Ann. of Appl. Math.
33:1(2017), 1-5

A SINGLE SPECIE MODEL WITH RANDOM PERTURBATION UNDER CONTRACEPTIVE CONTROL*†

Qiuying Li‡, Hanwu Liu, Jingqi Xin
(*Dept. of Math., Yuncheng University, Yuncheng 044000, Shanxi, PR China*)

Abstract

In this paper, we formulate a single-species model of contraception control with white noise on the death rate. Firstly, the uniqueness of global positive solution of the model is proved. Secondly, uniformly bounded mean of solution is obtained by using the Liyapunov function and Chebyshev inequality. Lastly, stochastic global asymptotic stability of zero equilibriums is analyzed.

Keywords contraceptive control; stochastic perturbation; stochastic mean ultimately bounded; stochastic global asymptotic stability

2000 Mathematics Subject Classification 60H10; 34D23

1 Introduction

Small mammalian pests pose major ecological and/or economic problems. A wide range of ways controlling small mammalian pest are available to farmers, such as physical, chemical, biological and cultural tools. Effectively controlling pests has become an increasingly complex issue over the past two decades. Farmers often catch pests by mechanical tools or poison pests by use of pesticides[1,2]. However, overuse of chemicals has created many ecological and sociological problems, hence chemical control now needs to be used reasonably. As an alternative to these methods, contraceptive control is based on reducing birth rates. There are many advantages of contraception for biological control. In recent years, many models have been formulated to investigate theoretically the potential function of contraception in the controlling of mammalian pests. [3] established the following model

$$\begin{cases} F'(t) = F(b - d_1 - k(F(t) + S(t)) - \mu), \\ S'(t) = \mu F(t) - d_1 S(t), \end{cases} \tag{1}$$

*Foundation: This work was supported by the National Natural Sciences Foundation of China (11371313) and the Sciences Foundation of Yuncheng University (XK2012003).

†Manuscript received January 24, 2016; Revised November 23, 2016

‡Corresponding author. E-mail: liqy-123@163.com

where $F(t)$ and $S(t)$ represent the densities of the fertile specie and the sterile specie at time t respectively. $b > 0$ and d_1 denote the birth and the death rates of the single specie respectively; μ stands for the rate from the fertile specie to the sterile specie. For (1), Liu and Li discussed the global stability of nonnegative equilibrium.

As well known, the growth of any species is interpreted by many small random factors such as sunlight, temperature, moisture, etc. These factors are intergraded as white noise effect. So, the stochastic differential equation ([4,5]) has more practical significance than the deterministic differential equation. This motivates us to investigate a model of single-species model with random perturbation under contraceptive control

$$\begin{cases} \mathrm{d}F(t) = F(b - d_1 - k(F(t) + S(t)) - \mu)\mathrm{d}t - \sigma F(t)\mathrm{d}B_t, \\ \mathrm{d}S(t) = [\mu F(t) - d_1 S(t)]\mathrm{d}t - \sigma S(t)\mathrm{d}B_t. \end{cases} \tag{2}$$

Here σ^2 is a constant which represents the environmental stochastic perturbation on the death rate. $B(t)$ is a real Wiener process defined on stochastic basis $(\Omega, \mathcal{F}, P)$.

The initial condition of (2) satisfies

$$(F(0), S(0)) = (\varphi_1(0), \varphi_2(0)) \in R_+^2.$$

2 Stochastically Ultimately Bounded

Theorem 1 *There exists a unique solution $(F(t), S(t))$ on $t \geq 0$ in R_+^2 with probability 1 for any given initial value $(F(0), S(0)) \in R_+^2$.*

Proof Let $x(t) = \ln F(t)$, $y(t) = S(t)$, $t \geq 0$, then (2) is transformed into

$$\begin{cases} \mathrm{d}x(t) = (b - d_1 - k(\mathrm{e}^{x(t)} + y(t)) - \mu)\mathrm{d}t - \sigma \mathrm{d}B_t, \\ \mathrm{d}y(t) = [\mu \mathrm{e}^{x(t)} - d_1 y(t)]\mathrm{d}t - \sigma y(t)\mathrm{d}B_t. \end{cases} \tag{3}$$

Obviously, (3) satisfies the linear and local Lipschitz conditions, then there is a unique local solution $(x(t), y(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Further, by Ito's formula, it is easy to obtain that $F(t) = \mathrm{e}^{x(t)}$, $S(t) = y(t)$, $t \in [0, \tau_e)$, is the unique positive local solution of (2) with an initial value $(F(0), S(0)) \in R_+^2$.

Next, we show that this solution is global, that is, $\tau_e = \infty$. Because $(F(0), S(0)) \in R_+^2$, we can choose an N_0 large enough such that $(F(0), S(0)) \in [\frac{1}{N_0}, N_0] \times [\frac{1}{N_0}, N_0]$. For every integer $N > N_0$, define the stopping time $\tau_N = \inf\{t \in [0, \tau_e) | F(t) \notin [\frac{1}{N}, N]$ or $S(t) \notin [\frac{1}{N}, N]\}$. Obviously, τ_N increases as to N. Let $\tau_\infty = \lim\limits_{N \to +\infty} \tau_N$, then we obtain $\tau_\infty < \tau_e$ a.s.. We can claim $\tau_\infty = \infty$. Assume $\tau_\infty \neq \infty$, then there exist constants $T > 0$ and $\delta \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \delta. \tag{4}$$

Therefore there exists a $N_1 > N_0$, satisfying $P\{\tau_N \le T\} > \delta$, $N > N_1$. Define a positive function $V(F,S) = \frac{1}{k}(F(t) - 1 - \ln F(t)) + \frac{1}{d_1}(S(t) - 1 - \ln S(t))$, where $(F(t), S(t)) \in R^2_+$. From Ito's formula, we get

$$\mathrm{d}V(F,S) = L(V)\mathrm{d}t - \frac{\sigma(F(t)-1)}{k}\mathrm{d}B_t - \frac{\sigma(S(t)-1)}{d_1}\mathrm{d}B_t. \tag{5}$$

Here

$$\begin{aligned}
L(V) = &\frac{1}{k}\frac{\partial V}{\partial F}F(t)(b - d_1 - k(F(t)+S(t)) - \mu) + \frac{1}{d_1}\frac{\partial V}{\partial S}(\mu F(t) - d_1 S(t)) \\
&+\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} -\sigma F(t) & 0 \\ 0 & -\sigma S(t)\end{pmatrix}\begin{pmatrix} \frac{1}{kF^2(t)} & 0 \\ 0 & \frac{1}{d_1 S^2(t)}\end{pmatrix}\begin{pmatrix} -\sigma F(t) & 0 \\ 0 & -\sigma S(t)\end{pmatrix}\right] \\
= &\frac{1}{k}\Big(1 - \frac{1}{F(t)}\Big)F(t)(b - d_1 - k(F(t)+S(t)) - \mu) \\
&+\frac{1}{d_1}\Big(1 - \frac{1}{S(t)}\Big)(\mu F(t) - d_1 S(t)) + \frac{\sigma^2(k+d_1)}{2kd_1} \\
\le &\, 1 + \frac{\sigma^2(k+d_1)}{2kd_1} + \frac{[d_1(b-d_1-\mu) + k(\mu+d_1)]^2}{4k^2 d_1^2} + \Big|\frac{b-d_1-\mu}{k}\Big|.
\end{aligned}$$

Integrating both sides of (5) from 0 to $\tau_N \wedge T$, by calculation we get

$$E(V(F(\tau_N \wedge T), S(\tau_N \wedge T))) \le V(F(0), S(0)) + M_1 T, \tag{6}$$

where

$$M_1 = \frac{2kd_1\sigma^2(k+d_1) + 4kd_1^2(k + 4kd_1^2|b-d_1-\mu|) + [d_1(b-d_1-\mu) + k(\mu+d_1)]^2}{4k^2 d_1^2}.$$

Define $\Omega_N = \{\tau_N \le T\}$, then by the inequality (4), we have $P\{\Omega_N\} > \delta$. From the definition of Ω_N and the stopping time, we derive that for any $\omega \in \Omega_N$,

$$V(F(\tau_N, \omega), S(\tau_N, \omega)) \ge g(N),$$

where

$$g(N) = \min\Big\{\frac{1}{k}, \frac{1}{d_1}\Big\}\min\Big\{N - 1 - \ln N, \frac{1}{N} - 1 - \ln\frac{1}{N}\Big\}.$$

Combining with (6), we have

$$V(F(0), S(0)) + M_1 T \ge E(I_{\Omega_N} V(F(\tau_N \wedge T), S(\tau_N \wedge T))) \ge \delta g(N). \tag{7}$$

When $N \to +\infty$, by taking the limit to both sides of inequality (7), we get that

$$V(F(0), S(0)) + M_1 T \ge E(I_{\Omega_N} V(F(\tau_N \wedge T), S(\tau_N \wedge T))) \ge +\infty,$$

which leads to a contradiction. Therefore, the assumption does not hold. That is $\tau_\infty = \infty$. Further, we have $\tau_e = \infty$. The proof of Theorem 1 is completed.

Theorem 2 *For the initial* $(F(0), S(0)) \in R_+^2$, *the mean of solution* $(F(t), S(t))$ *of* (2) *is uniformly bounded.*

Proof From Theorem 1, for any given initial $(F(0), S(0)) \in R_+^2$, $(F(t), S(t)) \in R_+^2$, $t \in R_+$.

Choosing a Lyapunov function $V(F(t), S(t)) = F(t) + S(t)$, by using Ito's formula, we obtain that

$$\mathrm{d}V = [(b - d_1 - k(F(t) + S(t)))F(t) - d_1 S(t)]\mathrm{d}t - \sigma(F(t) + S(t))\mathrm{d}B_t.$$

Then,

$$\begin{aligned}\mathrm{d}(\mathrm{e}^{d_1 t}V(F(t), S(t))) &= d_1 \mathrm{e}^{d_1 t}V(F, S)\mathrm{d}t + \mathrm{e}^{d_1 t}\mathrm{d}(V(F, S)) \\ &= \mathrm{e}^{d_1 t}[(bF(t) - k(F + S)F(t))\mathrm{d}t - \sigma(F(t) + S(t))\mathrm{d}B_t].\end{aligned}$$

Define a stopping time $\rho_N = \inf\{t \in R_+ : \sqrt{F^2(t) + S^2(t)} > N\}$, for a integer $N > \sqrt{F^2(0) + S^2(0)}$. Further, we have

$$\begin{aligned}E(\mathrm{e}^{d_1 t}V(F(t), S(t))) &< V(F(0), S(0)) + E\Big[\int_0^{t\wedge\rho_N} \mathrm{e}^{d_1 v}(bF(v) - k(F(v) + S(v))F(v))\mathrm{d}v\Big] \\ &\le V(F(0), S(0)) + E\Big[\int_0^{t\wedge\rho_N} \mathrm{e}^{d_1 v}\frac{b^2}{4k}\mathrm{d}v\Big].\end{aligned}$$

So, we obtain

$$\mathrm{e}^{d_1 t}E(V(F(t), S(t))) < V(0) + \frac{b^2}{4k}\mathrm{e}^{d_1 t}.$$

Therefore,

$$E(V(F(t), S(t))) < V(0)\mathrm{e}^{-d_1 t} + \frac{b^2}{4k}.$$

That is

$$E|(F(t), S(t))| \le E(V(F(t), S(t))) < V(0)\mathrm{e}^{-d_1 t} + \frac{b^2}{4k}.$$

Further, we get

$$\limsup_{t\to+\infty} E|(F(t), S(t))| \le \frac{b^2}{4k}.$$

The proof of Theorem 2 is completed.

Theorem 3 *For a given initial* $(F(0), S(0)) \in R_+^2$, *the solution* $(F(t), S(t))$ *of* (2) *is stochastically ultimately bounded.*

Proof By using Chebyshev inequality, we get

$$P\{|(F, S)| > \beta\} \le \frac{E|(F(t), S(t))|}{\beta}.$$

For any ϵ, we can choose $\beta = \frac{4k\epsilon}{b^2}$. Therefore, we obtain

$$P\{|(F, S)| > \beta\} \le \epsilon.$$

The proof of Theorem 3 is completed.

3 Extinction

Theorem 4 *When* $\mu > b - d_1 + \frac{\sigma^2}{2}$ *and* $d_1 - \frac{\sigma^2}{2} > 0$*, the trivial solution of* (2) *is globally asymptotically stable in probability.*

Proof Define a positive function $V(t) = \frac{1}{2}F^2(t) + \frac{c_1}{2}S^2(t)$. From Ito's formula, we obtain

$$\mathrm{d}V = L(V)\mathrm{d}t - \sigma(F^2(t) + S^2(t))\mathrm{d}B(t).$$

Here

$$\begin{aligned} L(V) &= \Big(b - d_1 - \mu + \frac{1}{2}\sigma^2\Big)F^2(t) - kF^2(t)(F(t) + S(t)) - \Big(c_1 d_1 - \frac{1}{2}\sigma^2 c_1\Big)S^2(t) + c_1\mu F(t)S(t) \\ &\leq \Big(b - d_1 - \mu + \frac{1}{2}\sigma^2\Big)F^2(t) - \Big(c_1 d_1 - \frac{1}{2}\sigma^2 c_1\Big)S^2(t) + c_1\mu F(t)S(t). \end{aligned}$$

We can choose a constant $c_1 = \frac{(d_1 + \mu - b - \frac{1}{2}\sigma^2)(2d_1 - \sigma^2)}{\mu^2}$, such that

$$L(V) \leq -\frac{(d_1 + \mu - b - \frac{1}{2}\sigma^2)}{\mu^2}[\mu F(t) - (2d_1 - \sigma^2)S(t)]^2 - \frac{3(2d_1 - \sigma^2)^2(d_1 + \mu - b - \frac{1}{2}\sigma^2)}{4\mu^2}S^2(t).$$

Further, if $\mu > b - d_1 + \frac{\sigma^2}{2}$ and $d_1 - \frac{\sigma^2}{2} > 0$, then $L(V)$ is negative definite. Therefore, the trivial solution of (2) is globally asymptotically stable in probability. Thus, the system is stochastically extinct in probability, when $\mu > b - d_1 + \frac{\sigma^2}{2}$ and $d_1 - \frac{\sigma^2}{2} > 0$. The proof of this theorem is completed.

References

[1] Zhibin Zang, The ecological fundematals of rodent control by contraception, *Acta Theriologica Sinicia*, **15**:3(1995),229-234.

[2] Xianli Zhang, Wei Tang, Zhenyun Gu, et. al., Experiment of population control with norgestrelem (NGT) on Gansu zokor (My osp alax cansus Lyon) in mountainous area of southern Ningxia, *Journal of Agricultural Sciences*, **26**:1(2005),37-42.

[3] Hanwu Liu, Qiuying Li, Dymical of single population under contraception, *Journal of Yuncheng University*, **27**:2(2009),9-12.

[4] Meng Liu, Chuanzhi Bai, Optimal harvesting policy for a stochastic predator-prey model, *Applied Mathematics Letters*, **34**(2014),22-26.

[5] Qiuying Lu, Stability of SIRS system with random perturbations, *Physica A*, **388**(2009), 3677-3686.

(*edited by Liangwei Huang*)