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ON ALMOST AUTOMORPHIC SOLUTIONS OF THIRD-ORDER NEUTRAL DELAY-DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT*†

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Abstract

We present some conditions for the existence and uniqueness of almost automorphic solutions of third order neutral delay-differential equations with piecewise constant of the form

$$(x(t)+px(t-1))'''=a_0x([t])+a_1x([t-1])+f(t),$$

where $[\cdot]$ is the greatest integer function, p, a_0 and a_1 are nonzero constants, and $f(t)$ is almost automorphic.

Keywords almost automorphic solutions; neutral delay equation; piecewise constant argument

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1 Introduction

In this paper we study certain functional differential equations of neutral delay type with piecewise constant argument of the form

$$(x(t)+px(t-1))'''=a_0x([t])+a_1x([t-1])+f(t), \tag{1}$$

here $[\cdot]$ is the greatest integer function, p, a_0 and a_1 are nonzero constants, and $f(t)$ is almost automorphic.

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By a solution $x(t)$ of (1) on $\mathbb{R}$ we mean a function continuous on $\mathbb{R}$, satisfying (1) for all $t \in \mathbb{R}$, $t \neq n \in \mathbb{Z}$, and such that the one sided third derivatives of $x(t) + px(t-1)$ exist at $n \in \mathbb{Z}$.

The concept of almost automorphic functions is more general than that of almost periodic functions, which were introduced by S. Bochner [1,2], for more details about this topics we refer to [3,4,6-9] and references therein.

Differential equations with piecewise constant argument (EPCA), which were firstly considered by Cooke and Wiener [11], and Shah and Wiener [12], describe the hybrid of continuous and discrete dynamical systems, which combine the properties of both differential equations and difference equations and have applications in certain biomedical models in the works of Busenberg and Cooke in [13]. Therefore, there are many papers concerning the differential equations with piecewise constant argument (see e.g. [14-20] and references therein). However, there are only a few works on the almost automorphy of solutions of EPCAs. To the best of our knowledge, only Minh et al [21] in 2006, Dimbour [22] in 2011 and Li [23] in 2013 studied in this line. They give sufficient conditions for the almost automorphy of bounded solutions of differential equation EPCAs.

Motivated by the above works, in this paper we investigate the existence of almost automorphy solutions of equation (1). The paper is organized as follows. In Section 2, some notation, preliminary definitions and lemmas are presented. The man result and its proofs is put in Sections 3.

2 Preliminary Definitions and Lemmas

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, real and complex numbers, respectively. $l^\infty(\mathbb{R})$ denotes the space of all bounded (two-sided) sequences $x : \mathbb{Z} \to \mathbb{R}$ with sup-norm. We always denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^k$ or $\mathbb{C}^k$, and by $BC(\mathbb{R}, \mathbb{R})$ the space of bounded continuous functions $u : \mathbb{R} \to \mathbb{R}$.

Definition 2.1 A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) = \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n)$$

for each $t \in \mathbb{R}$. The collection of such functions is denoted by $\mathcal{AA}(\mathbb{R})$.

It is clear that the function g in Definition 2.1 is bounded and measurable.

Remark 2.1 A classical example of an automorphic function given by [10] is defined as follows

$$f(t) = \sin\frac{1}{2+\cos\sqrt{2}t+\cos t}, \quad t \in \mathbb{R},$$

but $f(t)$ is not almost periodic as it is not uniformly continuous.

Some properties of the almost automorphic functions are listed below.

Proposition 2.1[3,4] *Let* $f, f_1, f_2 \in \mathcal{AA}(\mathbb{R})$. *Then the following statements hold:*

(i) $\alpha f_1 + \beta f_2 \in \mathcal{AA}(\mathbb{R})$ *for* $\alpha, \beta \in \mathbb{R}$.

(ii) $f_\tau := f(\cdot + \tau) \in \mathcal{AA}(\mathbb{R})$ *for every fixed* $\tau \in \mathbb{R}$.

(iii) $\breve{f} = f(-\cdot) \in \mathcal{AA}(\mathbb{R})$.

(iv) *The range* R_f *of* f *is precompact, so* f *is bounded.*

(v) *If* $\{f_n\} \subset \mathcal{AA}(\mathbb{R})$ *such that* $f_n \to f$ *uniformly on* $\mathbb{R}$, *then* $f \in \mathcal{AA}(\mathbb{R})$.

By (v) in Proposition 2.1, $\mathcal{AA}(\mathbb{R})$ is a Banach space equipped with the sup norm $\|f\| = \sup\limits_{t\in\mathbb{R}} |f(t)|$.

Definition 2.2[5] A sequence $x \in l^\infty(\mathbb{R})$ is said to be almost automorphic if for any sequence of integers $\{k_n'\}$, there exists a subsequence $\{k_n\}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} x_{p+k_n-k_m} = x_p,$$

for any $p \in \mathbb{Z}$. Denote by $\mathcal{AAS}(\mathbb{R})$ the set of all such sequences.

This limit means that

$$y_p = \lim_{n\to\infty} x_{p+k_n}$$

is well defined for each $p \in \mathbb{Z}$ and

$$x_p = \lim_{n\to\infty} y_{p-k_n}$$

for each $p \in \mathbb{Z}$.

It is obvious that $\mathcal{AAS}(\mathbb{R})$ is a closed subspace of $l^\infty(\mathbb{R})$, and the range of an almost automorphic sequence is precompact.

Proposition 2.2 $\{x(n)\} = \{(x_{n1}, x_{n2}, \cdots, x_{nk})\} \in \mathcal{AAS}(\mathbb{R}^k)$ (*resp.* $\mathcal{AAS}(\mathbb{C}^k)$) *if and only if* $\{x_{ni}\} \in \mathcal{AAS}(\mathbb{R})$ (*resp.* $\mathcal{AAS}(\mathbb{C})$), $i = 1, 2, \cdots, k$.

Lemma 2.1[10] *Let* B *be a bounded linear operator in* $\mathbb{R}^n$ *with* $\sigma_\Gamma(B)$ (*the part of the spectrum of* B *on the unit circle of the complex plane*) *being countable, and let* $\mathbb{R}^n$ *not contain any subspace isomorphic to* c_0. *Assume further that* $x = \{x_n\} \in l^\infty(\mathbb{R})$ *satisfies*

$$x_{n+1} = Bx_n + y_n, \quad n \in \mathbb{Z},$$

where $\{y_n\} \in \mathcal{AAS}(\mathbb{R})$. *Then* $x \in \mathcal{AAS}(\mathbb{R})$.

3 Main Results

We first rewrite equation (1) to the following equivalent system

$$(x(t)+px(t-1))'=y(t), \tag{2}$$
$$y'(t)=z(t), \tag{3}$$
$$z'(t)=a_0x([t])+a_1x([t-1])+f(t). \tag{4}$$

Let $(x(t),y(t),z(t))$ be a solution of (2)-(4) on $\mathbb{R}$, for $n\le t<n+1$, $n\in\mathbb{Z}$. Using (4) we obtain

$$z(t)=z(n)+a_0x(n)(t-n)+a_1x(n-1)(t-n)+\int_n^t f(v)\mathrm{d}v.$$

From this with (3) we obtain

$$y(t)=y(n)+z(n)(t-n)+\frac{1}{2}a_0x(n)(t-n)^2+\frac{1}{2}a_1x(n-1)(t-n)^2+\int_n^t\int_n^s f(v)\mathrm{d}v\mathrm{d}s.$$

This together with (2) we obtain

$$\begin{aligned}x(t)+px(t-1)=x(n)+px(n-1)+y(n)(t-n)+\frac{1}{2!}z(n)(t-n)^2+\frac{1}{6}a_0x(n)(t-n)^3\\+\frac{1}{6}a_1x(n-1)(t-n)^3+\int_n^t\int_n^s\int_n^\sigma f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s.\end{aligned}$$

Since $x(t)$ must be continuous at $n+1$, using the above equations we get for $n\in\mathbb{Z}$,

$$\begin{cases}x(n+1)=\Big(1-p+\dfrac{a_0}{3!}\Big)x(n)+y(n)+\dfrac{1}{2!}z(n)+\Big(p+\dfrac{a_1}{3!}\Big)x(n-1)+f_n^{(1)},\\ y(n+1)=\dfrac{a_0}{2!}x(n)+y(n)+z(n)+\dfrac{a_1}{2!}x(n-1)+f_n^{(2)},\\ z(n+1)=a_0x(n)+z(n)+a_1x(n-1)+f_n^{(3)},\end{cases} \tag{5}$$

where

$$f_n^{(1)}=\int_n^{n+1}\int_n^s\int_n^\sigma f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s,\quad f_n^{(2)}=\int_n^{n+1}\int_n^s f(v)\mathrm{d}v\mathrm{d}s,\quad f_n^{(3)}=\int_n^{n+1}f(v)\mathrm{d}v. \tag{6}$$

Next we express system (5) in terms of an equivalent system in $\mathbb{R}^4$ given by

$$v_{n+1}=Av_n+h_n, \tag{7}$$

where

$$A=\begin{pmatrix} 1-p+\dfrac{a_0}{3!} & 1 & \dfrac{1}{2!} & p+\dfrac{a_1}{3!} \\ \dfrac{a_0}{2!} & 1 & 1 & \dfrac{a_1}{2!} \\ a_0 & 0 & 1 & a_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

$v_n=(x(n),y(n),z(n),x(n-1))^{\mathrm{T}},\ h_n=(f_n^{(1)},f_n^{(2)},f_n^{(3)},0)^{\mathrm{T}}$.

Lemma 3.1 *If $f\in\mathcal{AA}(\mathbb{R})$, then the sequences $\{f_n^{(i)}\}_{n\in\mathbb{Z}}\in\mathcal{AAS}(\mathbb{R}),\ i=1,2,3$.*

Proof We typically consider $\{f_n^{(1)}\}$. Since $f(t)$ is almost automorphic, for any sequence $\{n'_k\}$, there exist a subsequence $\{n_k\}$ and a measurable function $g(t)$ such that

$$\lim_{k\to\infty}f(t+n_k)=g(t),\quad \lim_{k\to\infty}g(t-n_k)=f(t),\quad t\in\mathbb{R}.$$

Consequently, it follows from the Lebesgue dominated convergence theorem that, for each $n\in\mathbb{Z}$,

$$\begin{aligned} f_{n+n_k}^{(1)} &= \int_{n+n_k}^{n+1+n_k}\int_{n+n_k}^{s}\int_{n+n_k}^{\sigma}f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s=\int_n^{n+1}\int_n^s\int_n^{\sigma}f(v+n_k)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s \\ &\to \int_n^{n+1}\int_n^s\int_n^{\sigma}g(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s\triangleq\overline{g}_n, \\ \overline{g}_{n-n_k} &= \int_{n-n_k}^{n+1-n_k}\int_{n-n_k}^{s}\int_{n-n_k}^{\sigma}g(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s=\int_n^{n+1}\int_n^s\int_n^{\sigma}g(v-n_k)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s \\ &\to \int_n^{n+1}\int_n^s\int_n^{\sigma}f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s=f_n^{(1)}, \end{aligned}$$

as $k\to\infty$. So $\{f_n^{(1)}\}\in\mathcal{AAS}(\mathbb{R})$. In a manner similar to the above proof, we know that $\{f_n^{(2)}\},\{f_n^{(3)}\}\in\mathcal{AAS}(\mathbb{R})$. This completes the proof of Lemma 3.1.

Lemma 3.2 *Suppose that all eigenvalues of A are simple (denoted by $\lambda_1,\lambda_2,\lambda_3,\lambda_4$) and $|\lambda_i|\neq1$, $1\le i\le4$. Then there exists a unique almost automorphic solution $v_n:\mathbb{Z}\to\mathbb{C}^4$ of (7).*

Proof By Lemma 3.1 we have that $\{h_n\}=\{(f_n^{(1)},f_n^{(2)},f_n^{(3)},0)^{\mathrm{T}}\}\in\mathcal{AAS}(\mathbb{R}^4)$. It is clear that $\mathbb{R}^4$ does not contain any subspace isomorphic to c_0, and the bounded linear operator A on $\mathbb{R}^4$ has finite spectrum. So Lemma 2.1 implies that $\{v_n\}\in\mathcal{AA}(\mathbb{R}^4)$.

From our hypotheses, there exists a 4×4 nonsingular matrix P with in general complex entries such that $PAP^{-1}=\Lambda$ where $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Define $\overline{v}_n=Pv_n$; then (7) becomes

$$\overline{v}_{n+1}=\Lambda\overline{v}_n+\overline{h}_n, \tag{9}$$

where $\overline{h}_n=Ph_n$.

Suppose $\tilde{v}_n$ is another almost automorphic solution of (7) distinct from $\{v_n\}$, then $v_n - \tilde{v}_n$ solves the equation $v_{n+1} = Av_n$, and $Pv_n - P\tilde{v}_n$ solves the equation $\overline{v}_{n+1} = \Lambda\overline{v}_n$. Moreover, $Pv_n - P\tilde{v}_n$ would also be almost automorphic, but by our condition on Λ, it follows that each component of $\overline{u}_n$ must become unbounded either as $n \to \infty$ or as $n \to -\infty$, and that is impossible, since it must be almost automorphic (bounded). This proves the lemma.

Lemma 3.3 *For any solution $v_n = (x(n), y(n), z(n), x(n-1))^{\mathrm{T}}$, $n \in \mathbb{Z}$, of (7) there exists a solution $(x(t), y(t), z(t))$, $t \in R$, of (2)-(4) such that $x(n) = c_n$, $y(n) = d_n$, $z(n) = e_n$, $n \in \mathbb{Z}$.*

Proof Define

$$\begin{aligned} w(t) = {} & c_n + pc_{n-1} + d_n(t-n) + \frac{1}{2!}e_n(t-n)^2 + \frac{a_0}{3!}(t-n)^3 \\ & + \frac{a_1}{3!}c_{n-1}(t-n)^3 + \int_n^t\int_n^s\int_n^\sigma f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s, \end{aligned} \tag{10}$$

for $t \in [n, n+1)$, $n \in \mathbb{Z}$. It can easily be verified that $w(t)$ is continuous on $\mathbb{R}$. The rest proof is similar to that of Lemma 2 in [19], we omit the details.

Lemma 3.4 *Let $\{c_n\}, \{d_n\}, \{e_n\} \in \mathcal{AAS}(\mathbb{R})$, $f \in \mathcal{AA}(\mathbb{R})$ and $w(t)$ define as in (10) for $t \in [n, n+1)$, $n \in \mathbb{Z}$, then $w \in \mathcal{AA}(\mathbb{R})$.*

Proof The proof is divided into the following two steps.

Step 1 For any $\{n'_k\} \subset \mathbb{Z}$, there exist a subsequence $\{n_k\}$ of $\{n'_k\}$, three sequences $\{\widetilde{c}_n\}$, $\{\widetilde{d}_n\}$, $\{\widetilde{e}_n\}$ and a function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} & \lim_{k\to\infty} c_{n+n_k} = \widetilde{c}_n, \quad \lim_{k\to\infty} \widetilde{c}_{n-n_k} = c_n, \quad n \in \mathbb{Z}, \\ & \lim_{k\to\infty} d_{n+n_k} = \widetilde{d}_n, \quad \lim_{k\to\infty} \widetilde{d}_{n-n_k} = d_n, \quad n \in \mathbb{Z}, \\ & \lim_{k\to\infty} e_{n+n_k} = \widetilde{e}_n, \quad \lim_{k\to\infty} \widetilde{e}_{n-n_k} = e_n, \quad n \in \mathbb{Z}, \\ & \lim_{k\to\infty} f(t+n_k) = \widetilde{f}(t), \quad \lim_{k\to\infty} \widetilde{f}(t-n_k) = f(t), \quad t \in \mathbb{R}. \end{aligned} \tag{11}$$

Let

$$\begin{aligned} \widetilde{w}(t) = {} & \widetilde{c}_n + p\widetilde{c}_{n-1} + \widetilde{d}_n(t-n) + \frac{1}{2!}\widetilde{e}_n(t-n)^2 + \frac{1}{3!}a_0\widetilde{c}_n(t-n)^3 \\ & + \frac{1}{3!}a_1\widetilde{c}_{n-1}(t-n)^3 + \int_n^t\int_n^s\int_n^\sigma \widetilde{f}(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s \end{aligned} \tag{12}$$

for $t \in [n, n+1)$, $n \in \mathbb{Z}$. Noticing that f and $\widetilde{f}$ are bounded measurable, by (11) and (12),

$$
\begin{aligned}
&|w(t+n_k)-\widetilde{w}(t)|\\
\le\ &|c_{n+n_k}-\widetilde{c}_n|+|p||c_{n+n_k-1}-\widetilde{c}_{n-1}|+|d_{n+n_k}-\widetilde{d}_n|(t-n)+\frac{1}{2!}|e_{n+n_k}-\widetilde{e}_n|(t-n)^2\\
&+\frac{1}{3!}|a_0||c_{n+n_k}-\widetilde{c}_n|(t-n)^3+\frac{1}{3!}|a_1||c_{n+n_k-1}-\widetilde{c}_{n-1}|(t-n)^3\\
&+\Big|\int_{n+n_k}^{t+n_k}\int_{n+n_k}^{s}\int_{n+n_k}^{\sigma}f(r)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s-\int_n^t\int_n^s\int_n^\sigma\widetilde{f}(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s\Big|\\
\le\ &|c_{n+n_k}-\widetilde{c}_n|+|p||c_{n+n_k-1}-\widetilde{c}_{n-1}|+|d_{n+n_k}-\widetilde{d}_n|+\frac{1}{2!}|e_{n+n_k}-\widetilde{e}_n|\\
&+\frac{1}{3!}|a_0||c_{n+n_k}-\widetilde{c}_n|+\frac{1}{3!}|a_1||c_{n+n_k-1}-\widetilde{c}_{n-1}|\\
&+\int_n^t\int_n^s\int_n^\sigma|f(v+n_k)-\widetilde{f}(v)|\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s\\
\to\ &0\quad\text{as } k\to\infty.
\end{aligned}
$$

Similarly, we can show that $\lim\limits_{k\to\infty}\widetilde{w}(t-n_k)=w(t)$ for each $t\in\mathbb{R}$.

Step 2 We consider the general case where $\{s'_k\}_{k\in\mathbb{Z}}$ may not be an integer sequence. Let $n'_k=[s'_k]$ and $t'_k=s'_k-n'_k\in[0,1)$ for each k. Then by Step 1, there exist subsequences $\{t_k\},\{s_k\}$ and $\{n_k\}$ of $\{t'_k\},\{s'_k\}$ and $\{n'_k\}$, respectively, such that $t_k=s_k-n_k$, $k\in\mathbb{Z}$, $\lim\limits_{k\to\infty}t_k=\bar{t}\in[0,1]$, (11) holds and for each $t\in\mathbb{R}$,

$$
\lim_{k\to\infty}w(t+\bar{t}+n_k)=\widetilde{w}(t+\bar{t}),\quad \lim_{k\to\infty}\widetilde{w}(t+\bar{t}-n_k)=w(t+\bar{t}),\tag{13}
$$

where $\widetilde{w}$ is given by (12). Let $\widetilde{w}_1=\widetilde{w}(\cdot+\bar{t})$. Then it is sufficient to prove that

$$
\lim_{k\to\infty}w(t+s_k)=\widetilde{w}_1(t),\quad \lim_{k\to\infty}\widetilde{w}_1(t-s_k)=w(t),\quad \text{for each } t\in\mathbb{R}.\tag{14}
$$

Now there are two cases to be considered: $\bar{t}+t>[\bar{t}+t]$ and $\bar{t}+t=[\bar{t}+t]$. Assume that $\bar{t}+t>[\bar{t}+t]$. Then $[t+\bar{t}]=[t+t_k]$ for sufficiently large k. Noticing the boundedness of $f(t),\{c_n\},\{d_n\}$ and $\{e_n\}$, for sufficiently large k, we obtain

$$
\begin{aligned}
&|w(t+s_k)-w(t+\bar{t}+n_k)|\\
=\ &|w(t+t_k+n_k)-w(t+\bar{t}+n_k)|\\
\le\ &|d_{[t+\bar{t}]+n_k}||t_k-\bar{t}|+\frac{1}{2!}|e_{[t+\bar{t}]+n_k}||(t+t_k-[t+t_k])^2-(t+\bar{t}-[t+\bar{t}])^2|\\
&+\frac{a_0}{3!}|c_{[t+\bar{t}]+n_k}||(t+t_k-[t+t_k])^3-(t+\bar{t}-[t+\bar{t}])^3|\\
&+\frac{a_1}{3!}|c_{[t+\bar{t}]+n_k-1}||(t+t_k-[t+t_k])^3-(t+\bar{t}-[t+\bar{t}])^3|\\
&+\Big|\int_{[t+t_k]+n_k}^{t+t_k+n_k}\int_{[t+t_k]+n_k}^{s}\int_{[t+t_k]+n_k}^{\sigma}f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s-\int_{[t+\bar{t}]+n_k}^{t+\bar{t}+n_k}\int_{[t+\bar{t}]+n_k}^{s}\int_{[t+\bar{t}]+n_k}^{\sigma}f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le |d_{[t+\bar t]+n_k}||t_k-\bar t| + \frac{1}{2!}|e_{[t+\bar t]+n_k}|\,|(2t+t_k+\bar t-2[t+\bar t])(t_k-\bar t)| \\
&\quad +\frac{a_0}{3!}|c_{[t+\bar t]+n_k}|\,|(3t^2+t_k^2+t_k\bar t+\bar t^2-6t[t+\bar t]-3[t+\bar t](t_k+\bar t)+3[t+\bar t]^2)(t_k-\bar t)| \\
&\quad +\frac{a_1}{3!}|c_{[t+\bar t]+n_k-1}|\,|(3t^2+t_k^2+t_k\bar t+\bar t^2-6t[t+\bar t]-3[t+\bar t](t_k+\bar t)+3[t+\bar t]^2)(t_k-\bar t)| \\
&\quad +\int_{t+\bar t}^{t+t_k}\int_{[t+\bar t]}^{s}\int_{[t+\bar t]}^{\sigma}|f(v+n_k)|\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s \\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned}
$$

This together with (13) implies that $\lim\limits_{k\to\infty} w(t+s_k)=\widetilde{w}_1(t)$.

Assume that $t+\bar t=[t+\bar t]$, that is $t+\bar t\in\mathbb{Z}$. If $t+t_k\ge t+\bar t$, (14) can be proved by an argument similar to the above one, and we omit the details. If $t+t_k<t+\bar t$, $[t+t_k]=t+\bar t-1$ for sufficiently large k and $t+t_k-[t+t_k]\to 1$ as $k\to\infty$. Notice also that

$$
\begin{aligned}
w(m) = c_{m-1}+pc_{m-2}+d_{m-1}+\frac{1}{2!}e_{m-1}+\frac{1}{3!}a_0c_{m-1}+\frac{1}{3!}a_1c_{m-2} \\
+\int_{m-1}^{m}\int_{m-1}^{s}\int_{m-1}^{\sigma} f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s,
\end{aligned}
$$

for any $m\in\mathbb{Z}$. Then for sufficiently large k,

$$
\begin{aligned}
&|w(t+s_k)-w(t+\bar t+n_k)| \\
&= |w(t+t_k+n_k)-w(t+\bar t+n_k)| \\
&\le |d_{t+\bar t-1+n_k}||t_k-\bar t| + \frac{1}{2!}|e_{t+\bar t-1+n_k}||(t_k-\bar t+1)^2-1| \\
&\quad +\frac{a_0}{3!}|c_{t+\bar t-1+n_k}||(t_k-\bar t+1)^2-1| + \frac{a_1}{3!}|c_{t+\bar t+n_k-2}||(t_k-\bar t+1)^2-1| \\
&\quad +\Big|\int_{[t+t_k]+n_k}^{t+t_k+n_k}\int_{[t+t_k]+n_k}^{s}\int_{[t+t_k]+n_k}^{\sigma} f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s \\
&\quad -\int_{t+\bar t-1+n_k}^{t+\bar t+n_k}\int_{t+\bar t-1+n_k}^{s}\int_{t+\bar t-1+n_k}^{\sigma} f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s\Big| \\
&\le |d_{t+\bar t-1+n_k}||t_k-\bar t| + \frac{1}{2!}|e_{t+\bar t-1+n_k}||(t_k-\bar t+1)^2-1| \\
&\quad +\frac{a_0}{3!}|c_{t+\bar t-1+n_k}||(t_k-\bar t+1)^2-1| + \frac{a_1}{3!}|c_{t+\bar t+n_k-2}||(t_k-\bar t+1)^2-1| \\
&\quad +\int_{t+\bar t}^{t+t_k}\int_{t+\bar t-1}^{s}\int_{t+\bar t-1}^{\sigma}|f(v+n_k)|\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s \\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned}
$$

This together with (13) leads to $\lim\limits_{k\to\infty} w(t+s_k)=\widetilde{w}_1(t)$.

Similarly, we can prove that $\lim\limits_{k\to\infty}\widetilde{w}_1(t-s_k)=w(t)$ for each $t\in\mathbb{R}$, and then (14) is true.

Theorem 1 *If $|p| \neq 1$. Suppose that all eigenvalues of A are simple (denoted by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$) and $|\lambda_i| \neq 1$, $1 \leq i \leq 4$. Then equation (1) has a unique almost automorphic solution $x(t)$, which can, in fact be determined explicitly in terms of $w(t)$ as defined in the proof of Lemma 3.3.*

Proof From Lemma 3.2, we know that system (7) has a unique bounded solution $\{v_n\}_{n\in\mathbb{Z}} \in \mathcal{PAAS}(\mathbb{R}^4)$. Let (c_n, d_n, e_n) be the first three components of v_n, now it follows from Lemma 3.3 that (1) has a unique bounded solution $x(t)$ such that $x(n) = c_n$, $y(n) = d_n$, $z(n) = d_n$, $n \in \mathbb{Z}$, where $y(n)$ and $z(n)$ are defined in (2)-(4), and for $t \in [n, n+1)$, $n \in \mathbb{Z}$, and for $t \in \mathbb{R}$,

$$\begin{aligned} w(t) = c_n + pc_{n-1} + d_n(t-n) + \frac{1}{2!}e_n(t-n)^2 + \frac{1}{3!}a_0c_n(t-n)^3 \\ + \frac{1}{3!}a_1c_{n-1}(t-n)^3 + \int_n^t\int_n^s\int_n^\sigma f(v)\mathrm{d}v\mathrm{d}\sigma\mathrm{d}s. \end{aligned} \tag{15}$$

From Lemma 3.4, we have that $w \in \mathcal{AA}(\mathbb{R})$. It is easy to get

$$x(t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}(-p)^n w(t-n), & |p| < 1, \\ \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{n+1}} w(t+n+1), & |p| > 1. \end{cases} \tag{16}$$

Therefore $x \in \mathcal{AA}(\mathbb{R})$ by Proposition 2.1.

The uniqueness of $x(t)$ as an almost automorphic solution of (1) follows from the uniqueness of the almost automorphic solution $v_n : \mathbb{Z} \to \mathbb{R}^4$ of (7) given by Lemma 3.3, which determines the uniqueness of $w(t)$, and therefore from (16) the uniqueness of $x(t)$. This completes the proof.

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