d}s\\
\lesssim\ &\int_0^{\tau/2}(\tau-s)^{-\frac{q+n}{2q}}\cdot(\|\widetilde{u}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}+\|\nabla\widetilde{d}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)})\mathrm{d}s.
\end{aligned}\tag{4.22}
$$

Similarly to (4.13), we have

$$
\begin{aligned}
&\tau^{\frac{k}{2}}\int_{\tau/2}^{\tau}\|\nabla S(\tau-s)\mathbb{P}\nabla^k\cdot[\widetilde{u}\otimes\widetilde{u}+\nabla\widetilde{d}\otimes\nabla\widetilde{d}]\|_{L^q(\mathbb{R}^n)}\mathrm{d}s\\
\lesssim\ &\int_{\tau/2}^{\tau}s^{\frac{k}{2}}\sum_{l=0}^{k}(\|\nabla^{k-l}\widetilde{u}(\cdot,s)\nabla^l\widetilde{u}(\cdot,s)\|_{L^{\frac{q}{2}}(\mathbb{R}^n)}\\
&+\|\nabla^{k-l+1}\widetilde{d}(\cdot,s)\nabla^{l+1}\widetilde{d}(\cdot,s)\|_{L^{\frac{q}{2}}(\mathbb{R}^n)})\cdot(\tau-s)^{-\frac{q+n}{2q}}\mathrm{d}s\\
\lesssim\ &\int_{\tau/2}^{\tau}\sum_{l=0}^{k}(\|s^{l/2}\nabla^l\widetilde{u}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}+\|s^{l/2}\nabla^{l+1}\widetilde{d}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)})\cdot(\tau-s)^{-\frac{q+n}{2q}}\mathrm{d}s.
\end{aligned}\tag{4.23}
$$

From (4.21)-(4.23) and Young inequality, we have

$$
\begin{aligned}
&\|\tau^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^p(0,t;L^q(\mathbb{R}^n))}\\
\lesssim\ &\left\|\int_0^{\tau/2}(\tau-s)^{-\frac{q+n}{2q}}\cdot(\|\widetilde{u}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}+\|\nabla\widetilde{d}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)})\mathrm{d}s\right\|_{L^p(0,t)}\\
&+\left\|\int_{\tau/2}^{\tau}\sum_{l=0}^{k}(\|s^{l/2}\nabla^l\widetilde{u}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}+\|s^{l/2}\nabla^{l+1}\widetilde{d}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)})(\tau-s)^{-\frac{q+n}{2q}}\mathrm{d}s\right\|_{L^p(0,t)}\\
\lesssim\ &\|\tau^{-\frac{q+n}{2q}}\|_{L^1(0,t)}\Big(\big\|\|\widetilde{u}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}+\|\nabla\widetilde{d}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}\big\|_{L^p(0,t)}\\
&+\sum_{l=0}^{k}\big\|\|s^{l/2}\nabla^l\widetilde{u}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}+\|s^{l/2}\nabla^{l+1}\widetilde{d}(\cdot,s)\|^2_{L^q(\mathbb{R}^n)}\big\|_{L^p(0,t)}\Big)\\
\lesssim\ &t^{\frac{1}{p}}\|\widetilde{u}\|^{\frac{q-n}{q}}_{L^\infty(0,t;\mathbb{R}^n)}\|\widetilde{u}\|^{\frac{n}{q}}_{L^\infty(0,t;L^n(\mathbb{R}^n))}\|\widetilde{u}\|_{L^p(0,t;L^q(\mathbb{R}^n))}\\
&+t^{\frac{1}{p}}\|\nabla\widetilde{d}\|^{\frac{q-n}{q}}_{L^\infty(0,t;\mathbb{R}^n)}\|\nabla\widetilde{d}\|^{\frac{n}{q}}_{L^\infty(0,t;L^n(\mathbb{R}^n))}\|\nabla\widetilde{d}\|_{L^p(0,t;L^q(\mathbb{R}^n))}\\
&+t^{\frac{1}{p}}\sum_{l=0}^{k}\|t^{l/2}\nabla^l\widetilde{u}\|^{\frac{q-n}{q}}_{L^\infty(0,t;\mathbb{R}^n)}\|s^{l/2}\nabla^l\widetilde{u}\|^{\frac{n}{q}}_{L^\infty(0,t;L^q(\mathbb{R}^n))}\|s^{l/2}\nabla^l\widetilde{u}\|_{L^p(0,t;L^q(\mathbb{R}^n))}
\end{aligned}
$$

$$+t^{\frac{1}{p}}\sum_{l=0}^{k}\|s^{l/2}\nabla^{l+1}\widetilde{d}\|_{L^{\infty}(0,t;\mathbb{R}^n)}^{\frac{q-n}{q}}\|s^{l/2}\nabla^{l+1}\widetilde{d}\|_{L^p(0,t;L^q(\mathbb{R}^n))}^{\frac{n}{q}}\|s^{l/2}\nabla^{l+1}\widetilde{d}\|_{L^p(0,t;L^q(\mathbb{R}^n))}, \tag{4.24}$$

where we used the fact that $-\frac{q+n}{2q}=\frac{1}{p}-1$. Recalling (4.19), (4.20) and $\frac{q-n}{q}=\frac{2}{p}$, we have

$$\|\tau^{\frac{k}{2}}\nabla^k u^{(2)}\|_{L^p(0,t;L^q(\mathbb{R}^n))}\lesssim\theta^4(t). \tag{4.25}$$

Similarly to (4.21)-(4.25), we get

$$\|t^{\frac{k}{2}}\nabla^{k+1}d^{(2)}\|_{L^p(0,t;L^q(\mathbb{R}^n))}\lesssim\theta^4(t). \tag{4.26}$$

From (4.11)-(4.15) and (4.25)-(4.26), applying the summation $\sum\limits_{k=0}^{K}$, we get

$$\|(u,d)\|_{E_K^{[0,t]}}\le C_7\theta^2(t)(1+\theta^2(t)). \tag{4.27}$$

Repeat the progress as in (3.26)-(3.30), we can choose a $\delta_0>0$ small enough, such that for any $t\in[0,\delta_0]$, there holds

$$C_7\theta(t)(1+\theta^2(t))\le\frac{3}{4}, \tag{4.28}$$

then we finish the proof of Step 1.

Step 2 There exists a $\delta_1>0$ such that $\mathbb{T}$ is a contraction map on $\mathbb{E}_K^{[0,\delta_1]}$.

Let $(\widetilde{u},\widetilde{d})$ and $(\overline{u},\overline{d})$ be two pairs of functions in $E_K^{[0,\delta_1]}$, and we write $u^*=\widetilde{u}-\overline{u}$ and $d^*=\widetilde{d}-\overline{d}$. Similarly to the proof in (3.33)-(3.37), we have

$$\begin{aligned}&\||(\mathbb{T}_1(\widetilde{u},\widetilde{d})-\mathbb{T}_1(\overline{u},\overline{d}),\mathbb{T}_2(\widetilde{u},\widetilde{d})-\mathbb{T}_2(\overline{u},\overline{d}))\|_{\mathbb{E}_K^{[0,t]}}\\&\le C_8\theta(t)\big(\|d^*\|_{L^{\infty}(\mathbb{R}^n)}+\|(u^*,d^*)\|_{\mathbb{E}_K^{[0,t]}}\big).\end{aligned} \tag{4.29}$$

We choose a $\delta_1>0$ small enough, such that for any $t\in[0,\delta_1]$, there holds

$$C_8\theta(t)\le\frac{3}{4}. \tag{4.30}$$

Then we finish the proof of Step 2.

Taking $\delta=\min\{\delta_0,\delta_1\}$, we conclude that there exists a unique pair of solution $(u^*,d^*)\in\mathbb{E}_K^{[0,\delta]}$. By the uniqueness of the solution, for (u,d), the local solution to (1.1), we have $(u,d)=(u^*,d^*)$ in the time interval $[0,\delta]$.

We finish the proof of Proposition 4.1.

Remark 4.1 If we take the initial data $\|u_0\|_{L^n(\mathbb{R}^n)}$ and $\|\nabla d_0\|_{L^n(\mathbb{R}^n)}$ small enough, which implies the global existence, then after a slight modification of Proposition 4.1, we can prove the results of Proposition 4.1 on $t\in(0,+\infty)$. For this situation, we can get the following decay estimates immediately:

$$\|\nabla^k u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla^k \nabla d\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{k+1}{2}}, \tag{4.31}$$

for any $t > 0$ and integer $k \ge 0$.

Proof of (4.5) By Proposition 4.1, it is sufficient to prove Theorem 1.2 on $[\delta, T]$ with $T < \infty$ and $\delta \le T$.

Denote

$$\begin{cases} U(x,t) = u(x, t + \frac{\delta}{2}), \\ D(x,t) = d(x, t + \frac{\delta}{2}), \end{cases} \quad (x,t) \in \mathbb{R}^n \times \Big[0, T - \frac{\delta}{2}\Big]. \tag{4.32}$$

By the local existence in Theorem 1.1, (U, D) is the solution to (1.1) on $[\delta/2, T]$ with the initial data $(U_0, D_0) = (u, d)|_{t=\frac{\delta}{2}}$. Due to the results of Lin-Lin-Wang [24], we have

$$(U(x,t), D(x,t)) \in C^\infty\Big(\Big[0, T - \frac{\delta}{2}\Big], \mathbb{R}^n\Big). \tag{4.33}$$

We can write (U, D) as

$$\begin{cases} U = S(t)u\Big(x, \dfrac{\delta}{2}\Big) - \displaystyle\int_0^t S(t-\tau)\mathbb{P}\nabla \cdot [U \otimes U + \nabla D \otimes \nabla D](\cdot, \tau)\mathrm{d}\tau, \\ D = S(t)d\Big(x, \dfrac{\delta}{2}\Big) + \displaystyle\int_0^t S(t-\tau)\big(|\nabla D|^2 D - U \cdot \nabla D\big)(\cdot, \tau)\mathrm{d}\tau. \end{cases} \tag{4.34}$$

Similarly to (4.16), by using Hölder inequality and (4.25), we have

$$\begin{aligned} &\|t^{\frac{k}{2}} \nabla^k U\|^p_{L^p(0, T - \frac{\delta}{2}; L^q(\mathbb{R}^n))} \\ \lesssim\ & \Big(T - \frac{\delta}{2}\Big)^{\frac{pk}{2}} \|\nabla^k U\|^{p-(n+2)}_{L^\infty(0, T-\frac{\delta}{2}; L^n(\mathbb{R}^n))} \cdot \|\nabla^k U\|^{n+2}_{L^{n+2}(0, T - \frac{\delta}{2}; \mathbb{R}^n)} \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} &\|t^{\frac{k}{2}} \nabla^{k+1} D\|^p_{L^p(0, T - \frac{\delta}{2}; L^q(\mathbb{R}^n))} \\ \lesssim\ & \Big(T - \frac{\delta}{2}\Big)^{\frac{pk}{2}} \|\nabla^{k+1} D\|^{p-(n+2)}_{L^\infty(0, T-\frac{\delta}{2}; L^n(\mathbb{R}^n))} \|\nabla^{k+1} D\|^{n+2}_{L^{n+2}(0, T - \frac{\delta}{2}; \mathbb{R}^n)}. \end{aligned} \tag{4.36}$$

From (4.34), (4.33) and Proposition 4.1, for any integer $j \ge 0$, we have

$$\begin{aligned} &\|\nabla^j U\|_{L^\infty(0, T - \frac{\delta}{2}; L^n(\mathbb{R}^n))} \\ \lesssim\ & \|\nabla^j u(x, \delta/2)\|_{L^n(\mathbb{R}^n)} + \Big\| \int_0^t S(t-\tau)\mathbb{P}\nabla^{j+1} \cdot [U \otimes U + \nabla D \otimes \nabla D]\mathrm{d}\tau \Big\|_{L^n(\mathbb{R}^n)} \\ \lesssim\ & \|u_0\|_{L^n(\mathbb{R}^n)} + \sum_{l=0}^{j} \int_0^t (\|\nabla^l U\|_{L^n(\mathbb{R}^n)} + \|\nabla^{l+1} D\|_{L^n(\mathbb{R}^n)})\mathrm{d}\tau. \end{aligned} \tag{4.37}$$

Applying the summation $\sum\limits_{k=0}^{K}$, we get

$$\begin{aligned}&\sum_{k=0}^{K}\|\nabla^{k+2m}U\|_{L^\infty(0,T-\frac{\delta}{2};L^n(\mathbb{R}^n))}\\&\lesssim\|u_0\|_{L^n(\mathbb{R}^n)}+\sum_{k=0}^{K}\int_0^t(\|\nabla^k U\|_{L^n(\mathbb{R}^n)}+\|\nabla^{k+1}D\|_{L^n(\mathbb{R}^n)})\mathrm{d}\tau.\end{aligned}\tag{4.38}$$

Similarly to (4.37)-(4.38), we have

$$\begin{aligned}&\sum_{k=0}^{K}\|\nabla^{k+1}D\|_{L^\infty(0,T-\frac{\delta}{2};L^n(\mathbb{R}^n))}\\&\lesssim\|\nabla d_0\|_{L^n(\mathbb{R}^n)}+\sum_{k=0}^{K}\int_0^t(\|\nabla^{k+1} U\|_{L^n(\mathbb{R}^n)}+\|\nabla^{k+1}D\|_{L^n(\mathbb{R}^n)})\mathrm{d}\tau.\end{aligned}\tag{4.39}$$

By adding (4.38) and (4.39), and using Gronwall inequality, we get

$$\sum_{k=0}^{K}(\|\nabla^k U\|_{L^\infty(0,T-\frac{\delta}{2};L^n(\mathbb{R}^n))}+\|\nabla^{k+1}D\|_{L^\infty(0,T-\frac{\delta}{2};L^n(\mathbb{R}^n))})<\infty.\tag{4.40}$$

By Proposition 2.1, we have

$$\begin{aligned}&\|t^{k/2}\nabla^k U\|_{L^{n+2}(0,T-\frac{\delta}{2};\mathbb{R}^n)}\lesssim\|\nabla^k U\|_{L^{n+2}(0,T-\frac{\delta}{2};\mathbb{R}^n)}\\&\lesssim\Big\|\nabla^k u\Big(x,\frac{\delta}{2}\Big)\Big\|_{L^n(\mathbb{R}^n)}+\|\nabla^k(U\otimes U+\nabla D\otimes\nabla D)\|_{L^{\frac{n+2}{2}}(0,T-\frac{\delta}{2};\mathbb{R}^n)}\\&\lesssim\|u_0\|_{L^n(\mathbb{R}^n)}+\sum_{l=0}^{k}(\|\nabla^l U\|^2_{L^{n+2}(0,T-\frac{\delta}{2};\mathbb{R}^n)}+\|\nabla^{l+1}D\|^2_{L^{n+2}(0,T-\frac{\delta}{2};\mathbb{R}^n)})\\&\le C(k,T,\delta)\Big(\|u_0\|_{L^n(\mathbb{R}^n)}+\sum_{l=0}^{k}\|\nabla^l U\|^{\frac{2n}{n+2}}_{L^\infty(0,T-\frac{\delta}{2};L^n(\mathbb{R}^n))}\|\nabla^l U\|^{\frac{2}{n+2}}_{L^\infty(0,T-\frac{\delta}{2};\mathbb{R}^n)}\\&\quad+\sum_{l=0}^{k}\|\nabla^{l+1} D\|^{\frac{2n}{n+2}}_{L^\infty(0,T-\frac{\delta}{2};L^n(\mathbb{R}^n))}\|\nabla^{l+1} D\|^{\frac{2}{n+2}}_{L^\infty(0,T-\frac{\delta}{2};\mathbb{R}^n)}\Big).\end{aligned}\tag{4.41}$$

From (4.41), recalling that (4.33) and (4.40), we have

$$\sum_{k=0}^{K}\|t^{k/2}\nabla^k U\|_{L^{n+2}(0,T-\frac{\delta}{2};\mathbb{R}^n)}<\infty.\tag{4.42}$$

From (4.35), (4.40) and (4.42), we have

$$\sum_{k=0}^{K}\|t^{k/2}\nabla^k U\|_{L^p(0,T-\frac{\delta}{2};L^q(\mathbb{R}^n))}<\infty.\tag{4.43}$$

Similarly, we can also get

$$\sum_{k=0}^{K} \|t^{k/2}\nabla^{k+1} D\|_{L^p(0,T-\frac{\delta}{2};L^q(\mathbb{R}^n))} < \infty. \tag{4.44}$$

Combining Proposition 4.1 and the estimates (4.40), (4.42) and (4.44), we finish the proof of (4.5).

For the complete proof of Theorem 1.2, we use the induction as the explanation at the beginning of this Section.

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