Ann. of Appl. Math.
32:4(2016), 418-428

MULTIPLE POSITIVE SOLUTIONS FOR A FOURTH-ORDER NONLINEAR EIGENVALUE PROBLEM*†

Yong-Hui Zhou, Yun-Rui Yang‡, Li Liu

(*School of Math. and Physics, Lanzhou Jiaotong University, Lanzhou 730070, Gansu, PR China*)

Abstract

In this paper, by using the Guo-Krasnoselskii's fixed-point theorem, we establish the existence and multiplicity of positive solutions for a fourth-order nonlinear eigenvalue problem. The corresponding examples are also included to demonstrate the results we obtained.

Keywords positive solutions; eigenvalue problem; fixed point

2000 Mathematics Subject Classification 34B15

1 Introduction

In the past decades, an increasing interest in the existence and multiplicity of positive solutions for boundary value problems has been evolved by using some fixed-point theorems, for example, by the Krasnoselskii's fixed-point theorem, Ma [1] and Li [2] respectively established the existence and multiplicity of positive solutions for some fourth-order boundary value problems. Zhong [3] established the existence of at least one positive solution for the following four-point boundary value problem

$$\begin{cases} y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 \le t \le 1, \\ y(0) = y(1) = 0, \\ ay''(\xi_1) - by'''(\xi_1) = 0, \quad cy''(\xi_2) + dy'''(\xi_2) = 0. \end{cases}$$

In 2015, Wu [4] obtained some new results on the existence of at least one positive solution for the following fourth-order three-point nonlinear eigenvalue problem

*Supported by the NSF of China (11301241), Science and Technology Plan Foundation of Gansu Province (145RJYA250), and Young Scientist Foundation of Lanzhou Jiaotong University (2011029).

†Manuscript received June 12, 2016; Revised September 25, 2016

‡Corresponding author. E-mail: lily1979101@163.com

$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0, \\ au''(\eta) - bu'''(\eta) = 0, \quad cu''(1) + du'''(1) = 0. \end{cases}$$

Bai [5] obtained the existence of triple positive solutions via the Leggett-Williams fixed-point theorem [6]. There are other meaningful investigated results on the existence of positive solutions for some types of nonlinear differential equations, one can be referred to [1-4,7,8]. But to the best of our knowledge, there are not many results on the existence of multiple positive solutions for fourth-order nonlinear eigenvalue problems with multi-points boundary value condition.

Based on the fact, our purpose in this paper is to investigate the existence and multiplicity of positive solutions for the following fourth-order three-point eigenvalue problem

$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0, \\ au''(\xi) - bu'''(\xi) = 0, \quad cu''(1) + du'''(1) = 0, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $0<\frac{1}{4}<\xi<\frac{2}{3}<1$, a, b, c, d are nonnegative constants satisfying $ad+bc+ac>0$, $b-a\xi\ge 0$, $h(t)\in C[0,1]$, $f\in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty))$.

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we state and prove our main results on the existence and multiplicity of positive solutions for (1.1). At the same time, the corresponding examples are also included to demonstrate the results we obtained.

2 Preliminaries

For convenience, we first state some definitions and preliminary results which we need. Throughout this paper, we make the following assumptions:

(H1) $f \in C([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty))$ is continuous;

(H2) $h(t) \in C([0,1])$, $h(t) \le 0$ for all $t \in [0,\xi]$, $h(t) \ge 0$ for all $t \in [\xi,1]$, where $0 < \frac{1}{4} < \xi < \frac{2}{3} < 1$; and $h(t) \not\equiv 0$ for any subinterval of $[0,1]$.

Denote

$$\overline{f_0} = \limsup_{|u|+|v|\to 0^+} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \quad \overline{f_\infty} = \limsup_{|u|+|v|\to \infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \tag{2.1}$$

$$\underline{f_0} = \liminf_{|u|+|v|\to 0^+} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \quad \underline{f_\infty} = \liminf_{|u|+|v|\to \infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \tag{2.2}$$

and

$$A=\int_\xi^1 G_2(s,s)h(s)\mathrm{d}s,\ B=\min\Big\{\frac{a}{\Delta}\Big(\frac{1}{4}+\frac{3\xi}{4}\Big)\Big(\frac{c}{2}+d\Big)\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s,\ \frac{a}{2\Delta}\Big(\frac{c}{4}+d\Big)\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s\Big\},$$

where $\Delta = ad + bc + ac(1-\xi) > 0$.

Our main results in this paper mainly depend on the following Guo-Krasnoselskii's fixed-point theorem.

Theorem 2.1[9] *Let E be a Banach space, $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ is a completely continuous operator such that either*

(i) $\|Tu\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|,\ u \in K \cap \partial\Omega_2$*; or*

(ii) $\|Tu\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|,\ u \in K \cap \partial\Omega_2$

holds, then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let $C[0,1]$ be endowed with the maximum norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|,$$

and $C^2[0,1]$ be endowed with the norm

$$\|u\|_2 = \|u\| + \|u''\| = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u''(t)|.$$

Let $G_1(t,s)$ and $G_2(t,s)$ be the Green's functions of the following boundary value problems

$$\begin{cases} -u'' = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} -v'' = 0, \quad t \in (0,1), \\ av(\xi) - bv'(\xi) = 0, \quad cv(1) + dv'(1) = 0, \end{cases}$$

respectively. In particular,

$$G_1(t,s) = \begin{cases} (1-s)t, & 0 \leq t \leq s \leq 1, \\ (1-t)s, & 0 \leq s \leq t \leq 1; \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{1}{\Delta}(a(t-\xi)+b)(c(1-s)+d), & 0 \leq t \leq s \leq 1,\ \xi \leq s \leq 1, \\ \dfrac{1}{\Delta}(a(s-\xi)+b)(c(1-t)+d), & 0 \leq s \leq t \leq 1,\ \xi \leq s \leq 1. \end{cases}$$

It is easy to check that

$$0 \leq G_1(t,s) \leq G_1(s,s), \quad 0 \leq t \leq s \leq 1, \tag{2.3}$$

$$G_1(t,s) \geq \frac{1}{4} G_1(s,s), \quad s \in [0,1],\ t \in \left[\frac{1}{4}, \frac{3}{4}\right], \tag{2.4}$$

$$0 \leq G_2(t,s) \leq G_2(s,s), \quad (t,s) \in [0,1] \times [\xi,1], \tag{2.5}$$

$$G_2(t,s) \geq \frac{1}{4} G_2(s,s), \quad s \in [\xi,1],\ t \in \left[\frac{1}{4} + \frac{3\xi}{4}, \frac{3}{4} + \frac{\xi}{4}\right]. \tag{2.6}$$

Define a cone K in $C^2[0,1]$ by

$$K=\Big\{u\in C^2[0,1]: u\geq 0,\ u''\leq 0,\ \min_{t\in[\frac14,\frac34]}u(t)\geq\frac14\|u\|,\ \min_{t\in[\frac14+\frac{3\xi}{4},\frac34+\frac{\xi}{4}]}[-u''(t)]\geq\frac14\|u''\|\Big\}. \tag{2.7}$$

Define an integral operator $T:K\to C^2[0,1]$ by

$$(Tu)(t)=\int_0^1\Big[\int_\xi^1 G_1(t,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s. \tag{2.8}$$

Therefore,

$$(Tu)''(t)=-\int_\xi^1 G_2(t,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\leq 0, \tag{2.9}$$

and

$$|(Tu)''(t)|=\int_\xi^1 G_2(t,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau.$$

It is easy to check that

$$|u(t)|+|u''(t)|\geq\frac14\|u\|_2,\quad u\in K,\ t\in\Big[\frac14+\frac{3\xi}{4},\frac34\Big]. \tag{2.10}$$

Obviously, $u(t)$ is a solution for the BVP (1.1) if and only if $u(t)$ is a fixed point of the operator T.

Lemma 2.1 *Assume that* (H1) *and* (H2) *hold. If* $b\geq a\xi$, *then* $T:K\longrightarrow K$ *is completely continuous.*

Proof Denote

$$\begin{aligned}(Tu)(t)&=\int_0^1\Big[\int_\xi^1 G_1(t,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s\\&=\int_0^1 G_1(t,s)\Big[\int_\xi^s(\tau-s)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\&\quad+\frac{1}{\Delta}\int_\xi^1(b-a(\xi-s))(c(1-\tau)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s\\&=\int_0^1 G_1(t,s)(Qu)(s)\mathrm{d}s,\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}(Qu)(s)&=\int_\xi^s(\tau-s)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\&\quad+\frac{1}{\Delta}\int_\xi^1(b-a(\xi-s))(c(1-\tau)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau.\end{aligned} \tag{2.12}$$

Next, for each $t\in[0,1]$, we consider the following two cases:

<u>Case 1</u> When $t\in[0,\xi]$, for any $u\in K$, from (2.12), (H1), (H2) and $b\geq a\xi$, we have

$$\begin{aligned}(Qu)(t)=&\int_t^{\xi}(t-\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\&+\frac{1}{\Delta}\int_{\xi}^1(b-a(\xi-t))(c(1-\tau)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\ge 0.\end{aligned}\tag{2.13}$$

Case 2 When $t\in[\xi,1]$, for any $u\in K$, from (2.12), (H1), (H2) and $b\ge a\xi$, we have

$$\begin{aligned}(Qu)(t)=&\int_{\xi}^{t}(\tau-t)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\&+\frac{1}{\Delta}\int_{\xi}^t(b-a(\xi-t))(c(1-\tau)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\&+\frac{1}{\Delta}\int_{t}^1(b-a(\xi-t))(c(1-\tau)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\=&\frac{1}{\Delta}\int_{\xi}^t(b+a(\tau-\xi))(c(1-t)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\&+\frac{1}{\Delta}\int_{t}^1(b+a(t-\xi))(c(1-\tau)+d)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\ge 0.\end{aligned}\tag{2.14}$$

From (2.13) and (2.14), we get

$$(Qu)(t)\ge 0\quad\text{and}\quad (Tu)''(t)=-(Qu)(t)\le 0,\quad t\in[0,1].$$

Moreover, for any $u\in K$, from (2.13), (2.14), (H1), (H2), and $b\ge a\xi$, we have

$$\begin{aligned}\|Tu\|&=\max_{t\in[0,1]}|(Tu)(t)|\\&=\max_{t\in[0,1]}\Big|\int_0^1\Big[\int_{\xi}^1 G_1(t,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s\Big|\\&\le\int_0^1\Big[\int_{\xi}^1 G_1(s,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s,\end{aligned}$$

and

$$\|(Tu)''\|\le\int_{\xi}^1 G_2(\tau,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau.$$

On the other hand,

$$\begin{aligned}\min_{t\in[\frac14,\frac34]}(Tu)(t)&=\min_{t\in[\frac14,\frac34]}\int_0^1\Big[\int_{\xi}^1 G_1(t,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s\\&\ge\frac14\int_0^1\Big[\int_{\xi}^1 G_1(s,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s\ge\frac14\|Tu\|,\end{aligned}$$

$$\begin{aligned}\min_{t\in[\frac14+\frac{3\xi}{4},\frac34+\frac{\xi}{4}]}[-(Tu)''(t)] &= \min_{t\in[\frac14+\frac{3\xi}{4},\frac34+\frac{\xi}{4}]}(Qu)(t)\\ &= \min_{t\in[\frac14+\frac{3\xi}{4},\frac34+\frac{\xi}{4}]}\int_\xi^1 G_2(t,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\\ &\ge \frac14\int_\xi^1 G_2(\tau,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau \ge \frac14\|(Tu)''\|.\end{aligned}$$

Consequently $T: K \longrightarrow K$. Furthermore, it is not difficult to check that the operator T is completely continuous by Arzela-Ascoli theorem. This completes the proof.

Lemma 2.2 *Suppose that* (H1) *and* (H2) *hold. If $b \ge a\xi$, then for the operator $T: K \longrightarrow K$, the following conclusions hold:*

(i) *If $\overline{f_0} < \frac{6}{7\lambda A}$, $r > 0$ is small enough, then $\|Tu\|_2 \le \|u\|_2$ for any $u \in K$ with $\|u\|_2 = r$;*

(ii) *if $\underline{f_0} > \frac{4}{\lambda B}$, $r > 0$ is small enough, then $\|Tu\|_2 \ge \|u\|_2$ for any $u \in K$ with $\|u\|_2 = r$;*

(iii) *if $\overline{f_\infty} < \frac{6}{7\lambda A}$, $R > 0$ is sufficiently large, then $\|Tu\|_2 \le \|u\|_2$ for any $u \in K$ with $\|u\|_2 = R$;*

(iv) *if $\underline{f_\infty} > \frac{4}{\lambda B}$, $R > 0$ is sufficiently large, then $\|Tu\|_2 \ge \|u\|_2$ for any $u \in K$ with $\|u\|_2 = R$.*

Proof We only prove (iii) and (iv), since the proofs of (i) and (ii) are similar to those of (iii) and (iv) respectively.

(iii) If $\overline{f_\infty} < \frac{6}{7\lambda A}$, we obtain from (2.1) that there exists a number $R_0 > 0$ satisfying $|u| + |v| \ge R_0$, such that $f(t,u,v) \le \frac{6}{7\lambda A}(|u|+|v|)$. Let $R_1 \gg R_0$, then $\max\{f(t,u,v) : |u|+|v| \le R_0\} \le \frac{6}{7\lambda A}R_1$. Choose a $R > R_1$ satisfying $|u|+|v| \le R$ such that $f(t,u,v) \le \frac{6}{7\lambda A}R$.

For any $t \in [\xi,1]$, $u \in K$, $\|u\|_2 = R$, we get $f(t,u,v) \le \frac{6}{7\lambda A}R = \frac{6}{7\lambda A}\|u\|_2$. By (2.3), (2.5) and (2.8), we have

$$\begin{aligned}\|Tu\| &\le \int_0^1\Big[\int_\xi^1 G_1(s,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s\\ &\le \frac{6}{7\lambda A}\lambda\|u\|_2\int_0^1\Big[\int_\xi^1 G_1(s,s)G_2(s,\tau)h(\tau)\mathrm{d}\tau\Big]\mathrm{d}s\\ &\le \frac{6}{7\lambda A}\lambda\|u\|_2\int_0^1\Big[\int_\xi^1 G_1(s,s)G_2(\tau,\tau)h(\tau)\mathrm{d}\tau\Big]\mathrm{d}s\\ &= \frac{\frac{6}{7\lambda A}\lambda\|u\|_2}{6}\int_\xi^1 G_2(\tau,\tau)h(\tau)\mathrm{d}\tau = \frac17\|u\|_2.\end{aligned}$$

By (2.5) and (2.9), we get

$$\begin{aligned}\|(Tu)''\| &\le \int_\xi^1 G_2(s,s)\lambda h(s)f(s,u(s),u''(s))\mathrm{d}s\\ &\le \frac{6}{7\lambda A}\lambda\|u\|_2\int_\xi^1 G_2(s,s)h(s)\mathrm{d}s \le \frac{6}{7}\|u\|_2.\end{aligned}$$

Therefore, $\|Tu\|_2 = \|Tu\| + \|(Tu)''\| \le \frac{1}{7}\|u\|_2 + \frac{6}{7}\|u\|_2 = \|u\|_2$.

(iv) If $\underline{f_\infty} > \frac{4}{\lambda B}$, we obtain from (2.2) that there exists a number $R_0 > 0$, such that $f(t,u,v) \ge \frac{4}{\lambda B}(|u|+|v|)$ for any $t \in [0,1]$ and $|u|+|v| \ge R_0$. For any $t \in [\frac{1}{4}+\frac{3\xi}{4}, \frac{3}{4}]$, $u \in K$, $R \ge 4R_0$, and $\|u\|_2 = R$, we have $|u(t)|+|u''(t)| \ge \frac{1}{4}\|u\|_2 \ge R_0$. Thus, $f(t,u(t),u''(t)) \ge \frac{4}{\lambda B}(|u(t)|+|u''(t)|) \ge \frac{1}{\lambda B}\|u\|_2$ for $t \in [\frac{1}{4}+\frac{3\xi}{4}, \frac{3}{4}]$. Thus, we consider the following two cases:

Case 1 For $0 \le s \le t \le 1$, we have

$$\begin{aligned}\Big|(Tu)''\Big(\frac{1}{2}\Big)\Big| &= \int_\xi^1 G_2\Big(\frac{1}{2},s\Big)\lambda h(s)f(s,u(s),u''(s))\mathrm{d}s\\ &\ge \int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s)f(s,u(s),u''(s))\mathrm{d}s\\ &\ge \frac{1}{\lambda B}\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s)\mathrm{d}s\cdot\|u\|_2\\ &\ge \frac{\lambda}{\lambda B}\frac{1}{\Delta}a\Big(\frac{1}{4}+\frac{3\xi}{4}\Big)\Big(\frac{c}{2}+d\Big)\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s\cdot\|u\|_2 \ge \|u\|_2.\end{aligned}$$

Case 2 For $0 \le t \le s \le 1$, we have

$$\begin{aligned}\Big|(Tu)''\Big(\frac{1}{2}\Big)\Big| &= \int_\xi^1 G_2\Big(\frac{1}{2},s\Big)\lambda h(s)f(s,u(s),u''(s))\mathrm{d}s\\ &\ge \int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s)f(s,u(s),u''(s))\mathrm{d}s\\ &\ge \frac{1}{\lambda B}\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s)\mathrm{d}s\cdot\|u\|_2\\ &\ge \frac{1}{B}\frac{1}{\Delta}\Big(\frac{c}{4}+d\Big)\frac{a}{2}\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s\cdot\|u\|_2 \ge \|u\|_2.\end{aligned}$$

Therefore, $\|Tu\|_2 \ge |(Tu)''(\frac{1}{2})| \ge \|u\|_2$. The proof is completed.

3 Main Results

Theorem 3.1 *Suppose* (H1) *and* (H2) *hold. Furthermore, f satisfies either*
(i) $\overline{f_0} < \frac{6}{7\lambda A}$, $\underline{f_\infty} > \frac{4}{\lambda B}$, *for any* $\lambda \in \big(\frac{4}{B\underline{f_\infty}}, \frac{6}{7A\overline{f_0}}\big)$; *or*

(ii) $\underline{f_0} > \frac{4}{\lambda B}$, $\overline{f_\infty} < \frac{6}{7\lambda A}$, *for any* $\lambda \in \left(\frac{4}{B\underline{f_0}}, \frac{6}{7A\overline{f_\infty}}\right)$,
then the BVP (1.1) *has at least one positive solution* $u = u(t)$.

Proof Let $E = C^2[0,1]$, $\Omega_1 = \{u \in E : \|u\|_2 < r\}$, $\Omega_2 = \{u \in E : \|u\|_2 < R\}$, where $0 < r < R$. By Lemma 2.1, we know that the operator $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \longrightarrow K$ is completely continuous, then the condition (i) or (ii) of Lemma 2.2 is satisfied. Applying Theorem 2.1, it follows that T has a fixed point $u_0 \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Thus u_0 is the solution for the BVP (1.1) and satisfies

$$u_0 \geq 0, \quad u_0'' \leq 0, \quad r \leq \|u\|_2 \leq R.$$

For any $t \in (0,1)$, take $\varepsilon \in (0, \frac{1}{2})$, such that when $t \in [\varepsilon, 1-\varepsilon]$,

$$G_1(t,s) \geq \begin{cases} (1-s)\varepsilon, & t \leq s \leq 1, \\ \varepsilon s, & 0 \leq s \leq t. \end{cases}$$

Therefore, $G_1(t,s) \geq \varepsilon s(1-s)$ for any $s \in [0,1]$.

By (2.8), we have

$$\begin{aligned} u_0(t) = (Tu_0)(t) &= \int_0^1 \Big[\int_\xi^1 G_1(t,s)G_2(s,\tau)\lambda h(\tau) f(\tau, u_0(\tau), u_0''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s \\ &\geq \varepsilon \int_0^1 \Big[\int_\xi^1 G_1(s,s)G_2(s,\tau)\lambda h(\tau) f(\tau, u_0(\tau), u_0''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s \\ &\geq \varepsilon\|Tu_0\| = \varepsilon\|u_0\| > 0. \end{aligned}$$

That is $u_0(t) > 0$. This completes the proof.

If

$$f_0 = \lim_{|u|+|v|\to 0} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|} \quad \text{and} \quad f_\infty = \lim_{|u|+|v|\to\infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}$$

exist, then $\overline{f_0} = \underline{f_0} = f_0$, $\overline{f_\infty} = \underline{f_\infty} = f_\infty$.

Corollary 3.1 *Suppose* (H1) *and* (H2) *hold. Suppose further that* f *satisfies either*

(i) $f_0 = 0$, $f_\infty = \infty$ (*superlinear*); *or* (ii) $f_0 = \infty$, $f_\infty = 0$ (*sublinear*),
then the BVP (1.1) *has at least one positive solution.*

Example 3.1 Consider the following boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t), u''(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ u''\Big(\dfrac{1}{2}\Big) - 2u'''\Big(\dfrac{1}{2}\Big) = 0, \quad 2u''(1) + u'''(1) = 0, \end{cases} \tag{3.1}$$

where $h(t) = t - \frac{1}{2}$, $f = \frac{[1000(|u|+|v|)+1](|u|+|v|)}{1+|u|+|v|}$, $a = 1, b = 2, c = 2, d = 1, \xi = \frac{1}{2}$, then $\overline{f_0} = 1$, $\underline{f_\infty} = 1000$ and $\Delta = 6$, $A = \frac{37}{576}$, $B = \min\{\frac{3}{1024}, \frac{5}{1024}\} = \frac{3}{1024}$. Therefore, if

$\frac{512}{375} < \lambda < \frac{3456}{259}$, then Theorem 3.1 (i) guarantees the existence of one positive solution for the BVP (3.1).

In order to discuss the multiplicity of positive solutions for the BVP (1.1), we further assume

(H3) $f(t,u,v) > 0$, for any $t \in [0,1]$ and $|u|+|v| > 0$.

Theorem 3.2 *Suppose* (H1) *and* (H2) *hold. If one of the following two conditions holds:*

(i) (H3) *holds,* $\overline{f_0}, \overline{f_\infty} < \frac{6}{7\lambda A}$ *and there exists a positive constant* $R_0 > 0$ *satisfying* $r \ll R_0 \ll R$ *such that* $\lambda \geq \frac{R_0}{m(R_0)B}$, *where* $r > 0$ *is small enough,* $R > 0$ *is large enough and* $m(R) = \min\{f(t,u,v) : \frac{R}{4} \leq |u|+|v| \leq R, t \in [\frac{1}{4}+\frac{3\xi}{4}, \frac{3}{4}]\}$;

(ii) $\underline{f_0}, \underline{f_\infty} > \frac{4}{\lambda B}$ *and there exists a positive constant* $R_0 > 0$ *satisfying* $r \ll R_0 \ll R$ *such that* $\lambda \leq \frac{6R_0}{7M(R_0)A}$, *where* $r > 0$ *is small enough,* $R > 0$ *is large enough and* $M(R) = \max\{f(t,u,v) : |u|+|v| \leq R, t \in [\xi,1]\}$,

then the BVP (1.1) *has at least two positive solutions.*

Proof (i) Let $E = C^2[0,1]$, $\Omega_1 = \{u \in E : \|u\|_2 < r\}$, $\Omega_2 = \{u \in E : \|u\|_2 < R\}$. By the condition (i) or (iii) of Lemma 2.2, we get $\|Tu\|_2 \leq \|u\|_2$ when $u \in K \cap \partial\Omega_1$ or $u \in K \cap \partial\Omega_2$. Choose $\Omega_3 = \{u \in E : \|u\|_2 < R_0\}$ such that $\overline{\Omega_1} \subset \Omega_3$, $\overline{\Omega_3} \subset \Omega_2$. By (2.10), we know that $\frac{R_0}{4} \leq |u(t)|+|u''(t)| \leq R_0$ for any $t \in [\frac{1}{4}+\frac{3\xi}{4}, \frac{3}{4}]$ and $u \in K \cap \partial\Omega_3$. Thus, it follows the following two cases:

<u>Case 1</u> For $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned}
\|Tu\|_2 &\geq \Big|(Tu)''\Big(\frac{1}{2}\Big)\Big| = \int_\xi^1 G_2\Big(\frac{1}{2},s\Big)\lambda h(s) f(s,u(s),u''(s))\mathrm{d}s\\
&\geq \int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s) f(s,u(s),u''(s))\mathrm{d}s \geq \frac{1}{\lambda B}\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s)\mathrm{d}s\cdot\|u\|_2\\
&\geq \frac{\lambda}{\lambda B}\frac{1}{\Delta}a\Big(\frac{1}{4}+\frac{3\xi}{4}\Big)\Big(\frac{c}{2}+d\Big)\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s\cdot\|u\|_2 = \lambda m(R_0)B \geq R_0 = \|u\|_2.
\end{aligned}$$

<u>Case 2</u> For $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned}
\|Tu\|_2 &\geq \Big|(Tu)''\Big(\frac{1}{2}\Big)\Big| = \int_\xi^1 G_2\Big(\frac{1}{2},s\Big)\lambda h(s) f(s,u(s),u''(s))\mathrm{d}s\\
&\geq \int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s) f(s,u(s),u''(s))\mathrm{d}s \geq \frac{1}{\lambda B}\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} G_2\Big(\frac{1}{2},s\Big)\lambda h(s)\mathrm{d}s\cdot\|u\|_2\\
&\geq \frac{1}{B}\frac{1}{\Delta}\Big(\frac{c}{4}+d\Big)\frac{a}{2}\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s\cdot\|u\|_2 = \lambda m(R_0)B \geq R_0 = \|u\|_2.
\end{aligned}$$

Therefore, $\|Tu\|_2 > \|u\|_2$, $u \in K \cap \partial\Omega_3$. By Theorem 2.1, the BVP (1.1) has two positive solutions $u_1 \in K \cap (\overline{\Omega_3} \setminus \Omega_1)$, $u_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_3)$. Therefore

$$r \le \|u_1\|_2 < R_0 < \|u_2\|_2 \le R, \quad u_1, u_2 \notin \partial\Omega_3. \tag{3.2}$$

Together with Theorem 3.1, it follows that the BVP (1.1) has two distinct positive solutions u_1 and u_2.

(ii) Let $E = C^2[0,1]$, $\Omega_1 = \{u \in E : \|u\|_2 < r\}$, $\Omega_2 = \{u \in E : \|u\|_2 < R\}$. By the condition (ii) or (iv) of Lemma 2.2, we get $\|Tu\|_2 \ge \|u\|_2$ when $u \in K \cap \partial\Omega_1$ or $u \in K \cap \partial\Omega_2$. Choose $\Omega_3 = \{u \in E : \|u\|_2 < R_0\}$ such that $\overline{\Omega_1} \subset \Omega_3$, $\overline{\Omega_3} \subset \Omega_2$. For any $t \in [\xi, 1]$, $u \in K \cap \partial\Omega_3$, then $|u(t)| + |u''(t)| \le R_0$. By combining (2.5) with (2.8) and (2.9), we have

$$\begin{aligned}
\|Tu\| &= \int_0^1 \Big[\int_\xi^1 G_1(t,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s \\
&\le \int_0^1 \Big[\int_\xi^1 G_1(s,s)G_2(s,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s \\
&\le \lambda M(R_0)\int_0^1 \Big[\int_\xi^1 s(1-s)G_2(\tau,\tau)h(\tau)\mathrm{d}\tau\Big]\mathrm{d}s \\
&< \frac{1}{6}\lambda M(R_0)\int_\xi^1 G_2(\tau,\tau)h(\tau)\mathrm{d}\tau \le \frac{1}{7}R_0, \\
\|(Tu)''\| &= \int_\xi^1 G_2(\tau,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau \\
&\le \int_\xi^1 G_2(\tau,\tau)\lambda h(\tau)f(\tau,u(\tau),u''(\tau))\mathrm{d}\tau \\
&\le \lambda M(R_0)\int_\xi^1 G_2(\tau,\tau)h(\tau)\mathrm{d}\tau \le \frac{6}{7}R_0.
\end{aligned}$$

Hence, we get $\|Tu\|_2 = \|Tu\| + \|(Tu)''\| < \frac{1}{7}R_0 + \frac{6}{7}R_0 = R_0 = \|u\|_2$.

Applying the Theorem 3.1 again, the operator T has two fixed points $u_1 \in K \cap (\overline{\Omega}_3 \backslash \Omega_1)$ and $u_2 \in K \cap (\overline{\Omega}_2 \backslash \Omega_3)$ satisfying (3.2), which are the two different positive solutions for the BVP (1.1). The proof is completed.

Corollary 3.2 *Suppose* (H1)-(H3) *hold. Suppose further that f and λ satisfies either*

(i) $f_0 = f_\infty = 0$, $0 < \lambda \le \frac{1}{Bm(1)}$; *or* (ii) $f_0 = f_\infty = \infty$, $\lambda \ge \frac{6}{7AM(1)}$, *then BVP* (1.1) *has at least two positive solutions.*

Example 3.2 Consider the following boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda h(t)f(t,u(t),u''(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ u''\Big(\dfrac{1}{2}\Big) - 6u'''\Big(\dfrac{1}{2}\Big) = 0, \quad u'''(1) = 0, \end{cases} \tag{3.3}$$

where $h(t)=t-\frac{1}{2}$, $f=\frac{(|u|+|v|)^{\frac{1}{2}}}{10}+\frac{(|u|+|v|)^3}{60}$, $a=1$, $b=6$, $c=0$, $d=1$, $\xi=\frac{1}{2}$, then we get $\underline{f_0}=\infty$, $\underline{f_\infty}=\infty$, $\Delta=1$, $A=\frac{19}{24}$, $B=\min\{\frac{15}{1024},\frac{3}{256}\}=\frac{3}{256}$. Moreover,

$$f=\frac{(|u|+|v|)^{\frac{1}{2}}}{10}+\frac{(|u|+|v|)^3}{60}\le\frac{(|u|+|v|)^{\frac{1}{2}}}{10}+\frac{(|u|+|v|)^{\frac{1}{2}}}{60}=7\frac{(|u|+|v|)^{\frac{1}{2}}}{60},\quad t\in\Big[\frac{1}{2},1\Big].$$

Let $R_0=1$ such that $|u|+|v|\le 1$, then $M(R_0)=\max\limits_{t\in[\frac{1}{2},1]}f(t,u,v)=\frac{7}{60}$. Hence, if $0<\lambda\le\frac{6R_0}{7M(R_0)A}=\frac{8640}{931}$, then Theorem 3.2 (ii) guarantees the existence of two positive solutions for the BVP (3.3).

References

[1] R. Ma, Multiple positive solutions for a semipositone fourth-order boundary value problem, *J. Math. Anal. Appl.*, **33**(2003),217-227.

[2] Y. Li, Existence and multiplicity of positive solutions for fourth-order three-point boundary value problem, *Comput. Math. Appl.*, **26**(2003),109-116.

[3] Y.L. Zhong, S.H. Chen and C.P. Wang, Existence results for a fourth-order differential equation with a four-point boundary condition, *Appl. Math. Lett.*, **21**(2008),465-470.

[4] H. Wu, Positive solutions to fourth-order three-point nonlinear eigenvalue problem, *Ann. Math. Anal. Appl.*, **31**(2015),96-104.

[5] Ch. Bai, Triple positive solutions of three-point boundary value problems of fourth-order differential equations, *Comput. Math. Appl.*, **56**(2008),1364-1371.

[6] R.I. Leggett and L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math.J.* , **28**(1979),673-688.

[7] B. Liu, Positive solutions of fourth-order three-point boundary value problem *Comput. Math. Appl.*, **148**(2001),313-322.

[8] B. Liu, Positive solutions of fourth-order two-point boundary value problems, *Appl. Math. Comput.*, **148**(2004),407-420.

[9] D. Guo and V. Lskshmikanthan, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.

(*edited by Mengxin He*)