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CONTROLLABILITY OF QUASI-LINEAR IMPULSIVE FUNCTIONAL BOUNDARY VALUE PROBLEMS*†

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Abstract

By employing the Schauder fixed-point theorem, we establish new sufficient conditions for the controllability of impulsive functional boundary value problems.

Keywords controllability; boundary value problems (BVPs); fixed points

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1 Introduction

In practical control systems, impulses exist widely involving almost all fields such as medicine, biology, economics, electronics and etc. And hence this kind of systems has attracted considerable interest during the past decades. In general, as reported in Lakshmikanthan, Bainov and Simeonov [1], impulsive systems combine continuous evolution with instantaneous state jumps or resets. These systems provide a natural framework for mathematical modeling of many real world evolutionary processes where the states undergo abrupt changes at certain instants or at variable instants.

The concept of controllability plays an important role in control theory and engineering, and the problem of controllability of boundary value problems represented by functional differential equations has been extensively studied (see Han and Park [2], Akhmetov, Perestyuk and Tleubergenova [3], Akhmetov and Zafer [4] and Balachandran, Dauer [5]). In Lando [6], a method was suggested for solving problems of control over linear systems based on the normal solvability of boundary value problems. Akhmetov and Zafer [4] developed the above ideas for impulsive system

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$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) = A(t)x(t) + C(t)u + f(t) + \mu g(t, x, u, \mu), \quad t \neq \theta_i, \\ \Delta x|_{t=\theta_i} = B_i x + D_i v_i + J_i + \mu W_i(x, v_i, \mu), \\ x(\alpha) = a, \quad x(\beta) = b, \end{cases} \tag{1.1}$$

and obtained the controllability of system (1.1) by contraction mapping principle.

For fixed real numbers α and β with $\alpha < \beta$ and fixed positive integers r and p, denote by $L_2^r[\alpha, \beta]$ the set of all square integrable functions $\eta : [\alpha, \beta] \to \mathbb{R}^r$ and by $D^r[1, p]$ the set of all finite sequences $\{\xi_i\}$, $\xi_i \in \mathbb{R}^r$, $i = 1, \cdots, p$. We define a space $\Pi_p^r = L_2^r \times D^r$ whose elements are denoted by $\{\eta, \xi\}$ and let

$$\langle\{\eta, \xi\}, \{\omega, v\}\rangle = \int_\alpha^\beta (\eta, \omega)\mathrm{d}t + \sum_{i=1}^p (\xi_i, v_i)$$

be an inner product in Π_p^r, where $(\cdot, \cdot)$ is the Euclidean scalar product in $\mathbb{R}^r$.

Set

$$PC(I, \mathbb{R}^n) = \{x : I \to \mathbb{R}^n,\ x(t) \text{ is continuous everywhere expect for a finite number of points } \widetilde{t} \text{ at which } x(\widetilde{t}^-), x(\widetilde{t}^+) \text{ exist and } x(\widetilde{t}^-) = x(\widetilde{t})\}.$$

If $x \in PC([-\tau, T], \mathbb{R}^n)$, then for each $t \in [0, T]$, we define $x_t \in PC([-\tau, 0], \mathbb{R}^n)$ by $x_t(\theta) = x(t + \theta)$ for $-\tau \leq \theta \leq 0$.

In this paper, we investigate the controllability of the following impulsive functional boundary value problems

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t) - h(t, x_t)] = A(t)x(t) + C(t)u + f(t, x_t) + \mu g(t, x_t, u), \quad t \neq \theta_i, \\ \Delta x(t) = B_i x(t) + D_i v_i + J_i + \mu W_i(x, v_i, \mu), \quad t = \theta_i, \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \\ x(T) = b, \end{cases} \tag{1.2}$$

where μ is a small positive parameter, both τ and T are positive constants, $x \in \mathbb{R}^n$, the symbol $\Delta(\theta)$ means $x(\theta^+) - x(\theta^-)$ with $x(\theta^+) = \lim\limits_{t\to\theta+0} x(t)$ and $x(\theta^-) = \lim\limits_{t\to\theta-0} x(t)$, $\phi \in C_\tau = PC([-\tau, 0], \mathbb{R}^n)$, $A(t)$ and $C(t)$ are the known matrices of the sizes $(n\times n)$ and $(n\times m)$, respectively, the elements of which belong to $L_2^1[0, T]$, both B_i and D_i are constant matrices of size $(n \times n)$ with $\det(I_i + B_i) \neq 0$ $(i = 1, \cdots, p)$, $\{\theta_i\}$ $(i = 1, \cdots, p)$ is a strictly increasing sequence of real numbers in $(0, T)$ with $0 < \theta_1 < \theta_2 < \cdots < \theta_p < \theta_{p+1} = T$, $b, J_i, v_i \in \mathbb{R}^n$ are all constant vectors, $J = \{J_i\}$, $v = \{v_i\}$ $(i = 1, \cdots, p)$. $f : [0, T] \times C_\tau \to \mathbb{R}^n$, $g : [0, T] \times C_\tau \times \mathbb{R}^m \to \mathbb{R}^n$, $h : [0, T] \times C_\tau \to \mathbb{R}^n$, $W_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ $(i = 1, \cdots, p)$, $\{f, J\} \in \Pi_p^n[0, T]$. Set $G_0 = [0, \theta_1]$, $G_k = (\theta_k, \theta_{k+1}]$ $(k = 1, \cdots, p)$.

Definition 1.1 The boundary value problem (1.2) is said to be controllable, if for any $\phi\in C_\tau$, $b\in\mathbb{R}^n$, there exists a $\{u, v\}\in\Pi_p^m$ for which the boundary value problem

(1.2) has a nontrivial solution.

We denote by $X(t)$ with $X(0)=I$, a fundamental matrix of system

$$\dot{x}(t)=A(t)x, \quad t\in[0,T]$$

and define

$$\Psi(t)=\int_0^t Q(t)Q^{\mathrm{T}}(t)\mathrm{d}t+\sum_{0<\theta_i<t}P_iP_i^{\mathrm{T}},$$

where

$$Q(t)=X^{-1}(t)C(t), \quad P_i=X^{-1}(\theta_i)D_i.$$

Define a space

$$\begin{aligned}\mathbb{B}=\{x:[-\tau,T]\to\mathbb{R}^n,\ x\in PC([0,T],\mathbb{R}^n),\ x(\theta_k^+),x(\theta_k^-)\\ \text{exist and } x(\theta_k^-)=x(\theta_k),\ x_0\in C_\tau,\ k=1,\cdots,p\}\end{aligned}$$

with the norm

$$\|x\|_{\mathbb{B}}=\|x_0\|_\tau+\|x\|,$$

and a space

$$\mathbb{B}'=\{x\in\mathbb{B},x_0=0\in C_\tau\}$$

with the norm

$$\|x\|_{\mathbb{B}'}=\|x_0\|_\tau+\|x\|=\|x\|,$$

where $\|x_t\|_\tau=:\sup\limits_{t\in[t-\tau,t]}\|x(t)\|$, $\|x\|=:\sup\limits_{t\in[0,T]}\|x(t)\|$, then both $(\mathbb{B},\|\cdot\|_{\mathbb{B}})$ and $(\mathbb{B}',\|\cdot\|_{\mathbb{B}'})$ are Banach spaces.

Definition 1.2 $x:[-\tau,T]\to\mathbb{R}^n$ is said to be a mild solution of system (1.2), if $x(t)$ has the following properties:

(i) $x_0=\phi\in C_\tau$;

(ii) $\Delta x(\theta_i)=B_ix(\theta_i)+D_iv_i+J_i+\mu W_i(x(\theta_i),v_i,\mu)$;

(iii) the restriction of $x(\cdot)$ to the interval G_k $(k=0,1,\cdots,p+1)$ is continuous and the following integral equation is verified:

$$\begin{aligned}x(t)=&X(t)\left[\phi(0)-h(0,\phi)\right]+h(t,x_t)\\&+X(t)\int_0^t[Q(s)u+X^{-1}(s)f(s,x_s)+\mu X^{-1}(s)g(s,x_s,u)]\mathrm{d}s\\&+X(t)\sum_{0<\theta_i<t}\big(P_iv_i+X^{-1}(\theta_i)J_i+\mu X^{-1}(\theta_i)W_i(x(\theta_i),v_i,\mu)\big);\end{aligned}$$

(iv) $x(T)=b$.

Definition 1.3[7] A set $\mathcal{M}\subset\mathbb{R}^n$ is said to be quasi-equicontinuous in $[-\tau,T]$ if for any $\varepsilon>0$, there exists a constant $\delta>0$ such that if $y\in\mathcal{M}$, $\tau_1,\tau_2\in[-\tau,t_1]$ or $\tau_1,\tau_2\in G_k$, $k=1,2,\cdots,p+1$ and $|\tau_1-\tau_2|<\delta$, then $\|y(\tau_1)-y(\tau_2)\|<\varepsilon$.

Lemma 1.1[8] (Schauder fixed-pint theorem) *Let $\mathbb{E}$ be a Banach space and $\mathbb{B} \subset \mathbb{E}$ be a bounded, closed and convex set. If $\mathcal{T} : \mathbb{B} \to \mathbb{B}$ is completely continuous, then the operator $\mathcal{T}$ has a fixed point $x^* \in \mathbb{B}$, that is, $\mathcal{T}x^* = x^*$.*

Lemma 1.2[7] *The set $\mathcal{M} \subset \mathbb{R}^n$ is relatively compact if*

(a) *$\mathcal{M}$ is uniformly bounded, that is, $\|x\|_{\mathcal{M}} \le B$ for each $x \in \mathcal{M}$ and some positive constant B;*

(b) *$\mathcal{M}$ is quasi-equicontinuous in $[t_0, T]$.*

2 Main Results

In this section, we investigate the controllability of the boundary value problem (1.2). For convenience, we introduce the following hypotheses and notations.

(H_1) There exist constants $M_i > 0$ ($i = 1, \cdots, 4$), such that for $t \in [0, T]$, we have

$$\|X(t)\| \le M_1, \quad \|X^{-1}(t)\| \le M_2, \quad \|\Psi(t)\| \le M_3, \quad \|\Psi^{-1}(t)\| \le M_4;$$

(H_2) (i) for almost all $t \in [0, T]$, $f : [0, T] \times C_\tau \to \mathbb{R}^n : (t, \phi) \to f(t, \phi)$ is continuous on ϕ;

(ii) for any constant $r > 0$, there exists a function $\alpha_r(t)$, such that for $t \in [0, T]$, we have

$$\sup_{\|\phi\|_\tau \le r} \|f(t, \phi)\| \le \alpha_r(t)$$

and

$$\liminf_{r \to +\infty} \frac{1}{r} \int_0^T \alpha_r(t) \mathrm{d}t = l_1 < +\infty;$$

(H_3) (i) the function $h : [0, T] \times C_\tau \to \mathbb{R}^n$ is Lipschitz continuous, that is, there exists a positive constant l_2, such that

$$\|h(t_1, \phi_1) - h(t_2, \phi_2)\| \le l_2(|t_1 - t_2| + \|\phi_1 - \phi_2\|_\tau);$$

(ii) there exists a positive constant l_3, such that for $(t, \phi) \in [0, T] \times C_\tau$, we have

$$\|h(t, \phi)\| \le l_3(1 + \|\phi\|_\tau);$$

(H_4) (i) for almost all $t \in [0, T]$, the function g is continuous;

(ii) for any constant $r > 0$ and $u \in \mathbb{R}^m$, there exists a function $\beta_r(t)$, such that for $t \in [0, T]$, we have

$$\sup_{\|\phi\|_\tau \le r} \|g(t, \phi, u)\| \le \beta_r(t)$$

and

$$\liminf_{r \to +\infty} \frac{1}{r} \int_0^T \beta_r(t) \mathrm{d}t = l_4 < +\infty;$$

(H_5) (i) the functions W_i $(i = 1, \cdots, p)$ are all continuous;

(ii) for any constant $r > 0$, $v_i \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^+$, there exist functions $\rho_i(r)$, such that

$$\sup_{\|y\| \le r} \|W_i(y, v_i, \mu)\| \le \rho_i(r) \ (i = 1, \cdots, p)$$

and

$$\liminf_{r \to +\infty} \frac{1}{r} \sum_{i=1}^{p} \rho_i(r) = l_5 < +\infty.$$

Theorem 2.1 *Assume that* $\det(I + B_i) \neq 0$ $(i = 1, \cdots, p)$ *and conditions* (H_1)-(H_5) *hold, then system* (1.2) *is controllable provided that the following equation is satisfied:*

$$l_1 M_1 M_2 (1 + M_3 M_4) + l_3 (1 + M_1 M_2 M_3 M_4) + \mu M_1 M_2 (1 + M_3 M_4)(l_4 + l_5) < 1. \quad (2.1)$$

Proof Assume that the input control $\{u, v\}$ is as follows:

$$\begin{cases} u(t) = Q^{\mathrm{T}}(t) c + \widehat{u}(t), \quad t \in [0, T], \\ v_i = P_i^{\mathrm{T}} c + \widehat{v}_i, \quad i = 1, \cdots, p, \end{cases} \quad (2.2)$$

where $c \in \mathbb{R}^n$ is a constant vector, $\{\widehat{u}, \widehat{v}\} \in \Pi_p^m$ is orthogonal to all columns of $[Q^{\mathrm{T}}, P_i^{\mathrm{T}}]$, then the controllability problem of system (1.2) is equivalent to

$$x(t) = \begin{cases} \phi(t), \quad t \in [-\tau, 0], \\ X(t)\left[\phi(0) - h(0, \phi)\right] + h(t, x_t) \\ +X(t) \displaystyle\int_0^t [Q(s) u + X^{-1}(s) f(s, x_s) + \mu X^{-1}(s) g(s, x_s, u)] \mathrm{d}s \\ +X(t) \displaystyle\sum_{0 < \theta_i < t} \left(P_i v_i + X^{-1}(\theta_i) J_i + \mu X^{-1}(\theta_i) W_i(x(\theta_i), v_i, \mu)\right), \quad t \in (0, T), \\ b, \quad t = T. \end{cases} \quad (2.3)$$

Substituting equation (2.2) into equation (2.3), we obtain that the vector c as

$$\begin{aligned} c = {} & \Psi^{-1}(T) \left[X^{-1}(T) b - \phi(0) + h(0, \phi) - X^{-1}(T) h(T, x_T)\right] \\ & - \Psi^{-1}(T) \int_0^T X^{-1}(s) \left(f(s, x_s) + \mu g(s, x_s, u)\right) \mathrm{d}s \\ & - \Psi^{-1}(T) \sum_{i=1}^{p} X^{-1}(\theta_i) \left(J_i + \mu W_i(x(\theta_i), v_i, \mu)\right). \end{aligned} \quad (2.4)$$

Define an operator $\Gamma : \mathbb{B} \to \mathbb{B}$ as

$$(\Gamma x)(t)=\begin{cases}\phi(t), \quad t\in[-\tau,0],\\ X(t)\left[\phi(0)-h(0,\phi)\right]+h(t,x_t)\\ +X(t)\displaystyle\int_0^t [Q(s)u+X^{-1}(s)f(s,x_s)+\mu X^{-1}(s)g(s,x_s,u)]\mathrm{d}s\\ +X(t)\displaystyle\sum_{0<\theta_i<t}(P_iv_i+X^{-1}(\theta_i)J_i+\mu X^{-1}(\theta_i)W_i(x(\theta_i),v_i,\mu)), \quad t\in[0,T],\end{cases} \tag{2.5}$$

where u is defined by equations (2.2) and (2.4).

Now we should prove that the operator Γ defined in equation (2.5) has one fixed point $x\in\mathbb{B}$, then system (1.2) has a mild solution $x(t,u,v)$ with respect to $\{u,v\}\in\Pi_p^m$ which implies that system (1.2) is controllable. Thus, the problem to discuss the controllability of system (1.2) can be reduced into the existence of the fixed point of the operator Γ.

Define

$$\widehat{\phi}(t)=\begin{cases}\phi(t), \quad t\in[-\tau,0],\\ X(t)\phi(0), \quad t\in[0,T],\end{cases} \tag{2.6}$$

then $\widehat{\phi}\in\mathbb{B}$. Let

$$x(t)=y(t)+\widehat{\phi}(t), \quad t\in[-\tau,T].$$

Then $y\in\mathbb{B}'$ and for $t\in[0,T]$,

$$\begin{aligned} y(t)=&-X(t)h(0,\phi)+h(t,y_t+\widehat{\phi}_t)\\ &+X(t)\int_0^t \big[Q(s)u+X^{-1}(s)f(s,y_s+\widehat{\phi}_s)+\mu X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u)\big]\mathrm{d}s\\ &+X(t)\sum_{0<\theta_i<t}\big(P_iv_i+X^{-1}(\theta_i)J_i+\mu X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\big),\end{aligned}$$

for $t\in[-\tau,T]$, $x(t)=(\Gamma x)(t)$.

Define an operator $\Phi:\mathbb{B}'\to\mathbb{B}'$ as

$$(\Phi y)(t)=\begin{cases}0, \quad t\in[-\tau,0],\\ X(t)\Big[-h(0,\phi)+\Psi(t)\Psi^{-1}(T)k-\Psi(t)\Psi^{-1}(T)\displaystyle\int_0^T X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ -\Psi(t)\Psi^{-1}(T)X^{-1}(T)h(T,y_T+\widehat{\phi}_T)+\displaystyle\int_0^t X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ +\displaystyle\sum_{0<\theta_i<t}X^{-1}(\theta_i)J_i\Big]+h(t,y_t+\widehat{\phi}_t)-\mu X(t)\Big[\Psi(t)\Psi^{-1}(T)\Big(\displaystyle\int_0^t X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ +\displaystyle\sum_{i=1}^{p}X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big)-\displaystyle\int_0^t X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u)\mathrm{d}s\\ -\displaystyle\sum_{0<\theta_i<t}X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big], \quad t\in[0,T],\end{cases} \tag{2.7}$$

where $k = X^{-1}(T)b - \phi(0) + h(0,\phi) - \sum\limits_{i=1}^{p} X^{-1}(\theta_i)J_i$.

By simple computation, we know that the operator $\Gamma : \mathbb{B} \to \mathbb{B}$ has a fixed point if and only if the operator $\Phi : \mathbb{B}' \to \mathbb{B}'$ has a fixed point. So it turns out to prove that $\Phi : \mathbb{B}' \to \mathbb{B}'$ has a fixed point.

The proof is given in the following several steps.

Step 1 $\Phi(\mathbb{B}'_q) \subset \mathbb{B}'_q$ for some $q > 0$, where $\mathbb{B}'_q = \{y \in \mathbb{B}', \|y\|_{\mathbb{B}'} \le q\}$. Otherwise, for any positive constant q, there exists a $y^q \in \mathbb{B}'_q$, such that $\Phi(y^q) \notin \mathbb{B}'_q$, that is, $\|\Phi(y^q)\|_{\mathbb{B}'} > q$. By the definition of the operator Φ, we have

$$
\begin{aligned}
q &< \|(\Phi y^q)\|_{\mathbb{B}'} \\
&\le M_1\Big[\|h(0,\phi)\| + \|\Psi\|\|\Psi^{-1}(T)\||k| + M_2(1+M_3M_4)\int_0^T \|f(s, y_s + \widehat{\phi}_s)\|\mathrm{d}s \\
&\quad + M_2\|\Psi\|\|\Psi^{-1}(T)\|\|h(T, y_T + \widehat{\phi}_T)\| + M_2\sum_{i=1}^{p}\|J_i\|\Big] + l_3(1 + \|y_t + \widehat{\phi}_t\|_\tau) \\
&\quad + \mu M_1M_2\Big[\|\Psi\|\|\Psi^{-1}(T)\|\Big(\int_0^T \|g(s, y_s + \widehat{\phi}_s, u)\|\mathrm{d}s + \sum_{i=1}^{p}\|W_i(y(\theta_i) + \widehat{\phi}(\theta_i), v_i, \mu)\|\Big) \\
&\quad + \int_0^T \|g(s, y_s + \widehat{\phi}_s, u)\|\mathrm{d}s + \sum_{i=1}^{p}\|W_i(y(\theta_i) + \widehat{\phi}(\theta_i), v_i, \mu)\|\Big] \\
&\le M_1\Big[l_3(1 + \|\phi\|_\tau) + M_3M_4|k| + M_2\sum_{i=1}^{p}\|J_i\|\Big] + l_3(1 + M_1M_2M_3M_4)(1+q') \\
&\quad + M_1M_2(1+M_3M_4)\int_0^T \alpha_{q'}(s)\mathrm{d}s + \mu M_1M_2(1+M_3M_4)\Big(\int_0^T \beta_{q'}(s)\mathrm{d}s + \sum_{i=1}^{p}\rho_i(q')\Big),
\end{aligned}
$$

where $q' = q + (1+M_1)\|\phi\|_\tau$. Dividing both sides of this by q and considering conditions (H_1)-(H_5), we obtain that

$$
l_1M_1M_2(1+M_3M_4) + l_3(1 + M_1M_2M_3M_4) + \mu(l_4 + l_5)M_1M_2(1+M_3M_4) \ge 1,
$$

which contradicts equation (2.1). Hence for some positive constant q, $\Phi(\mathbb{B}'_q) \subset \mathbb{B}'_q$.

Step 2 $\Phi : \mathbb{B}' \to \mathbb{B}'$ is continuous.

Let $\{y^{(n)}, y\} \subset \mathbb{B}'$ with $\|y^{(n)} - y\|_{\mathbb{B}'} \to 0$ for $n \to \infty$. Then there exists some positive constant q, such that

$$
\|y^{(n)}\|_{\mathbb{B}'} \le q, \quad \|y\|_{\mathbb{B}'} \le q.
$$

That is, $\{y^{(n)}, y\} \subset \mathbb{B}'_q$.

$$
\begin{aligned}
&\big\|\Phi y^{(n)}-\Phi y\big\|_{\mathbb{B}'}\\
=&\Big\|X(t)\Big[-\Psi(t)\Psi^{-1}(T)\int_0^T X^{-1}(s)\big(f(s,y_s^{(n)}+\widehat{\phi}_s)-f(s,y_s+\widehat{\phi}_s)\big)\mathrm{d}s\\
&-\Psi(t)\Psi^{-1}(T)X^{-1}(T)\big(h(T,y_T^{(n)}+\widehat{\phi}_T)-h(T,y_T+\widehat{\phi}_T)\big)\\
&+\int_0^t X^{-1}(s)\big(f(s,y_s^{(n)}+\widehat{\phi}_s)-f(s,y_s+\widehat{\phi}_s)\big)\mathrm{d}s\Big]+\big(h(t,y_t^{(n)}+\widehat{\phi}_t)-h(t,y_t+\widehat{\phi}_t)\big)\\
&+\mu X(t)\Big[-\Psi(t)\Psi^{-1}(T)\int_0^t X^{-1}(s)\big(f(s,y_s^{(n)}+\widehat{\phi}_s)-f(s,y_s+\widehat{\phi}_s)\big)\mathrm{d}s\\
&-\sum_{i=1}^{p}X^{-1}(\theta_i)\big(W_i(y^{(n)}(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)-W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\big)\\
&+\int_0^t X^{-1}(s)\big(g(s,y_s^{(n)}+\widehat{\phi}_s,u(s))-g(s,y_s+\widehat{\phi}_s,u(s))\big)\mathrm{d}s\\
&+\sum_{0<\theta_i<t}X^{-1}(\theta_i)\big(W_i(y^{(n)}(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)-W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\big)\Big]\Big\|_{\mathbb{B}'}\\
\le&((\mu+1)M_1M_2M_3M_4+M_1M_2)\int_0^T\big\|f(s,y_s^{(n)}+\widehat{\phi}_s)-f(s,y_s+\widehat{\phi}_s)\big\|\mathrm{d}s\\
&+M_1M_2M_3M_4\big\|h(T,y_T^{(n)}+\widehat{\phi}_T)-h(T,y_T+\widehat{\phi}_T)\big\|\\
&+\sup_{t\in[0,T]}\big\|h(t,y_t^{(n)}+\widehat{\phi}_t)-h(t,y_t+\widehat{\phi}_t)\big\|\\
&+2\mu M_1M_2\sum_{i=1}^{p}\big\|W_i(y^{(n)}(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)-W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\big\|\\
&+\mu M_1M_2\int_0^T\big\|g(s,y_s^{(n)}+\widehat{\phi}_s,u(s))-g(s,y_s+\widehat{\phi}_s,u(s))\big\|\mathrm{d}s.
\end{aligned}
$$

In view of conditions (H_2)-(H_5) and $\|y_t^{(n)}+\widehat{\phi}_t\|_\tau\le q'$, we have

$$
\begin{aligned}
&\|f(t,y_t^{(n)}+\widehat{\phi}_t)-f(t,y_t+\widehat{\phi}_t)\|\le 2\alpha_{q'}(t),\\
&\|g(t,y_t^{(n)}+\widehat{\phi}_t,u(t))-g(t,y_t+\widehat{\phi}_t,u(t))\|\le 2\beta_{q'}(t),
\end{aligned}
$$

and hence by dominated convergence theorem, we obtain that

$$\lim_{n\to\infty}\|\Phi y^{(n)}-\Phi y\|_{\mathbb{B}'}\to 0.$$

That is, $\Phi:\mathbb{B}'\to\mathbb{B}'$ is continuous.

Step 3 $\Phi_\varepsilon(\mathbb{B}'_q)$ is quasi-equicontinuous in $[-\tau,T]$, where $\Phi_\varepsilon:\mathbb{B}'\to\mathbb{B}'$ is an operator defined by

$$(\Phi_\varepsilon y)(t)=\begin{cases}0, \quad t\in[-\tau,0],\\ X(t)\Big[-h(0,\phi)+\Psi(t)\Psi^{-1}(T)k-\Psi(t)\Psi^{-1}(T)\int_0^T X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ -\Psi(t)\Psi^{-1}(T)X^{-1}(T)h(T,y_T+\widehat{\phi}_T)+\int_0^{t-\varepsilon}X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ +\sum\limits_{0<\theta_i<t-\varepsilon}X^{-1}(\theta_i)J_i\Big]+h(t,y_t+\widehat{\phi}_t)\\ -\mu X(t)\Big[\Psi(t)\Psi^{-1}(T)\Big(\int_0^{t-\varepsilon}X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ +\sum\limits_{i=1}^{p}X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big)-\int_0^{t-\varepsilon}X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u)\mathrm{d}s\\ -\sum\limits_{0<\theta_i<t-\varepsilon}X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big], \quad t\in[0,T],\end{cases}$$

where $0<\varepsilon<t$ is an arbitrary positive constant.

For any $t_1,t_2\in G_k$ $(k=0,1,\cdots,p)$ with $t_1>t_2$,

$$\begin{aligned}&\|(\Phi_\varepsilon y)(t_1)-(\Phi_\varepsilon y)(t_2)\|_{\mathbb{B}'}\\ =&\Big\|\left(X(t_2)-X(t_1)\right)h(0,\phi)+\left(X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\right)\Psi^{-1}(T)k\\ &-(X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2))\Psi^{-1}(T)\int_0^T X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ &-\left(X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\right)\Psi^{-1}(T)X^{-1}(T)h(T,y_T+\widehat{\phi}_T)\\ &+\left(X(t_1)-X(t_2)\right)\int_0^{t_2-\varepsilon}X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s+X(t_1)\int_{t_2-\varepsilon}^{t_1-\varepsilon}X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ &+\left(X(t_1)-X(t_2)\right)\sum_{0<\theta_i<t_2-\varepsilon}X^{-1}(\theta_i)J_i+X(t_1)\sum_{t_2-\varepsilon\le\theta_i<t_1-\varepsilon}X^{-1}(\theta_i)J_i\\ &+h(t_1,y_{t_1}+\widehat{\phi}_{t_1})-h(t_2,y_{t_2}+\widehat{\phi}_{t_2})-\mu X(t_1)\Psi(t_1)\Psi^{-1}(T)\int_{t_2-\varepsilon}^{t_1-\varepsilon}X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ &+\mu\left(X(t_2)\Psi(t_2)-X(t_1)\Psi(t_1)\right)\Psi^{-1}(T)\int_0^{t_2-\varepsilon}X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\\ &+\mu\left(X(t_2)\Psi(t_2)-X(t_1)\Psi(t_1)\right)\Psi^{-1}(T)\sum_{i=1}^{p}X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\\ &+\mu\left(X(t_1)-X(t_2)\right)\int_0^{t_2-\varepsilon}X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u(s))\mathrm{d}s\\ &+\mu X(t_1)\int_{t_2-\varepsilon}^{t_1-\varepsilon}X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u(s))\mathrm{d}s\\ &+\mu\left(X(t_1)-X(t_2)\right)\sum_{0<\theta_i<t_2-\varepsilon}X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\end{aligned}$$

$$
\begin{aligned}
&+\mu X(t_1)\sum_{t_2-\varepsilon\le\theta_i<t_1-\varepsilon} X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big\|_{\mathbb{B}'}\\
\le{}& \|X(t_2)-X(t_1)\|\|h(0,\phi)\|+\|X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\|\|\Psi^{-1}(T)k\|\\
&+\|X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\|\|\Psi^{-1}(T)\|\Big\|\int_0^T X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\Big\|\\
&+\|X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\|\|\Psi^{-1}(T)\|\|X^{-1}(T)\|\|h(T,y_T+\widehat{\phi}_T)\|\\
&+\|X(t_1)-X(t_2)\|\Big\|\int_0^{t_2-\varepsilon} X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\Big\|\\
&+\|X(t_1)\|\Big\|\int_{t_2-\varepsilon}^{t_1-\varepsilon} X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\Big\|\\
&+\|X(t_1)-X(t_2)\|\Big\|\sum_{0<\theta_i<t_2-\varepsilon} X^{-1}(\theta_i)J_i\Big\|+\|X(t_1)\|\Big\|\sum_{t_2-\varepsilon\le\theta_i<t_1-\varepsilon} X^{-1}(\theta_i)J_i\Big\|\\
&+\|h(t_1,y_{t_1}+\widehat{\phi}_{t_1})-h(t_2,y_{t_2}+\widehat{\phi}_{t_2})\|\\
&+\mu\|X(t_1)\|\|\Psi(t_1)\|\|\Psi^{-1}(T)\|\Big\|\int_{t_2-\varepsilon}^{t_1-\varepsilon} X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\Big\|\\
&+\mu\|X(t_2)\Psi(t_2)-X(t_1)\Psi(t_1)\|\|\Psi^{-1}(T)\|\Big\|\int_0^{t_2-\varepsilon} X^{-1}(s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}s\Big\|\\
&+\mu\|X(t_2)\Psi(t_2)-X(t_1)\Psi(t_1)\|\|\Psi^{-1}(T)\|\Big\|\sum_{i=1}^{p} X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big\|\\
&+\mu\|X(t_1)-X(t_2)\|\Big\|\int_0^{t_2-\varepsilon} X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u(s))\mathrm{d}s\Big\|\\
&+\mu\|X(t_1)\|\Big\|\int_{t_2-\varepsilon}^{t_1-\varepsilon} X^{-1}(s)g(s,y_s+\widehat{\phi}_s,u(s))\mathrm{d}s\Big\|\\
&+\mu\|X(t_1)-X(t_2)\|\Big\|\sum_{0<\theta_i<t_2-\varepsilon} X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big\|\\
&+\mu\|X(t_1)\|\Big\|\sum_{t_2-\varepsilon\le\theta_i<t_1-\varepsilon} X^{-1}(\theta_i)W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\Big\|\\
\le{}& l_3(1+\|\phi\|_\tau)\|X(t_2)-X(t_1)\|+\|(X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\|\|\Psi^{-1}(T)k\|\\
&+\|X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\|\|\Psi^{-1}(T)\|\|X^{-1}(s)\|\int_0^T \alpha_{q'}(s)\mathrm{d}s\\
&+\|X(t_1)\Psi(t_1)-X(t_2)\Psi(t_2)\|\|\Psi^{-1}(T)\|\|X^{-1}(T)\|l_3(1+\|y_T+\widehat{\phi}_T\|_\tau)\\
&+\|X(t_1)-X(t_2)\|\|X^{-1}(s)\|\int_0^{t_2-\varepsilon}\alpha_{q'}(s)\mathrm{d}s+\|X(t_1)\|\|X^{-1}(s)\|\int_{t_2-\varepsilon}^{t_1-\varepsilon}\alpha_{q'}(s)\mathrm{d}s\\
&+\|X(t_1)-X(t_2)\|\sum_{0<\theta_i<t_2-\varepsilon}\|X^{-1}(\theta_i)J_i\|+l_2(|t_1-t_2|+\|y_{t_1}-y_{t_2}\|_\tau+\|\widehat{\phi}_{t_1}-\widehat{\phi}_{t_2}\|_\tau)
\end{aligned}
$$

$$+\mu\|X(t_2)\Psi(t_2)-X(t_1)\Psi(t_1)\|\|\Psi^{-1}(T)\|\|X^{-1}(s)\|\int_0^{t_2-\varepsilon}\alpha_{q'}(s)\mathrm{d}s$$
$$+\mu\|X(t_1)\|\|\Psi(t_1)\|\|\Psi^{-1}(T)\|\|X^{-1}(s)\|\int_{t_2-\varepsilon}^{t_1-\varepsilon}\alpha_{q'}(s)\mathrm{d}s$$
$$+\mu\|X(t_2)\Psi(t_2)-X(t_1)\Psi(t_1)\|\|\Psi^{-1}(T)\|\|X^{-1}(\theta_i)\|\sum_{i=1}^{p}\|W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\|$$
$$+\mu\|X(t_1)-X(t_2)\|\|X^{-1}(s)\|\bigg\|\int_0^{t_2-\varepsilon}\beta_{q'}(s)\mathrm{d}s+\mu\|X(t_1)\bigg\|\|X^{-1}(s)\|\int_{t_2-\varepsilon}^{t_1-\varepsilon}\beta_{q'}(s)\mathrm{d}s$$
$$+\mu\|X(t_1)-X(t_2)\|\|X^{-1}(\theta_i)\|\sum_{0<\theta_i<t_2-\varepsilon}\|W_i(y(\theta_i)+\widehat{\phi}(\theta_i),v_i,\mu)\|.$$

The right-hand side is independent of $y\in\mathbb{B}'_q$ and tends to zero as $t_1-t_2\to 0$. Thus, $\Phi_\varepsilon(\mathbb{B}'_q)$ is quasi-equicontinuous. The quasi-equicontinuities for the cases $t_2<t_1\le 0$ and $t_2<0<t_1$ are obvious.

Note that by using the same method as in Step 1, we can show that the operator Φ_ε is uniformly bounded, which implies by Lemma 1.2 that $\Phi_\varepsilon(\mathbb{B}'_q)$ is a relatively compact set in $\mathbb{B}'$ for any $\varepsilon\in(0,t)$. On the other side, we can easily prove that $\|(\Phi y)(t)-(\Phi_\varepsilon y)(t)\|_{\mathbb{B}'}\to 0$ as $\varepsilon\to 0+$. By Lemma 1.1, the operator Φ has a fixed point $\bar{y}\in\mathbb{B}'$. And hence the operator Γ has a fixed point $\bar{x}=\bar{y}+\widehat{\phi}\in\mathbb{B}$. By Definition 1.1, the boundary value problem (1.2) is controllable.

3 An Example

As an application, we consider the following impulsive functional boundary value problems

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}[x(t)-h(t,x_t)]=A(t)x(t)+C(t)u+f(t,x_t)+\mu g(t,x_t,u), & t\ne\theta_i,\\ \Delta x(t)=B_ix(t)+D_iv_i+J_i+\mu W_i(x,v_i,\mu), & t=\theta_i,\\ x(t)=\phi(t), & t\in[-\tau,0],\\ x(1)=b,\end{cases}\tag{3.1}$$

where

$$A(t)=\begin{pmatrix}-2t^4+2t^2-t & 2t^3+1\\ -2t^5+4t^3-t^2-2t-1 & 2t^4-2t^2+t\end{pmatrix},$$
$$C=\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix},\quad D_i=\frac{1}{3}\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix},\quad B_i=\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix},$$
$$J_i=\big((-1)^i,(-1)^{i+1}\big)^{\mathrm{T}},\quad \theta_i=i/3,\quad i=1,2,\quad \mu=1/360,\quad u=(u_1,u_2)^{\mathrm{T}}.$$

For $\phi\in C_\tau$, $s=\|\phi\|_\tau$,

$$g(t,\phi,u)=\frac{1}{10}\big(\cos u_1+s\cos t,\sin u_2+s\sin t\big)^{\mathrm{T}},$$
$$W_i(y,v_i,\mu)=\frac{1}{10}(\sin\mu v_{i1}+s,\cos\mu v_{i2}+s)^{\mathrm{T}},$$
$$f(t,\phi)=\frac{1}{360}\big(\sin\sqrt{1+s^2},\cos\sqrt{1+s^2}\big)^{\mathrm{T}},\quad h(t,\phi)=\frac{1}{360}(\sin t,\sin s)^{\mathrm{T}}.$$

By computation, we have

$$X(t)=\begin{pmatrix}1-t^2, & t\\ t^4-t^2-t, & 1-t^3\end{pmatrix},\quad X^{-1}(t)=\begin{pmatrix}1-t^3, & -t\\ t+t^2-t^4, & 1-t^2\end{pmatrix},$$

$$Q(t)=X^{-1}(t),\quad P_1=X^{-1}(\theta_1)D_1=\begin{pmatrix}\frac{26}{81} & -\frac{1}{9}\\ \frac{35}{243} & \frac{8}{27}\end{pmatrix},\quad P_2=X^{-1}(\theta_2)D_2=\begin{pmatrix}\frac{19}{81} & -\frac{2}{9}\\ \frac{74}{243} & \frac{5}{27}\end{pmatrix},$$

$$\sup_{t\in[0,1]}\|X(t)\|<2,\quad \sup_{t\in[0,1]}\|X^{-1}(t)\|<3,\quad \sup_{t\in[0,1]}\|\Psi(t)\|<9.1,\quad \sup_{t\in[0,1]}\|\Psi^{-1}(t)\|<2.9,$$

$$\|f(t,\phi)\|\le\frac{\sqrt{2}}{360}\sqrt{1+s^2},\quad l_1=\frac{\sqrt{2}}{360},\quad \|h(t,\phi)\|\le 1+s,\quad l_3=\frac{1}{360},$$

$$\|g(t,\phi,u)\|\le\frac{1}{5}+\frac{\sqrt{2}}{10}s,\quad l_4=\frac{\sqrt{2}}{10},\quad \|W_i\|\le\frac{1+s}{5}\ (i=1,2),\quad l_5=\frac{2}{5},$$

where $\|x\|=\sum\limits_{i=1}^{n}|x_i|$ for $x\in\mathbb{R}^n$ and $\|A\|=\max\limits_{1\le j\le n}\sum\limits_{i=1}^{n}|a_{ij}|$ for $A\in\mathbb{R}^{n\times n}$. We know that conditions (H_1)-(H_5) and (2.1) are satisfied, then by Theorem 2.1, system (3.1) is controllable.

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