Ann. of Appl. Math.
33:2(2017), 111-129

GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR GENERALIZED QUANTUM MHD EQUATION*

Boling Guo
(*Institute of Applied Physics and Computational Math., China Academy of Engineering Physics, Beijing, 100088, PR China*)
Binqiang Xie†
(*Graduate School of China Academy of Engineering Physics, Beijing, 100088, PR China*)

Abstract

We prove the existence of a weak solution for a generalized quantum MHD equation in a 2-dimensional periodic box for large initial data. The existence of a global weak solution is established through a three-level approximation, energy estimates, and weak convergence for the adiabatic exponent $\gamma > 1$.

Keywords weak solutions; MHD equation; quantum hydrodynamic

2010 Mathematics Subject Classification 76W05; 35Q35; 35D05; 76X05

1 Introduction

The evolution of quantum MHD equations in $\Omega = T^2$ is described by the following system

$$\partial_t n + \mathrm{div}(nu) = 0, \tag{1.1a}$$

$$\begin{aligned}&\partial_t(nu) + \mathrm{div}(nu \otimes u) + \nabla(P(n) + P_c(n)) - 2\mathrm{div}(\mu(n)D(u))\\ &-\nabla(\lambda(n)\mathrm{div}u) - \tfrac{\hbar^2}{2} n\nabla(\varphi'(n)\Delta\varphi(n)) - (\nabla \times B) \times B = 0,\end{aligned} \tag{1.1b}$$

$$\partial_t B - \nabla \times (u \times B) + \nabla(\nu_b(\rho)\nabla \times B) = 0, \tag{1.1c}$$

$$n(x,0) = n_0(x),\ nu(x,0) = m_0, \tag{1.1d}$$

$$B(x,0) = B_0(x),\ \mathrm{div}B_0 = 0, \tag{1.1e}$$

where the functions n, u and B represent the mass density, the velocity field and the magnetic field respectively. $P(n) = n^\gamma$ stands for the pressure, P_c is a singular continuous function and called cold pressure. $\mu(n)$ and $\lambda(n)$ denote the fluid viscosity coefficient. $\hbar > 0$ is the quantum plank constant, ν_b is the magnetic viscosity coefficient.

---

*Manuscript received March 13, 2017

†Corresponding author. E-mail: xbq211@163.com

Our analysis is based on the following physically grounded assumptions:

[A1] The viscosity coefficient is determined by the Newton's rheological law

$$\mu(n) = \mu_0 n^\alpha, \quad 0 < \alpha \le 1, \quad \lambda(n) = 2(n\mu'(n) - \mu(n)), \tag{1.2}$$

where μ and λ are respectively the shear and bulk constant viscosity coefficients, and the dispersion term φ satisfies

$$\varphi(n) = n^\alpha. \tag{1.3}$$

[A2] The cold pressure P_c obeys the following growth assumption:

$$\lim_{n\to 0} P_c(n) = +\infty. \tag{1.4}$$

More precisely, we assume

$$P_c'(n) = \begin{cases} c_1 n^{-\gamma^- -1}, & n \le 1, \\ c_2 n^{\gamma-1}, & n > 1, \end{cases} \tag{1.5}$$

where $\gamma^-, \gamma \ge 1,\ c_1, c_2 > 0$.

[A3] The positive coefficient ν_b is supposed to be a continuous function of the density, bounded from above and taking large values for small and large densities. More precisely, we assume that there exist $B > 0$, positive constants d_0, d_0', d_1, d_1' large enough, $2 \le a < a' < 3$ and $b \in [0, \infty]$ such that

$$\text{for any } s < B, \ \frac{d_0}{s^a} \le \nu_b(s) \le \frac{d_0'}{s^{a'}} \quad \text{and} \quad \text{for any } s \ge B, \ d_1 \le \nu_b(s) \le d_1' s^b. \tag{1.6}$$

Define functions $H(n)$ and $\xi(n)$ as follows:

$$\begin{cases} nH'(n) - H(n) = P(n), \quad nH_c'(n) - H_c(n) = P_c(n), \\ n\xi'(n) = \mu'(n). \end{cases} \tag{1.7}$$

The quantum fluid models have lots of applications, for instance, quantum semiconductor [6], weakly interacting Bose gases [12], superfluids [20]. More recently, dissipative quantum fluid models have been proposed by Jüngel [16], the quantum ideal magnetohydrodynamic model was derived by Hass [13]. To get the weak solution for these quantum model, it is often to introduce the damping terms $-r_0 u - r_1 n|u|^2 u$ or the singular pressure term $P_c(n)$. These terms allow us to get the compactness of the velocity field when dealing the degenerate viscosity case. In this paper, we adopt the cold pressure form, in fact, the global existence of weak solutions can be obtained by replacing the cold pressure by a drag pressure.

There is a large amount work on the global existence of weak solutions for the compressible Navier-Stokes equation, in the constant viscosity coefficients case, one of the main result is due to P.L. Lions [18], who proved the global existence of weak solutions for the compressible Navier-Stokes system in the case of barotropic equations of state. Later, this result was extended to the somehow optimal case $\gamma > n/2$ in [7] using oscillation defect measures on density sequences associated with suitable approximation solutions. Bresch-Desjardins [1] achieved some improvement in the case of viscosity coefficients depending on the density ρ. Under some structure constraint on the viscosity coefficients, they discovered a new entropy inequality (called BD entropy) which can yield global in time integrability properties on density gradient. This new structure was used in the framework of capillary fluid [2]. Later on, they founded that this BD entropy inequality also can be applied in the compressible Navier-Stokes equation without capillarity [3]. By this new BD entropy inequality, they succeeded in obtaining the global existence of weak solutions in the barotropic fluids with some additional drag terms. However, there are some difficulties without any additional drag term, as lack of estimates for the velocity. To deal with this obstacle, Mellet-Vasseur [21] obtained a new logatithmic velocity estimate. Unfortunately, they cannot construct smooth approximation solutions, only the stability of solutions for barotropic compressible Navier-Stokes equations were proved. Li and Xin [19] recently constructed a suitable approximate system which has smooth solutions satisfying the energy inequality, the BD entropy inequality, and the Mellet-Vasseur type estimate, therefore they completely solved an open problem. D. Bresch and B. Desjardins [4] also used this new entropy to obtain the global existence of weak solutions for the Navier-Stokes equations for viscous compressible and heat conducting fluids when the viscosity coefficients depend on the density. However, they have to add the cold pressure term into the usual pressure term.

In this paper we study the global existence of weak solutions for quantum MHD model (1.1)-(1.7). For this system, when the generalized Bohm potential $\varphi'(n)\Delta\varphi(n)$ reduces to common form $\frac{\Delta\sqrt{n}}{\sqrt{n}}$, without the singular pressure term $P_c(n)$, Jüngle [15] used the test function of the form $n\varphi$ to handle the convection term, thus the author proved the existence of such a particular weak solution. Gisxlon and Violet [11] proved the existence of weak solutions for the quantum Navier-Stokes with singular pressure, where the authors adopt some arguments to make use of the cold pressure for compactness. There is little results about quantum MHD model. Therefore we give some result for this quantum MHD model.

Now, we give the definition of a weak solution for (1.1)-(1.7).

Definition 1.1 We call (n, u, B) to be a weak solution for problem (1.1)-(1.7), if the following conditions are satisfied:

(1) The density n is a non-negative function satisfying the internal identity

$$\int_0^T\int_\Omega (n\partial_t\phi + nu\cdot\nabla\phi)\mathrm{d}x\mathrm{d}t + \int_\Omega n_0\phi(0)\mathrm{d}x = 0, \tag{1.8}$$

for any test function $\phi \in C^\infty([0,T]\times\overline{\Omega})$, $\phi(T) = 0$;

(2) the momentum equation in (1.1b) holds in $D'((0,T)\times\Omega)$ (in the sense of distributions), that means,

$$\begin{aligned}&\int_\Omega m_0\phi(0)\mathrm{d}x+\int_0^T\int_\Omega[nu\cdot\partial_t\phi+n(u\otimes u):\nabla\phi+P\mathrm{div}\phi]\mathrm{d}x\mathrm{d}t\\ =&\frac{\hbar^2}{2}\int_0^T\int_\Omega[\varphi'(n)\Delta\varphi(n)\nabla n\phi+n\phi'\Delta\varphi(n)\mathrm{div}\phi]\mathrm{d}x\mathrm{d}t+2\int_0^T\int_\Omega\mu(n)D(u)\nabla\phi\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega\lambda(n)\mathrm{div}u\mathrm{div}\phi\mathrm{d}x\mathrm{d}t-\nu_b\int_0^T\int_\Omega(\nabla\times B)\times B\cdot\phi\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{1.9}$$

for any test function $\phi\in C^\infty([0,T]\times\overline{\Omega})$, $\phi(T)=0$.

(3) the magnetic field B is a non-negative function satisfying

$$\int_\Omega B_0\phi(0)\mathrm{d}x=\int_0^T\int_\Omega[B\cdot\partial_t\phi+(u\times B)\cdot(\nabla\times\phi)-\nu_b\nabla B:\nabla\phi]\mathrm{d}x\mathrm{d}t,\tag{1.10}$$

for any test function $\phi\in C^\infty([0,T]\times\overline{\Omega})$, $\phi(T)=0$.

Remark 1.1 If $\mu(n)=0$, $\lambda(n)=0$, $\alpha=1/2$, $B=0$, $P_c(n)=0$, then the quantum hydrodynamic equation (1.1) becomes

$$\partial_t n+\mathrm{div}(nu)=0,\tag{1.11a}$$

$$\partial_t(nu)+\mathrm{div}(nu\otimes u)+\nabla P(n)-\frac{\hbar^2}{2}n\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big)=0.\tag{1.11b}$$

If $\mu(n)=0$, $\lambda(n)=0$, $\alpha=1$, $B=0$, $P_c(n)=0$, $\nu=0$, then the quantum hydrodynamic equation (1.1) becomes

$$\partial_t n+\mathrm{div}(nu)=0,\tag{1.12a}$$

$$\partial_t(nu)+\mathrm{div}(nu\otimes u)+\nabla P(n)-\frac{\hbar^2}{2}n\nabla\Delta n=0.\tag{1.12b}$$

Now, we are ready to formulate the main result of this paper.

Theorem 1.1(global existence for the quantum Euler model) *Let $\Omega=T^2$ be a periodic box. Assume $T>0$. Let the initial data satisfy*

$$\begin{cases}\displaystyle\int_{T^2}\Big(\frac{|m|^2}{2n_0}+[H(n_0)+H_c(n_0)]+\frac{\hbar^2}{2}|\nabla\varphi(n_0)|^2+|B_0|^2\Big)\mathrm{d}x\le C,\\ \displaystyle\frac{\nabla\mu(n_0)}{\sqrt{n_0}}\in L^2(\Omega).\end{cases}\tag{1.13}$$

Then problem (1.1)-(1.7) posses at least one global weak solution n,u,B.

This paper is organized as follows. In Section 2, we establish the global existence of solutions to the Faedo-Galerkin approximation for (1.1). In Section 3 we deduce the B-D entropy energy estimates, which is a key part in the analysis process. In Sections 4 and 5, we use the uniform estimates to recover the original system by vanishing the artificial viscosity and artificial pressure respectively, therefore the main theorem is proved by using the weak convergence method.

2 Faedo-Galerkin Approximation

In this section, we prove the existence of solutions to approximate solutions for quantum MHD equations by the Faedo-Galerkin method. Motivated by the work of Feireisl, Novotný, and Petzeltová [7], we proceed similarly as in Zatorska [22].

2.1 Local existence of solutions

Let $T>0$, and (e_k) be an orthonormal basis of $L^2(T^2)$ which is also an orthogonal basis of $H^1(T^2)$. Introduce the finite-dimensional space $X_N=\mathrm{span}\{e_1,e_2,\cdots,e_N\}$, $N\in\mathbb{N}$. Let $(n_0,u_0,B_0)\in C^\infty(T^2)^3$ be some initial data satisfying $n_0\geq\delta>0$ for $x\in T^2$ for some $\delta>0$, and let the velocity $u\in C^0([0,T];X_N)$ be given. We notice that u can be written as

$$u(x,t)=\sum_{j=1}^{N}\lambda_j(t)e_j(x),\quad (x,t)\in T^2\times[0,T], \tag{2.1}$$

for some function $\lambda_i(t)$, and the norm of u in $C^0([0,T];X_N)$ can be formulated as

$$\|u\|_{C^0([0,T];X_N)}=\max_{t\in[0,T]}\left|\sum_{j=1}^{N}\lambda_j(t)\right|.$$

As a consequence, u can be bounded in $C^0([0,T];C^k(T^2))$ for any $k\in\mathbb{N}$, and there exists a constant $C>0$ depending on k such that

$$\|u\|_{C^0([0,T];C^k(T^2))}\leq C\|u\|_{C^0([0,T];L^2(T^2))}. \tag{2.2}$$

Therefore there exists a solution operator $F:C^0([0,T];X_N)\to C^0([0,T];C^3(T^2))$ such that $n=F(u)$ is the classical solution for

$$n_t+\mathrm{div}(nu)=\varepsilon\Delta n,\quad n(x,0)=n_0\quad \text{in } (0,T)\times T^2. \tag{2.3}$$

The maximum principle provides the lower and upper bounds

$$\begin{aligned}&\inf_{x\in T^d}n_0(x)\exp\Big(-\int_0^t\|\mathrm{div}u\|_{L^\infty(T^d)}\mathrm{d}s\Big)\\ &\leq n(x,t)\leq\sup_{x\in T^d}n_0(x)\exp\Big(\int_0^t\|\mathrm{div}u\|_{L^\infty(T^d)}\mathrm{d}s\Big),\quad \text{for } (x,t)\in[0,T]\times T^2.\end{aligned} \tag{2.4}$$

Since the equation is linear with respect to n, the operator F is Lipschitz continuous in the following sense:

$$\|F(v_1)-F(v_2)\|_{C^0([0,T];C^k(T^d))}\leq C\|v_1-v_2\|_{C^0([0,T];L^2(T^d))}. \tag{2.5}$$

Since we assumed that $n_0(x)\geq\delta>0$, $n(t,x)$ is strictly positive. In view of (2.1), for $\|v\|_{C^0([0,T];L^2(T^d))}\leq c$, there exist constants $\underline{n}(c)$ and $\overline{n}(c)$ such that

$$0<\underline{n}(c,\varepsilon)\leq n(x,t)\leq\overline{n}(c,\varepsilon). \tag{2.6}$$

Next, we wish to obtain the solvability of the magnetic field on the space X_N. To this end, for given u above, we are looking for a unique function B satisfying

$$\partial_t B - \nabla\times(u\times B) + \nabla(\nu_b(\rho)\nabla\times B) = 0, \tag{2.7a}$$

$$\mathrm{div} B = 0, \tag{2.7b}$$

$$B(x,0) = B_0(x), \tag{2.7c}$$

which is a linear parabolic-type problem in B. Therefore, by the standard Faedo-Galerkin methods, there exists a solution

$$B \in L^2([0,T];H^1(T^3)) \cap L^\infty([0,T];L^2(T^3)) \tag{2.8}$$

for (2.7). Further, there exists a continuous solution operator $G : C^0([0,T];X_N) \to L^2([0,T];H^1(T^3)) \cap L^\infty([0,T];L^2(T^3))$ by $G(v) = B$.

Now, for all test function $\psi \in C([0,T];X_N)$ satisfying $\psi(\cdot,T) = 0$, we wish to solve the momentum equation on the space X_N. To this end, for given $n = F(u)$, $B = G(u)$, we are looking for a function $u \in C^0([0,T];X_N)$ such that

$$\begin{aligned}
&-\int_\Omega n_0 u_0 \psi(\cdot,0)\mathrm{d}x\\
=&\int_0^T\!\!\int_{T^d}[nu\cdot\psi_t+(nu\otimes u):\nabla\psi+P(n)\mathrm{div}\psi]\mathrm{d}x\mathrm{d}t-\lambda\int_0^T\!\!\int_{T^d}\Delta^{s+1}(nu):\Delta^s(n\psi)\mathrm{d}x\mathrm{d}t\\
&-\frac{\hbar^2}{2}\int_0^T\!\!\int_{T^d}[\varphi'(n)\Delta\varphi(n)\nabla n\psi+n\varphi'(n)\Delta\varphi(n)\mathrm{div}\psi]\mathrm{d}x\mathrm{d}t\\
&-\lambda\int_0^T\!\!\int_{T^d}\Delta^s(\mathrm{div}(n\psi)):\Delta^{s+1}n\mathrm{d}x\mathrm{d}t-2\int_0^T\!\!\int_{T^d}\mu(n)D(u)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\
&-\varepsilon\int_0^T\!\!\int_{T^d}(\nabla n\cdot\nabla)u\cdot\psi\mathrm{d}x\mathrm{d}t-\int_0^T\!\!\int_{T^d}\lambda(n)\mathrm{div}u\cdot\mathrm{div}\psi\mathrm{d}x\mathrm{d}t\\
&+\mu_b\int_0^T\!\!\int_{T^d}(\nabla\times B)\times B\cdot\psi\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{2.9}$$

we will apply Banach's fixed point theorem to prove the local-in-time existence of solutions for the above equation. The regularization yields the H^1 regularity of u which is needed to conclude the global existence of solutions.

To solve (2.9), we follow from [6] and consider a family of linear operators, given a function $\rho \in L^1(T^d)$ with $\rho \geq \underline{\rho} > 0$,

$$M[\rho] : X_N \to X_N^\star, \quad \langle M[n]v, u\rangle = \int_{T^d} nv\cdot u\mathrm{d}x, \quad v, u \in X_N,$$

where the symbol $X_N^\star$ stands for the dual space of X_N. It is easy to see that the operator M is invertible provided n is strictly positive on T^d, and

$$\|M^{-1}[n]\|_{L(X_N^\star,X_N)} \le \underline{\rho}^{-1},$$

where $L(X_N^\star, X_N)$ is the set of bounded linear mappings from $X_N^\star$ to X_N. Moreover, the identity

$$M^{-1}[n_1] - M^{-1}[n_2] = M^{-1}[n_2](M[n_1] - M[n_2])M^{-1}[n_1]$$

can be used to get

$$\|M^{-1}[n_1] - M^{-1}[n_2]\|_{L(X_N^\star,X_N)} \le C(N, \underline{n})\|n_1 - n_2\|_{L^2(T^d)},$$

for any n_1, n_2 such that

$$\inf_{T^d} n_1 \ge n_0 > 0, \quad \inf_{T^d} n_2 \ge n_0 > 0.$$

So, M^{-1} is Lipschitz continuous in the sense of (2.8).

Now the integral equation (2.9) can be rephrased as an ordinary differential equation on the finite-dimensional space X_N

$$\frac{\mathrm{d}}{\mathrm{d}t}(M[n(t)u(t)]) = N[v, u, n, B],$$

where $n = F(u)$, $B = G(u)$ and

$$\begin{aligned}
&\langle N[v, u, n, B]\rangle \\
&= \int_0^T\int_{T^d} \Big\{[n_N u \otimes u : \nabla\psi + P(n_N) + P_c(n_N)]\mathrm{div}\psi - \frac{\hbar^2}{2}(\varphi'(n_N)\Delta\psi(n_N)\nabla n_N\psi \\
&\quad +n_N\varphi'\Delta\varphi(n_N)\mathrm{div}\psi)\Big\}\mathrm{d}x\mathrm{d}t + \lambda\int_0^T\int_{T^d} n_N\nabla\Delta^{2s+1}(n_N u)\psi\mathrm{d}x\mathrm{d}t \\
&\quad +\lambda\int_0^T\int_{T^d} n_N\Delta^s\mathrm{div}(n_N\psi)\cdot\Delta^{s+1}n_N\mathrm{d}x\mathrm{d}t - 2\int_0^T\int_{T^d}\mu(n_N)D(u)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t \qquad (2.10)\\
&\quad -\int_0^T\int_{T^d}\lambda(n_N)\mathrm{div}n_N\cdot\mathrm{div}\psi\mathrm{d}x\mathrm{d}t + \nu_b\int_0^T\int_{T^d}(\nabla\times B_N)\times B_N\cdot\psi\mathrm{d}x\mathrm{d}t, \quad \psi\in X_N.
\end{aligned}$$

The operator $N[v, u, n, B]$, defined for every $t \in [0, T]$ as an operator from X_N to $X_N^\star$ is continuous in time. Then the existence of a unique solution for (2.9) can be obtained by using standard theory for systems of ordinary equations. In other words, for given u, there exists a unique solution $u \in C^1([0, T]; X_N)$ for (2.7). Integrating (2.9) over $(0, t)$ yields the following nonlinear equation:

$$u = M^{-1}[F(u)](t)\Big(M[n_0]u_0 + \int_0^t N(u, u(s), n_N, B_N)\mathrm{d}s\Big) \qquad (2.11)$$

in X_N. Because the operators F, G, M^{-1} is Lipschitz continuous, this equation can be solved by evoking the fixed-pointed theorem of Banach on a short time interval $[0, T']$, where $T' \le T$, in the space $C^0([0, T']; X_N)$. In fact, we have even $u \in C^0([0, T']; X_N)$. Thus, there exists a unique local-in-time solution (n_N, u, B_N) to (2.2), (2.7) and (2.4).

2.2 Global existence of solutions

In order to prove that the solution (n_N, u_N, B_N) constructed above exists on the whole time interval $[0,T]$, it is sufficient to show that u_N is bounded in X_N on $[0,T']$ by employing the energy estimate.

Lemma 2.1 *Let $T' \le T$, and $n_N \in C^1([0,T'];C^3(T^d))$, $u_N \in C^1([0,T'];X_N)$ and $B_N \in L^2([0,T'];H^1(T^d)) \cap L^\infty([0,T'];L^2(T^d))$ be a local-in-time solution to* (2.2), (2.7), *and* (2.4) *with $n = n_N$, $u = u_N$, $B = B_N$. Then*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E(n_N,u_N,B_N) + 2\int_{T^d}\mu(n_N)|\nabla u_N|^2\mathrm{d}x + \int_{T^d}\lambda(n_N)|\mathrm{div}u_N|^2\mathrm{d}x\\&+\varepsilon\int_{T^d}\frac{1}{n}(P'(n)+P'_c(n))|\nabla n|^2\mathrm{d}x + \nu_b\int_{T^d}|\nabla\times B_N|^2\mathrm{d}x + \lambda\int_{T^d}|\Delta^s\nabla(n_Nu_N)|^2\mathrm{d}x\\&+\lambda\varepsilon\int_{T^d}|\Delta^{s+1}n_N|^2\mathrm{d}x + \varepsilon\int_{T^d}\frac{\hbar^2}{2}\varphi'(n_N)\Delta\varphi(n_N)\Delta n_N\mathrm{d}x = 0, \qquad (2.12)\end{aligned}$$

where

$$\begin{aligned}E(n_N,u_N,B_N) =& \frac{1}{2}\int_{T^d}n_N|u_N|^2\mathrm{d}x+\int_{T^d}[H(n_N)+H_c(n_N)]\mathrm{d}x+\frac{\hbar^2}{2}\int_{T^d}|\nabla\varphi(u_N)|^2\mathrm{d}x\\&+\frac{1}{2}\int_{T^d}|B_N|^2\mathrm{d}x + \frac{1}{2}\int_{T^d}\frac{\lambda}{2}|\nabla^{2s+1}n_N|^2\mathrm{d}x, \qquad (2.13)\end{aligned}$$

Proof First we multipy (2.3) by $H'(n_N) - \frac{|u_N|^2}{2} - \frac{\hbar^2}{2}\varphi'(n_N)\Delta\varphi(n_N)$, integrate it over T^d, and integrate by parts, then we obtain

$$\begin{aligned}0 =& \int_{T^d}\Big(\partial_tH(n_N)-\frac{1}{2}|u_N|^2\partial_tn_N+\frac{\hbar^2}{2}\partial_t|\nabla\varphi(n_N)|^2-n_N(H''(n_N)+H''_c(n_N))\nabla n_N\cdot u_N\\&+n_Nu_N\cdot\nabla u_N\cdot u_N - \frac{\hbar^2}{2}\varphi'(n_N)\Delta\varphi(n_N)\mathrm{div}(n_Nu_N) + \varepsilon H''(n_N)|\nabla n_N|^2\\&-\varepsilon\nabla n_N\cdot\nabla u_N\cdot u_N + \varepsilon\frac{\hbar^2}{2}\varphi'(n_N)\Delta\varphi(n_N)\Delta n_N\Big)\mathrm{d}x. \qquad (2.14)\end{aligned}$$

Next, multipying the magnetic field equation (2.7) by B_N we deduce that

$$\int_{T^d}\nabla\times(u_N\times B_N)\cdot B_N\mathrm{d}x = \frac{1}{2}\int_{T^d}\frac{\mathrm{d}}{\mathrm{d}t}|B_N|^2\mathrm{d}x + \nu_b\int_{T^d}|\nabla\times B_N|^2\mathrm{d}x. \qquad (2.15)$$

Then using the test function $u = u_N$, $n = n_N$, $B = B_N = G(u_N)$ in (2.9) and integrating by parts leads to

$$\begin{aligned}0=&\int_{T^d}\Big(|u_N|^2\partial_tn_N+\frac{1}{2}n_N\partial_t|u_N|^2-n_Nu_N\otimes u_N:\nabla u_N+(P'(n_N)+P'_c(n_N))\nabla n_N\cdot u_N\\&+\frac{\lambda}{2}|\nabla^{2s+1}n|^2\Big)\mathrm{d}x - 2\int_{T^d}\mathrm{div}(\mu(n_N)D(u_N))u_N\mathrm{d}x\\&-\int_{T^d}\nabla(\lambda(n_N)\mathrm{div}u_N)\cdot u_N\mathrm{d}x - \frac{\hbar^2}{2}\int_{T^d}n_N\nabla(\varphi'\Delta\psi(n_N))u_N\mathrm{d}x\end{aligned}$$

$$-\nu_b\int_{T^d}(\nabla\times u_N)\times B_N\cdot B_N\mathrm{d}x+\lambda\int_{T^d}|\Delta^s\nabla(n_Nu_N)|^2\mathrm{d}x+\lambda\varepsilon\int_{T^d}|\Delta^{s+1}n_N|^2\mathrm{d}x. \quad (2.16)$$

Adding above three equations gives (2.12), since $n_NH''=p'(n_N)$. Thus the proof of Lemma 2.1 is finished.

From Lemma 2.1 we have the following estimates:

- the density estimates

$$\|n_N\|_{L^\infty(0,T;L^{\gamma^+}(\Omega))}+\|n_N\|_{L^\infty(0,T;L^{\gamma^-}(\Omega))}+\|\nabla\varphi(n_N)\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+\sqrt{\varepsilon}\Big\|\frac{1}{\sqrt{n_N}}\sqrt{\frac{\partial P_c}{\partial n_N}}\nabla n_N\Big\|_{L^2(0,T;L^2(\Omega))}\le C, \quad (2.17)$$

- the velocity estimates

$$\|\sqrt{n_N}u_N\|_{L^\infty(0,T;L^2(\Omega))}+\|\sqrt{n_N}D(u_N)\|_{L^2(0,T;L^2(\Omega))}$$
$$+\|\sqrt{\lambda}\Delta^s\nabla(n_Nu_N)\|_{L^2(0,T;L^2(\Omega))}\le C. \quad (2.18)$$

By a interpolation inequality we can get the density ρ is bounded from below by a positive constant

$$\begin{aligned}\|\rho^{-1}\|_{L^\infty((0,T)\times\Omega)}&\le\|\rho^{-1}\|_{L^\infty((0,T);H^2)}\\&\le C(1+\|\nabla^3\rho\|_{L^\infty((0,T);L^2(\Omega))})^3(1+\|\rho^{-1}\|_{L^\infty((0,T);L^2(\Omega))})^4\\&\le C(\lambda),\end{aligned} \quad (2.19)$$

where we require $\gamma^->4$ and $2s+1\ge3$.

Combing with (2.15) we deduce the uniform bound for $\mathbf{u}$, thus we get a global approximating solution.

The summarizing estimates (2.17),(2.18) are uniform with the dimension N, thus we can extract the weakly convergent subsequences and pass the limit passage $N\to\infty$ in the Galerkin approximation.

3 Passage to the Limit with N

This section is devoted to the limit passage $N\to\infty$. Using estimates from the previous subsection we can extract weakly subsequences, whose limits satisfy the approximate system.

3.1 Strong convergence of the density and passage to the limit in the continuity equation

From (2.17),(2.18) we deduce that

$$u_N\to u \ \text{ weakly in } L^2(0,T;W^{2s+1,2}(\Omega)) \quad (3.1)$$

and

$$n_N\to n \ \text{ weakly in } L^2(0,T;W^{2s+2,2}(\Omega)) \quad (3.2)$$

at least for a suitable subsequence. In addition the r.h.s. of the linear parabolic problem

$$\begin{gathered}\partial_t n + \operatorname{div}(nu) - \varepsilon\Delta n = 0,\\ \rho(0,x) = \rho_\lambda^0(x)\end{gathered} \tag{3.3}$$

is uniformly bounded in $L^2(0,T;W^{2s,2}(\Omega))$ and the initial condition is sufficiently smooth, thus, applying the $L^p - L^q$ theory to this problem we conclude that $\{\partial_t\rho_N\}_{n=1}^\infty$ is uniformly bounded in $L^2(0,T;W^{2s,2}(\Omega))$. Hence, the standard compact embedding implies $\rho_N \to \rho$ a.e. in $(0,T)\times\Omega$ and therefore passage to the limit in the approximate continuity equation is straightforward.

3.2 Passage to the limit in the momentum equation

Having the strong convergence of the density, we start to identify the limit for $N\to\infty$ in the nonlinear terms of the momentum equation.

The convective term. First, one observes that

$$\rho_N\mathbf{u}_N \to \rho\mathbf{u} \ \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega))$$

due to the uniform estimates (2.18) and the strong convergence of the density. Next, one can show that for any $\phi \in \bigcap\limits_{n=1}^{\infty} X_N$ the family of functions $\int_\Omega \rho_N\mathbf{u}_N\phi \mathrm{d}x$ is bounded and equi-continuous in $C(0,T)$, thus via the Arzela-Ascoli theorem and density of smooth functions in $L^2(\Omega)$ we get that

$$\rho_N\mathbf{u}_N \to \rho\mathbf{u} \ \text{ in } C([0,T];L^2_{weak}(\Omega)). \tag{3.4}$$

Finally, by the compact embedding $L^2(\Omega)\subset W^{-1,2}(\Omega)$ and the weak convergence of $\mathbf{u}_N$ we verify that

$$\rho_N\mathbf{u}_N\otimes\mathbf{u}_N \to \rho\mathbf{u}\otimes\mathbf{u} \ \text{ weakly in } L^2((0,T)\times\Omega). \tag{3.5}$$

The capillarity term. We write it in the form

$$\int_0^T\int_\Omega \rho_N\nabla\Delta^{2s+1}\rho_N\cdot\phi\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \Delta^s\operatorname{div}(\rho_N\phi)\Delta^{s+1}\rho_N\mathrm{d}x\mathrm{d}t.$$

Due to (2.18) and the boundedness of the time derivative of ρ_N, we infer that

$$\rho_N\to\rho \ \text{ strongly in } L^2(0,T;W^{2s+1,2}(\Omega)), \tag{3.6}$$

thus

$$\int_0^T\int_\Omega \Delta^s\operatorname{div}(\rho_N\phi)\Delta^{s+1}\rho_N\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega \Delta^s\operatorname{div}(\rho\phi)\Delta^{s+1}\rho\mathrm{d}x\mathrm{d}t,$$

for any $\phi\in C^\infty((0,T)\times\overline{\Omega})$.

The momentum term. We write it in the form

$$-\lambda\int_0^T\int_\Omega \rho_N\Delta^{2s+1}(\rho_N\mathbf{u}_N)\cdot\phi\mathrm{d}x\mathrm{d}t = -\lambda\int_0^T\int_\Omega \Delta^s\nabla(\rho_N\mathbf{u}_N):\Delta^s\nabla(\rho_N\phi)\mathrm{d}x\mathrm{d}t,$$

so the convergence established in (3.1) and (3.6) are sufficient to pass to the limit here.

Strong convergence of the density enables us to perform in the momentum equation (2.9) satisfied for any function $\phi \in C^1([0,T];(X_N))$ such that $\phi(T)=0$ and by the density argument we can take all such test functions from $C^1([0,T];W^{2s+1}(\Omega))$.

4 Derivation of the B-D Estimate

At this level we are left with only two parameters of approximation: ε and λ. From the so-far obtained a-priori estimates only the ones following from (2.17) and (2.18) were independent of these parameters. Now we will have get more enough estimates for density and velocity from the B-D entropy energy inequality, we will prove the following lemma.

Lemma 4.1(Bresch-Desjardins type estimate) *The following identity holds:*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{T^d}\Big(\frac{1}{2}n|u+\nabla\phi(n)|^2+H(n)+H_c(n)+\frac{\hbar^2}{2}|\nabla\varphi(n)|^2+\frac{1}{2}|B|^2\Big)\mathrm{d}x\\
&+\int_{T^d}2\mu(n)|A(u)|^2\mathrm{d}x+\frac{\hbar^2}{2}\int_{T^d}\varphi'(n)|\Delta\varphi(n)|^2\mathrm{d}x+\nu_b\int_{T^d}|\nabla\times B|^2\mathrm{d}x\\
&+2\int_{T^d}\mu'(n)(P'(n)+P_c'(n))\frac{|\nabla n|^2}{n}\mathrm{d}x+2\lambda\int_\Omega\Delta^{s+1}n\Delta^s\mu(n)\mathrm{d}x\\
=&-2\lambda\int_\Omega\Delta^s\nabla(nu):\Delta^s\nabla^2\mu(n)\mathrm{d}x-\varepsilon\int_\Omega\mathrm{div}(nu)\phi'(n)\Delta n\mathrm{d}x\\
&+\varepsilon\int_\Omega\frac{|\nabla\phi(n)|^2}{2}\Delta n\mathrm{d}x-\varepsilon\int_\Omega(\nabla n\cdot\nabla)u\cdot\nabla\phi(n)\mathrm{d}x\\
&+\varepsilon\int_\Omega n\nabla\phi(n)\cdot\nabla(\phi'(n)\Delta n)\mathrm{d}x+\int_{T^d}(\nabla\times B)\times B\cdot\nabla\phi(n)\mathrm{d}x,
\end{aligned}\tag{4.1}$$

in $\mathcal{D}'(0,T)$, *where* $\nabla\phi(n)=2\frac{\nabla\mu(n)}{n}$.

Proof The basic idea of the proof is to find the explicit form of the terms:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{1}{2}n|u|^2+nu\cdot\nabla\phi(n)+\frac{1}{2}n|\nabla\phi(n)|^2\Big)\mathrm{d}x.\tag{4.2}$$

The first term can be evaluated by means of the main energy inequality, that is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{T^d}\Big(\frac{1}{2}n|u|^2+[H(n)+H_c(n)]+\frac{\hbar^2}{2}|\nabla\varphi(u)|^2+\frac{1}{2}|B|^2+\frac{\lambda}{2}|\nabla^{2s+1}n|^2\Big)\mathrm{d}x\\
&+2\int_{T^d}\mu(n)|\nabla u|^2\mathrm{d}x+\int_{T^d}\lambda(n)|\mathrm{div}u|^2\mathrm{d}x+\varepsilon\int_{T^d}\frac{1}{n}(P'(n)+P_c'(n))|\nabla n|^2\mathrm{d}x\\
&+\nu_b\int_{T^d}|\nabla\times B|^2\mathrm{d}x+\lambda\int_{T^d}|\Delta^s\nabla(nu)|^2\mathrm{d}x+\lambda\varepsilon\int_{T^d}|\Delta^{s+1}n|^2\mathrm{d}x\\
&+\varepsilon\int_{T^d}\frac{\hbar^2}{2}\varphi'(n)\Delta\varphi(n)\Delta n\mathrm{d}x=0.
\end{aligned}\tag{4.3}$$

To get a relevant expression for third term in (4.2), we multiply the approximate continuity equation by $\frac{|\nabla\phi(n)|^2}{2}$ and we obtain the following sequence of equalities

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}n|\nabla\phi(n)|^2\mathrm{d}x &= \int_\Omega \Big(n\partial_t\frac{|\nabla\phi(n)|^2}{2} - \frac{|\nabla\phi(n)|^2}{2}\mathrm{div}(nu) + \varepsilon\frac{|\nabla\phi(n)|^2}{2}\Delta n\Big)\mathrm{d}x\\ &= \int_\Omega\Big(\rho\nabla\phi(n)\cdot\nabla(\phi'(n)\partial_t n) - \frac{|\nabla\phi(n)|^2}{2}\mathrm{div}(nu) + \varepsilon\frac{|\nabla\phi(n)|^2}{2}\Delta n\Big)\mathrm{d}x. \end{aligned} \tag{4.4}$$

Using the approximate continuity equation, we get

$$\begin{aligned}&\int_\Omega n\nabla\phi(n)\cdot\nabla(\phi'(n)\partial_t n)\mathrm{d}x\\ =&\int_\Omega \varepsilon n\nabla\phi(n)\cdot\nabla(\phi'(n)\Delta n)\mathrm{d}x - \int_\Omega \rho\nabla\mathbf{u} : \nabla\phi(\rho)\otimes\nabla\phi(n)\mathrm{d}x\\ &-\int_\Omega n\nabla\phi(n)\cdot\nabla(\phi'(n)n\mathrm{div}\mathbf{u})\mathrm{d}x - \int_\Omega n\mathbf{u}\otimes\nabla\phi(n) : \nabla^2\phi(n)\mathrm{d}x. \end{aligned}\tag{4.5}$$

Integrating by parts the two last terms from the r.h.s.

$$\begin{aligned}&\int_\Omega n\nabla\phi(n)\cdot\nabla(\phi'(n)\partial_t n)\mathrm{d}x\\ =&\int_\Omega \varepsilon n\nabla\phi(n)\cdot\nabla(\phi'(n)\Delta n)\mathrm{d}x - \int_\Omega n\nabla u : \nabla\phi(n)\otimes\nabla\phi(n)\mathrm{d}x\\ &+\int_\Omega n|\nabla\phi(n)|^2\mathrm{div}u\mathrm{d}x + \int_\Omega n^2\phi'(n)\Delta\phi(n)\mathrm{div}u\mathrm{d}x\\ &+\int_\Omega |\nabla\phi(n)|^2\mathrm{div}(nu)\mathrm{d}x + \int_\Omega nu\cdot\nabla(\nabla\phi(n))\cdot\nabla(\phi(n))\mathrm{d}x. \end{aligned}\tag{4.6}$$

Combining the three previous equalities we finally obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}n|\nabla\phi(n)|^2\mathrm{d}x\\ =&\int_\Omega \varepsilon n\nabla\phi(n)\cdot\nabla(\phi'(n)\Delta n)\mathrm{d}x - \int_\Omega n\nabla u : \nabla\phi(n)\otimes\nabla\phi(n)\mathrm{d}x\\ &+\int_\Omega n|\nabla\phi(n)|^2\mathrm{div}u\mathrm{d}x + \int_\Omega n^2\phi'(n)\Delta\phi(n)\mathrm{div}u\mathrm{d}x + \int_\Omega \varepsilon\frac{|\nabla\phi(n)|^2}{2}\Delta n\mathrm{d}x. \end{aligned}\tag{4.7}$$

In the above series of equalities, each one holds ponitwisely with respect to time due to the regularity of n and $\nabla\phi$. This is not the case of the middle integrant of (4.2), for which one should really think of weak in time formulation. Denote

$$V = W^{2s+1,2}(\Omega), \quad \text{and} \quad \mathbf{v} = n\mathbf{u}, \quad h = \nabla\phi. \tag{4.8}$$

We know that $\mathbf{v} \in L^2(0,T;V)$ and its weak derivative with respect to time variable $\mathbf{v}' \in L^2(0,T;V^*)$ where V^* denotes the dual space to V. Moreover, $h \in L^2(0,T;V)$,

$h' \in L^2(0,T;W^{2s-1,2}(\Omega))$. Now, let $\mathbf{v}_m$ and h_m denote the standard mollifications in time of $\mathbf{v}$ and h respectively. By the properties of mollifiers we know that

$$\mathbf{v}_m, \mathbf{v}'_m \in C^\infty(0,T;V), \quad h_m, h'_m \in C^\infty(0,T;V) \tag{4.9}$$

and

$$\begin{aligned}&\mathbf{v}_m \to v \ \ L^2(0,T;V), \quad h_m \to h \ \ L^2(0,T;V),\\ &\mathbf{v}'_m \to v' \ \ L^2(0,T;V^*), \quad h'_m \to h' \ \ L^2(0,T;V^*).\end{aligned} \tag{4.10}$$

For these regularized sequences we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{v}_m \cdot h_m \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}_m, h_m)_V = (\mathbf{v}'_m, h_m)_V + (\mathbf{v}_m, h'_m)_V, \quad \text{for any } \psi \in \mathcal{D}(0,T). \tag{4.11}$$

Using the Riesz representation theorem we verify that $\mathbf{v}'_m \in C^\infty(0,T;V)$ uniquely determines the functional $\Phi_{v'_m} \in V^*$ such that $(\mathbf{v}'_m, \psi)_V = (\Phi_{v'_m}, \psi)_{V^*,V} = \int_\Omega \mathbf{v}'_m \cdot \psi \mathrm{d}x$, for any $\psi \in V$; for the second term from the r.h.s. of (4.11) we can simply replace $V = L^2(\Omega)$ and thus we obtain

$$-\int_0^T (\mathbf{v}_m, h_m)_V \psi' \mathrm{d}t = \int_0^T (\mathbf{v}'_m, h_m)_{V^*,V} \psi \mathrm{d}t + \int_0^T (\mathbf{v}_m, h'_m)_{L^2(\Omega)} \psi \mathrm{d}t, \ \text{for any } \psi \in \mathcal{D}(0,T). \tag{4.12}$$

Observe that both integrands from the r.h.s. are uniformly bounded in $L^1(0,T)$, thus, using (4.10), we let $m \to \infty$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{v} \cdot h \mathrm{d}x = (\mathbf{v}', h)_V + (\mathbf{v}, h')_V, \quad \text{for any } \psi \in \mathcal{D}(0,T). \tag{4.13}$$

Coming with the original notation, this means that the operation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega nu \cdot \nabla\phi(n) \mathrm{d}x = \langle \partial_t(nu), \nabla\phi \rangle_{V^*,V} + \int_\Omega nu \cdot \partial_t \nabla\phi \mathrm{d}x \tag{4.14}$$

is well defined and is nothing but equality between two scalar distributions. By the fact that $\partial_t \nabla\phi$ exists a.e. in $(0,T) \times \Omega$ we may use approximation to write

$$\int_\Omega nu \cdot \partial_t \nabla\phi(n) \mathrm{d}x = \int_\Omega (\mathrm{div}(nu))^2 \phi'(n) \mathrm{d}x - \varepsilon \int_\Omega \mathrm{div}(nu)\phi'(n)\Delta n \mathrm{d}x, \tag{4.15}$$

whence the first term on the r.h.s. of (4.14) may be evaluated by testing the approximate momentum equation by ∇n

$$\begin{aligned}
&\langle \partial_t(n\mathbf{u}), \nabla\phi(n)\rangle_{V^*,V}\\
=&-\int_\Omega (2\mu(n)+\lambda(n))\Delta\phi(n)\mathrm{div}u\mathrm{d}x + 2\int_\Omega \nabla u : \nabla\phi(n)\otimes\nabla\mu(n)\mathrm{d}x\\
&-2\int_\Omega \nabla\phi(n)\cdot\nabla\mu(n)\mathrm{div}u\mathrm{d}x - \int_\Omega \nabla\phi(n)\cdot\nabla P\mathrm{d}x\\
&-\lambda\int_\Omega \Delta^{s+1}\mu(n)\Delta^s\mathrm{div}(n\nabla\phi(n))\mathrm{d}x - \lambda\int_\Omega \Delta^s\nabla(nu) : \Delta^s\nabla(n\nabla\phi(n))\mathrm{d}x\\
&-\int_\Omega \nabla\phi(n)\cdot\mathrm{div}(nu\otimes u)\mathrm{d}x - \varepsilon\int_\Omega (\nabla n\cdot\nabla)u\cdot\nabla\phi(n)\mathrm{d}x\\
&-\int_\Omega \varphi'(n)|\Delta\varphi(n)|^2\mathrm{d}x + \int_{T^d}(\nabla\times B)\times B\cdot\phi(n)\mathrm{d}x. \qquad (4.16)
\end{aligned}$$

Recalling the form of $\phi(n)$ it can be deduced that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(nu\cdot\nabla\phi(n)+\frac{1}{2}n|\nabla\phi(n)|^2\Big)\mathrm{d}x+\int_\Omega\nabla\phi(n)\cdot\nabla P\mathrm{d}x+\lambda\int_\Omega\mu'(n)\Delta\mu(n)\Delta^s\mu(n)\mathrm{d}x\\
=&-\int_\Omega \nabla\phi(n)\cdot\mathrm{div}(nu\otimes u)\mathrm{d}x+\int_\Omega(\mathrm{div}(nu))^2\phi'(n)\mathrm{d}x-2\lambda\int_\Omega\Delta^s\nabla(nu):\Delta^s\nabla^2\mu(n)\mathrm{d}x\\
&-\varepsilon\int_\Omega \mathrm{div}(nu)\phi'(n)\Delta n\mathrm{d}x + \varepsilon\int_\Omega\frac{|\nabla\phi(n)|^2}{2}\Delta n\mathrm{d}x - \varepsilon\int_\Omega(\nabla n\cdot\nabla)u\cdot\nabla\phi(n)\mathrm{d}x\\
&+\varepsilon\int_\Omega n\nabla\phi(n)\cdot\nabla(\phi'(n)\Delta n)\mathrm{d}x-\int_\Omega\varphi'(n)|\Delta\varphi(n)|^2\mathrm{d}x+\int_{T^d}(\nabla\times B)\times B\cdot\nabla\phi(n)\mathrm{d}x.
\end{aligned}$$
(4.17)

The first two terms from the r.h.s. of (4.17) can be transformed

$$\begin{aligned}
&\int_\Omega\big((\mathrm{div}(nu))^2\phi'(n) - \nabla\phi(n)\cdot\mathrm{div}(nu\otimes u)\big)\mathrm{d}x\\
=&\int_\Omega\big(n^2\phi^{'}(n)(\mathrm{div}u)^2 + n\phi^{'}\cdot\nabla n\mathrm{div}u - n\phi^{'}\nabla n(u\cdot\nabla u)\big)\mathrm{d}x\\
=&\,2\int_\Omega\mu(n)\partial_i u_j\partial_j u_i\mathrm{d}x = 2\int_\Omega\mu(n)|D(u)|^2\mathrm{d}x - 2\int_\Omega\mu(n)\Big(\frac{\partial_i u_j-\partial_j u_i}{2}\Big)^2\mathrm{d}x, \qquad (4.18)
\end{aligned}$$

thus, the assertion of Lemma 4.1 follows by adding (4.3) to (4.17). The proof is finished.

The main problem is to control the last term on the right hand side of (4.1), other terms can be easier to be controlled. For this obstacle, we estimate as follows

$$2\Big|\int_{T^d}(\nabla\times B)\times B\cdot\frac{\nabla\mu(n)}{n}\Big| \le \int_{T^d}\frac{|\nabla\times B|^2}{\varepsilon n^2}\mathrm{d}x + \varepsilon\int_{T^d}|\nabla\mu(n)\times B|^2\mathrm{d}x. \qquad (4.19)$$

The first term of the right hand side will sent to the left hand side of equation and will we compensated with the term related to the resistivity thanks to the profiles condition introduced in (1.6).

The dimension hypothesis appearing at this point, in a 2-dimensional space, insures $W^{1,1} \subset L^2$ and this will be the main tool to deal with the second term. We have

$$\begin{aligned}\|\nabla\mu(n)\times B\|^2_{L^2(T^d)} &\le C\|\nabla\mu(n)\times B\|^2_{W^{1,1}}\\ &\le C(\|\Delta\mu(n)\|^2_{L^2(T^d)}\|B\|^2_{L^2(T^d)}+\|\nabla\mu(n)\|^2_{L^2(T^d)}\|\nabla B\|^2_{L^2(T^d)}\\ &\quad +\|\nabla\mu(n)\times B\|^2_{L^1(T^d)}. \end{aligned}\tag{4.20}$$

But, from (2.12), we already know that $\|B\|_{L^2}$ and $\|\nabla\mu(n)\|_{L^2}$ are uniformly bounded by Λ_0, since

$$\|\nabla\mu(n)\times B\|^2_{L^2(T^d)} \le C(1+\|\Delta\mu(n)\|^2_{L^2}+\|\nabla\times B\|^2_{L^2}). \tag{4.21}$$

So we get, summing (4.20) and (4.21) and taking into account all these quantities, for ε small enough, we are considering here some coefficients $\varepsilon < 1/6$ and such that $\mu' - C\varepsilon$ still higher that a constant, say δ. It also appears the necessary conditions on the constants d_0 and d_1, to be high enough because we need to have $\eta(n) - \varepsilon^{-1}n^{-2} - C\varepsilon \ge 0$. To conclude, we apply a Gronwall's lemma, we will get B-D entropy energy estimates.

5 Estimates Independent of ε, λ, Passage to the Limit $\varepsilon, \lambda \to 0$

In this section we first present the new uniform bounds arising from the estimate of B-D entropy, performed in this section, and then we let the last two approximation parameters to 0. Note that the limit passage $\lambda \to 0$, $\varepsilon \to 0$ could be done in a single step.

We complete the set uniform bounds by

$$\begin{aligned}&\sqrt{\lambda}\|\Delta^{s+1}n_{\varepsilon,\lambda}\|_{L^2((0,T)\times\Omega)}+\|\nabla\phi(n_{\varepsilon,\lambda})\|_{L^2((0,T)\times\Omega)}\\ &+\left\|\sqrt{\frac{\mu'(n_{\varepsilon,\lambda})(P'(n_{\varepsilon,\lambda})+P'_c(n_{\varepsilon,\lambda}))}{n_{\varepsilon,\lambda}}}\nabla n\right\|_{L^2((0,T)\times\Omega)} \le C,\end{aligned}\tag{5.1}$$

moreover

$$\|\Delta\mu(n_{\varepsilon,\lambda})\|_{L^2((0,T)\times\Omega)} \le C. \tag{5.2}$$

The uniform estimates for the velocity vector field are

$$\sqrt{\lambda}\|\Delta^s\nabla(\rho u_{\varepsilon,\lambda})\|_{L^2((0,T)\times\Omega)}+\left\|\sqrt{\mu(n_{\varepsilon,\lambda})}\nabla A(u_{\varepsilon,\lambda})\right\|_{L^2((0,T)\times\Omega)} \le C, \tag{5.3}$$

and the constants from the r.h.s are independent of ε and λ.

We now present several additional estimates of $n_{\varepsilon,\lambda}$ and $u_{\varepsilon,\lambda}$ based on imbedding of Sobolev spaces and simple interpolation inequalities.

5.1 Further estimates of n

Lemma 5.1

$$n_{\varepsilon,\lambda}^{-1/2} \textit{ is uniformly bounded in } L^\infty(0,T;L^6_{loc}(\Omega))\cap L^2(0,T;H^1_{loc}(\Omega)), \tag{5.4}$$

$$n_{\varepsilon,\lambda} \textit{ is uniformly bounded in } L^\infty(0,T;L^p_{loc}(\Omega)), \textit{ for any } p<+\infty. \tag{5.5}$$

Proof On the one hand, from (2.10) we know that $H_c(n_{\varepsilon,\lambda})$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ which implies that $n_{\varepsilon,\lambda}^{-1/2}$ is bounded in $L^\infty(0,T;L^{2\gamma^-})$. On the other hand, there exist functions $\zeta(n)=n$ for $n<1$, $\zeta(n)=0$ for $n>1$ such that $\nabla\zeta(n)^{-1/2}$ is bounded in $L^2(0,T;L^2(\Omega))$. Then, noting that $\gamma^->1>\alpha$, we conclude that $\nabla n_{\varepsilon,\lambda}^{-1/2}$ is also bounded in $L^2(0,T;L^2(\Omega))$.

Since $\nabla\psi(n_{\varepsilon,\lambda})$ is bounded in $L^\infty(0,T;L^2(\Omega))$, and $H(n_{\varepsilon,\lambda})$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, we can use Sobolev embedding of $H^1(\Omega)$ in $L^p(\Omega)$ for all $p<+\infty$ in the two dimension. The proof is completed.

5.2 Estimate of the velocity vector field

Lemma 5.2

$$u_{\varepsilon,\lambda} \textit{ is uniformly bounded in } L^{q_1}(0,T;W^{1,q_2}_{loc}(\Omega)),\quad q_1>\frac{5}{3} \textit{ and } q_2>\frac{15}{8}. \tag{5.6}$$

Proof We use the Hölder inequality to write

$$\|\nabla u_{\varepsilon,\lambda}\|_{L^{q_1}(0,T;L^{q_3}(\Omega))}\le c(1+\|\zeta(n_{\varepsilon,\lambda})^{-\frac{\alpha}{2}}\|_{L^{2j}(0,T;L^{6j}(\Omega))})\|n_{\varepsilon,\lambda}^{\frac{\alpha}{2}}\nabla u_{\varepsilon,\lambda}\|_{L^2((0,T)\times\Omega)}, \tag{5.7}$$

where $j=\frac{\gamma^-+1-\alpha}{\alpha}$, $\frac{1}{q_1}=\frac{1}{2}+\frac{1}{2j}$, $\frac{1}{q_3}=\frac{1}{2}+\frac{1}{6j}$. Therefore, the Korn inequality together with the Sobolev imbedding implies the lemma.

5.3 Magnetic field

Thanks to estimates (2.10) and conditions on η that

$$B_{\varepsilon,\lambda} \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)). \tag{5.8}$$

By interpolation, we can also deduce the following result.

Lemma 5.3 *Let β be any parameter in $(0,1)$ and $p<+\infty$. Then*

$$B_{\varepsilon,\lambda} \textit{ is uniformly bounded in } L^{\frac{2}{\beta}}(0,T;L^{\frac{2}{(\frac{2}{p})\alpha+1}}(\Omega)). \tag{5.9}$$

5.4 Passage to the limit with $\varepsilon\to 0$ and $\lambda\to 0$

With the B-D estimate at hand, especially with the bound on $\Delta^{s+1}n_{\varepsilon,\lambda}$ in $L^2((0,T)\times\Omega)$, which is now uniform with respect to ε, we may perform the limit passage similarly as in previous step. Indeed, the uniform estimates allow us to extract subsequences, such that

$$\varepsilon\Delta^s\nabla u_{\varepsilon,\lambda},\varepsilon\nabla n_{\varepsilon,\lambda},\varepsilon\Delta^{s+1}n_{\varepsilon,\lambda}\to 0 \quad \textit{strongly in } L^2((0,T)\times\Omega), \tag{5.10}$$

therefore

$$\varepsilon\nabla n_{\varepsilon,\lambda}\nabla u_{\varepsilon,\lambda}\to 0 \quad \textit{strongly in } L^1((0,T)\times\Omega). \tag{5.11}$$

5.5 For $n_{\varepsilon,\lambda}$

We know from (5.5) that $n_{\varepsilon,\lambda}$ converges weakly to n in $L^\infty(0,T;L^q_{loc}(\Omega))$, for all $q<+\infty$. To prove strong convergence on the density, we shall use the transport equation satisfying $\mu(n)$:

$$\partial_t(\mu(n))+\mathrm{div}(\mu(n)u)+\frac{1}{2}\lambda(n)\mathrm{div}u=0.$$

Prove that $\partial_t(\phi\mu(n))$ is bounded in $L^2(0,T;H^{-\sigma_0}(\Omega))$ for any compactly supported ϕ, we then conclude that

$$n_{\varepsilon,\lambda}\to n \ \text{ in } C([0,T];L^q_{loc}(\Omega)), \quad \text{for any } q<+\infty. \tag{5.12}$$

From another point, to conclude a compactness for $n_{\varepsilon,\lambda}^{-1/2}$ in $C([0,T];L^q_{loc}(\Omega))$, for all $q<+\infty$, we must, in addition to (5.4), look at $\partial_t(n^{-1/2})$ and try to show a boundedness in a space $L^r(0,T;H^{-\sigma_0})$ with $r>1$. From the transport equation we find

$$\partial_t(n^{-1/2})-\frac{3}{2}n^{-1/2}\mathrm{div}u+\mathrm{div}(n^{-1/2}u)=0,$$

from which we can insure that $\partial_t(n^{-1/2})$ is bounded in $L^{5/3}(0,T;W^{-1,\frac{30}{11}}(\Omega))$. Then, from (5.4), we can deduce that

$$\begin{aligned} n_{\varepsilon,\lambda}^{-1/2}\to n^{-1/2} \quad & \text{in } L^p(0,T;L^q_{loc}(\Omega)), \ \text{ for any } p<+\infty,\ q<6,\\ & \text{in } L^2(0,T;L^q_{loc}(\Omega)), \ \text{ for any } q<+\infty, \end{aligned} \tag{5.13}$$

5.6 For $n_{\varepsilon,\lambda}u_{\varepsilon,\lambda}$

We know that $n_{\varepsilon,\lambda}u_{\varepsilon,\lambda}$ converges weakly to nu in $L^\infty(0,T;L^{s<2}_{loc}(\Omega))$ as the product of $n_{\varepsilon,\lambda}$ bounded in $L^\infty(0,T;L^{r<\infty}_{loc}(\Omega))$ and $\sqrt{n_{\varepsilon,\lambda}}u_{\varepsilon,\lambda}$ bounded in $L^\infty(0,T;L^2(\Omega))$. To have compactness on $n_{\varepsilon,\lambda}u_{\varepsilon,\lambda}$, we use the momentum equation to assure that $\partial_t(n_{\varepsilon,\lambda}u_{\varepsilon,\lambda})$ is bounded in $L^p_{loc}(0,T;H^{-\sigma_0}(\Omega))$ for $p>1$ and σ_0 large enough. To more precise on what is different in our system we shall forget the new term in the momentum equation related to the magnetic field, namely $\nabla B\times B$. Using (5.8), we know that $\nabla\times B$ is bounded in $L^2(0,T;L^2(\Omega))$, that is why we must have better than $L^2(0,T;L^2(\Omega))$ for B and it is time to use Lemma 5.3. Indeed, for any $0<\alpha<1$ we get the expected boundedness of B in $L^p(0,T;L^p(\Omega))$ with $p>2$ so that $(\nabla\times B)\times B$ is bounded in B in $L^q(0,T;L^q(\Omega))$ with $q>1$. Thus, we get

$$n_{\varepsilon,\lambda}u_{\varepsilon,\lambda}\to nu \ \text{ in } L^p(0,T;W^{-1,q}_{loc}(\Omega)), \quad \text{for any } p<+\infty,\ q<6. \tag{5.14}$$

(5.14) together with Lemma 5.2 implies the strong convergence of $\int_B n_{\varepsilon,\lambda}|u_{\varepsilon,\lambda}|^2$ to

$\int_B n|u|^2$, for all subset B in Ω. Moreover, since $\sqrt{n_{\varepsilon,\lambda}}u_{\varepsilon,\lambda}$ converges weakly to $\sqrt{n}u$ in $L^\infty(0,T;L^2_{loc}(\Omega))$, we insure that

$$\sqrt{n_{\varepsilon,\lambda}}u_{\varepsilon,\lambda} \to \sqrt{n}u \quad \text{in } L^2(0,T;L^2_{loc}(\Omega)). \tag{5.15}$$

5.7 For the magnetic field $B_{\varepsilon,\lambda}$

We already know that the sequence $B_{\varepsilon,\lambda}$ weakly converges to the limit B in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$. Now deal with $\partial_t B$ in order to insure a strong convergence statement. Equation (1.1c) leads to bound $u\times B$ and $(\xi_b)\nabla\times B$. For the first one, from Lemmas 5.2 and 5.3, we get $u\times B$ is bounded in $L^p_{loc}(0,T;L^p)$ with $p>1$ what is enough comfortable. For the second, we write $\xi_b\nabla\times B=\sqrt{\xi_b}\sqrt{\xi_b}\nabla\times B$. We know that the term $\sqrt{\xi_b}\nabla\times B$ is bounded in $L^2(0,T;L^2(\Omega))$. From conditions (1.6) and the bounds (37) or (38), $\sqrt{\xi_b}$ is bounded in $L^2(0,T;L^2(\Omega))$. This is just enough to conclude that B_t is bounded in $L^1(0,T;W^{-1,1}(\Omega))$. Then we get

$$B_{\varepsilon,\lambda}\to B \ \text{ in } L^p(0,T;L^2(\Omega)), \quad \text{for any } p<+\infty. \tag{5.16}$$

6 Convergences

The limit mass conservation equation holds because of the strong convergence of $n_{\varepsilon,\lambda}$ in (5.12) and the strong convergence of $\sqrt{n_{\varepsilon,\lambda}}u_{\varepsilon,\lambda}$ in (5.15).

In the momentum equation, the strong convergence of $n_{\varepsilon,\lambda}u_{\varepsilon,\lambda}$ in (5.14) and $\sqrt{n_{\varepsilon,\lambda}}u_{\varepsilon,\lambda}$ in (5.15) allows to pass to the limit in the sense of distributions in the first two terms. As for the pressure term, we use the strong convergence of $n_{\varepsilon,\lambda}$ in (5.12) and the boundness of $n_{\varepsilon,\lambda}$ in (5.5). The viscous flux terms can be handled similar to [4]. Pass to the limit in the quantum term, we need to split several term which is similar to [15]. The strong convergence of $B_{\varepsilon,\lambda}$ in (5.16) and the boundness of $B_{\varepsilon,\lambda}$ in (5.8) are ensure to pass the limit in the magnetic term in the momentum equation.

In the magnetic field equation, the difficult is the last term $\nabla\times(\nu_b(\rho_{\varepsilon,\lambda})B_{\varepsilon,\lambda}\times B_{\varepsilon,\lambda})$, it can be written as the product $\sqrt{\nu_b(\rho_{\varepsilon,\lambda})}\nabla B_{\varepsilon,\lambda}$, weakly convergence in $L^2(0,T;L^2_{loc}(\Omega))$ and $\sqrt{\nu_b(\rho_{\varepsilon,\lambda})}$ strongly convergence in $L^2(0,T;L^2_{loc}(\Omega))$.

The proof is complete.

References

[1] D. Bresch, B. Desjardins, Some diffusive capillary models for Korteweg type, *C. R. Math. Acad. Sci. Paris*, (2002),1079-1084.

[2] D. Bresch, B. Desjardins, C.K. Lin, On some compressible uid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations*, **28**(2003),843-868.

[3] D. Bresch, B. Desjardins, and C.-K. Lin, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrohic model, *Comm. Math. Phys.*, **238**(2003),211-223.

[4] D. Bresch, B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting uids, *J. Math. Pures Appl.*, **87**(2007),57-90.

[5] J. Dong, A note on barotropic compressible quantum Navier-Stokes equations, *Nonlinear Analysis: Real World Applications*, **73**(2010),854-856.

[6] D. Ferry and J.R. Zhou, Form of the quantum potential for use in hydrodynamic equations for semiductor device modeling, *Phys. Rev.*, **48**:3B(2009),536-551.

[7] E. Feireisl, A. Novotny, H. Petzeltova, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Flu. Mech*, **3**(2001),358-392.

[8] E. Firesel, Dynamics of viscous compressible fliuds, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004.

[9] E. Firesel, A. Novotny, Singular Limits in Thermodynamics of Viscous Fluids, Birkhauser Verlag, Basel, 2009.

[10] C. Gardner, The quantum hydrodynamic model for semiconductor devices, *SIAM J. Appl. Math.*, **4**(1994),409-427.

[11] M. Gisclon, I. Lacroix, About the barotropic compressible quantum Navier-Stokes equations, http://arxiv.org/abs/1412.1332v1.

[12] J. Grant, Pressure and stress tensor expressions in the fluid mechanical formulation of the Bose condensate equations, *J. Phys. A*, **6**(1973),L151-L153.

[13] F. Haas, Quantum Plasmas, Springer.

[14] A. Jungel, D. Matthes, J.P. Milisic, Derivation of new quantum hydrodynamic equations using entropy minimization, *SIAM J. Appl. Math.*, **3**(2006),46-68.

[15] A. Jungel, Global weak solutions to compressible Navier-Stokes equations for quantum uids, *SIAM J. Math. Anal.*, **42**(2010),1025-1045.

[16] A. Jüngel, Dissipative quantum fluid models, *Riv. Math. Univ. Parma (N.S.)*, **3**(2012), 217-290.

[17] F. Jiang, A remark on weak solutions to the barotropic compressible quantum Navier-Stokes equations, *Nonlinear Analysis: Real World Applications*, **12**(2011),1733-1735.

[18] P.L. Lions, Mathematical topics in fluid dynamics, Vol. 2, Compressible models, Oxford Science Publication, Oxford, 1998.

[19] J. Li, Z.P. Xin, Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities, http://arxiv.org/abs/1504.06826v1 .

[20] M. Loffredo and L. Morato, On the creation of quantum vortex lines in rotating He II.Il Nouvo Cimento, **108B**(1993),205-215.

[21] A. Mellet, A. Vasseur, On the isentropic compressible Navier-Stokes equations, *Mathematics*, **32**:3(2005),431-452.

[22] P.B. Mucha, M. Pokorny, E. Zatorska, Heat-conducting, compressible mixtures with multicomponent diffusion: construction of a weak solution, *SIAM J. Math. Anal.*, **47**(2015),3747-3797.

(*edited by Liangwei Huang*)