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# SOLUTION FOR TWO-POINT BOUNDARY VALUE PROBLEM OF THE SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATION*†

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**Abstract**

In this paper, we establish the existence result of solution and positive solution for two-point boundary value problem of a semilinear fractional differential equation by using the Leray-Schauder fixed-point theorem. The discussion is based on the system of integral equations on a bounded region.

**Keywords** boundary value problem; Green's function; Leray-Schauder fixed point theorem; system of integral equations

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## 1 Introduction

Fractional differential equations have received increasing attention during the past decades. It has attracted a lot of attention of researchers to promote the continuous development of methods, theories and applications in the field of small area estimation (see [1-3]). Fractional derivative is divided into two categories: standard Riemann-Liouville derivative and Caputo fractional derivative.

The aim of this paper is to study the existence result of solution and positive solution for the following two-point boundary value problem of the semilinear fractional differential equation

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\alpha-1}u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = 0, \quad u(1) = B, \quad D^{\alpha-1}u(0) = C, \end{cases} \tag{1.1}$$

where $2 < \alpha \leqslant 3$ and $A, B, C$ are real numbers, $D^{\alpha}$ is the standard Riemann-Liouville derivative, and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous on its domain. Such a

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nonlinearity term $f(t, u(t), D^{\alpha-1}u(t))$ has been studied widely in [6,7]. In [6], by means of the Schauder fixed point theorem and the Banach contraction principle the authors investigated the existence and uniqueness of solutions for a class of nonlinear multi-point boundary value problems for fractional differential equations

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\beta}u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = 0, \quad D^{\beta}u(1) - \sum\limits_{i=1}^{m-2} \xi_i D^{\beta}u(\xi_i) = u_0. \end{cases}$$

In [7], by means of a fixed point theorem on a cone, the authors investigated the existence of positive solutions for the following singular fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\mu}u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = u(1) = 0. \end{cases}$$

The difference between [6] and [7], the system of integral equations is adopted skillfully in this paper. In the literature of [8], $A = 0$ is the special case of this paper.

## 2 Preliminaries

For convenience, we present here the necessary definitions and some lemmas from fractional calculus theory.

**Definition 2.1**[4] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s$$

provided the right side is pointwise defined on $(0, \infty)$.

**Definition 2.2**[4] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number $\alpha$, provided the right side integral is pointwise defined on $[0, 1)$.

**Lemma 2.1**[4] *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

*has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in \mathbb{R}$, $i = 1, 2, \cdots, N$, which is a unique solution, where $N$ is the smallest integer greater than or equal to $\alpha$.*

**Lemma 2.2**[4] *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N},$$

*for some $C_i\in\mathbb{R}$, $i=1,2,\cdots,N$, where $N$ is the smallest integer greater than or equal to $\alpha$.*

**Property 2.1**[4] *Let $\alpha\geqslant 0$, $m\in\mathbb{N}$ and $D=\frac{\mathrm{d}}{\mathrm{d}x}$. If the fractional derivatives $(D_{0+}^{\alpha}y)(x)$ and $(D_{0+}^{\alpha+m}y)(x)$ exist, then*

$$(D^mD_{0+}^{\alpha}y)(x)=(D_{0+}^{\alpha+m}y)(x).$$

**Lemma 2.3** *Let $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a given function. Then the boundary value problem*

$$\begin{cases}D^{\alpha}u(t)+f(t,u(t),D^{\alpha-1}u(t))=0, \quad 0\leqslant t\leqslant 1,\\ u(0)=0, \quad u(1)=B, \quad D^{\alpha-1}u(0)=C\end{cases} \tag{2.1}$$

*is equivalent to the system of integral equations*

$$\begin{cases}v(t)=C-\displaystyle\int_0^t f(s,u(s),v(s))\mathrm{d}s,\\ u(t)=Bt^{\beta-1}-\displaystyle\int_0^1 G(t,s)v(s)\mathrm{d}s,\end{cases} \tag{2.2}$$

*where*

$$G(t,s)=\begin{cases}\dfrac{[t(1-s)]^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & s\leqslant t;\\ \dfrac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & t\leqslant s.\end{cases}$$

*Here $G(t,s)$ is called the Green function of boundary value problem* (2.1), *$\beta=\alpha-1$.*

**Proof** Let $v(t)=D^{\alpha-1}u(t)$. Then problem (2.1) is equivalent to the system of ordinary differential equations

$$\begin{cases}D^{\alpha-1}u(t)=v(t), \quad 0\leqslant t\leqslant 1,\ 2<\alpha\leqslant 3,\\ v'(t)=-f(t,u(t),v(t)),\\ u(0)=0, \quad u(1)=B, \quad v(0)=C.\end{cases} \tag{2.3}$$

Let $\beta=\alpha-1$, $1<\beta\leqslant 2$. Then problem (2.3) is equivalent to the system of ordinary differential equations

$$\begin{cases}D^{\beta}u(t)=v(t), \quad 0\leqslant t\leqslant 1,\ 2<\alpha\leqslant 3,\\ v'(t)=-f(t,u(t),v(t)),\\ u(0)=0, \quad u(1)=B, \quad v(0)=C.\end{cases}$$

We find that

$$v(t) = C - \int_0^t f(s, u(s), v(s))\mathrm{d}s,$$

$$u(t) = I^\beta v(t) + C_1 t^{\beta-1} + C_2 t^{\beta-2} = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s)\mathrm{d}s + C_1 t^{\beta-1} + C_2 t^{\beta-2}.$$

On the one hand, the boundary condition $u(0) = 0$ implies that $C_2 = 0$.

On the other hand, by applying boundary condition $u(1) = B$, we find that

$$B = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} v(s)\mathrm{d}s + C_1,$$

so it implies that

$$C_1 = B - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} v(s)\mathrm{d}s.$$

Consequently,

$$\begin{aligned}
v(t) &= C - \int_0^t f(s, u(s), v(s))\mathrm{d}s, \\
u(t) &= Bt^{\beta-1} - \Big[ \int_0^1 \frac{(1-s)^{\beta-1} t^{\beta-1}}{\Gamma(\beta)} v(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} v(s)\mathrm{d}s \Big] \\
&= Bt^{\beta-1} - \Big[ \int_0^t \frac{[t(1-s)]^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} v(s)\mathrm{d}s + \int_t^1 \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)} v(s)\mathrm{d}s \Big] \\
&= Bt^{\beta-1} - \int_0^1 G(t,s) v(s)\mathrm{d}s.
\end{aligned}$$

**Lemma 2.4**[5] *The function $G(t,s)$ defined by Lemma 2.3 satisfies:*

(1) $G(t,s) > 0, \quad t, s \in (0,1)$;

(2) *there exists a positive function $\gamma \in C(0,1)$ such that*

$$\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} G(t,s) \geqslant \gamma(s) \max_{0 \leqslant t \leqslant 1} G(t,s) = \gamma(s) G(s,s), \quad for\ 0 < s < 1.$$

# 3 Main Results and Proofs

Let $C[0,1]$ be the Banach space endowed with the max norm $\|u\| = \max\limits_{0 \leqslant t \leqslant 1} |u(t)|$ and $\eta = \max\{|B|, |C|\}$,

$$T_1(u,v)(t) = Bt^{\beta-1} - \int_0^1 G(t,s) v(s)\mathrm{d}s,$$

$$T_2(u,v)(t) = C - \int_0^t f(s, u(s), v(s))\mathrm{d}s,$$

$$(T(u,v)) = (T_1(u,v), T_2(u,v)).$$

Then, problem (2.2) is equivalent to the following equation

$$T(u,v)=(u,v),\quad (u,v)\in C[0,1]\times C[0,1]. \tag{3.1}$$

That is, every solution of (2.2) is also a fixed-point of (3.1).

**Theorem 3.1** *Suppose that* $f:[0,1]\times R\times R\to R$ *holds. If there exist* $d>0$ *and* $\frac{1}{2}\leqslant k\leqslant\frac{1}{2m}$*, with* $m=\int_0^1 G(s,s)\mathrm{d}s$ *such that*

$$\max\{|f(t,u,v)|:t\in[0,1],|u|\leqslant 2\eta+d,|v|\leqslant k(2\eta+d)\}\leqslant(2k-1)\eta+kd. \tag{3.2}$$

*Then, problem* (1.1) *has at least one solution* $u^*\in C[0,1]$ *satisfying* $\|u^*\|\leqslant 2\eta+d$ *and* $\|D^{\alpha-1}u^*\|\leqslant k(2\eta+d)$.

**Proof** Let $C[0,1]\times C[0,1]$ be the Banach space endowed with the norm $\|(u,v)\|=\max\{\|u\|,\frac{\|v\|}{k}\}$, $R=2\eta+d$, $V_R=\{(u,v)\in C[0,1]\times C[0,1]:\|(u,v)\|\leqslant R\}$. Then $V_R$ is a convex closed set in $C[0,1]\times C[0,1]$. If $(u,v)\in V_R$, then $\|u\|\leqslant R$, and $\|v\|\leqslant kR$. So $|u(t)|\leqslant R$, $|v(t)|\leqslant kR$, $0\leqslant t\leqslant 1$. By the condition (3.2) of Theorem 3.1, we obtain $|f(t,u,v)|\leqslant(2k-1)\eta+kd$, $0\leqslant t\leqslant 1$. Thus

$$\begin{aligned}\|T_1(u,v)\|&\leqslant\max_{0\leqslant t\leqslant 1}|Bt^{\beta-1}|+\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)|v(s)|\mathrm{d}s\\&\leqslant\eta+kR\int_0^1 G(s,s)\mathrm{d}s=(1+2km)\eta+kmd,\\\|T_2(u,v)\|&\leqslant\max_{0\leqslant t\leqslant 1}|C|+\max_{0\leqslant t\leqslant 1}\int_0^t|f(s,u(s),v(s))|\mathrm{d}s\\&\leqslant\eta+(2k-1)\eta+kd=2k\eta+kd.\end{aligned}$$

In view of the above, we see that

$$\begin{aligned}\|(T_1(u,v),T_2(u,v))\|&=\max\Big\{\|T_1(u,v)\|,\frac{1}{k}\|T_2(u,v)\|\Big\}\\&\leqslant\max\{(1+2km)\eta+kmd,2\eta+d\}=2\eta+d.\end{aligned}$$

Therefore, $T:V_R\to V_R$. Then, we can easily prove that $T:C[0,1]\times C[0,1]\to C[0,1]\times C[0,1]$ is completely continuous by Arzela-Ascoli theorem. Therefore, according to Leray-Schauder fixed-point theorem, the operator $T$ has a fixed point $(u^*,v^*)\in V_R$.

**Theorem 3.2** *Suppose that* $B\geqslant 0$*,* $C\leqslant 0$ *and* $f:[0,1]\times R_+\times R_-\to R_+$*. If there exist* $d>0$ *and* $\frac{1}{2}\leqslant k\leqslant\frac{1}{2m}$*, with* $m=\int_0^1 G(s,s)\mathrm{d}s$ *such that*

$$\max\{f(t,u,v):t\in[0,1],0\leqslant u\leqslant 2\eta+d,-k(2\eta+d)\leqslant v\leqslant 0\}\leqslant(2k-1)\eta+kd.$$

*Then, problem* (1.1) *has at least one solution* $u^*\in C[0,1]$ *satisfying* $\|u^*\|\leqslant 2\eta+d$ *and* $\|D^{\alpha-1}u^*\|\leqslant k(2\eta+d)$.

**Proof** Set

$$f_1(t,u,v)=\begin{cases} f(t,u,v), & (t,u,v)\in[0,1]\times R_+\times R_-,\\ f(t,u,0), & (t,u,v)\in[0,1]\times R_+\times R_+,\end{cases}$$

$$f_2(t,u,v)=\begin{cases} f_1(t,u,v), & (t,u,v)\in[0,1]\times R_+\times R,\\ f_1(t,0,v), & (t,u,v)\in[0,1]\times R_-\times R.\end{cases}$$

Obviously, $f_2:[0,1]\times R\times R\to R_+$ is continuous and

$$\begin{aligned}&\max\{|f_2(t,u,v)|:t\in[0,1],|u|\leqslant 2\eta+d,|v|\leqslant k(2\eta+d)\}\\ =&\max\{f(t,u,v):t\in[0,1],0\leqslant u\leqslant 2\eta+d,-k(2\eta+d)\leqslant v\leqslant 0\}\\ \leqslant&(2k-1)\eta+kd.\end{aligned}$$

Applying Theorem 3.1, the problem

$$\begin{cases} D^{\alpha}u(t)+f_2(t,u(t),D^{\alpha-1}u(t))=0, & 0\leqslant t\leqslant 1,\\ u(0)=0,\quad u(1)=B,\quad D^{\alpha-1}u(0)=C\end{cases}$$

has at least one solution $u^*\in C[0,1]$ satisfying $\|u^*\|\leqslant 2\eta+d$ and $\|D^{\alpha-1}u^*\|\leqslant k(2\eta+d)$.

If $C\leqslant 0$, then

$$D^{\alpha-1}u^*(t)=C-\int_0^t f_2(s,u^*(s),D^{\alpha-1}u^*(s))\mathrm{d}s\leqslant 0,\quad 0\leqslant t\leqslant 1. \tag{3.3}$$

Consider that

$$u^*(t)=Bt^{\beta-1}-\int_0^1 G(t,s)D^{\alpha-1}u^*(s)\mathrm{d}s\geqslant 0,\quad 0\leqslant t\leqslant 1. \tag{3.4}$$

From (3.3) and (3.4), we get

$$f_2(s,u^*(s),D^{\alpha-1}u^*(s))=f(s,u^*(s),D^{\alpha-1}u^*(s)).$$

This proves that $u^*$ is a solution of problem (1.1).

If $C<0$, then

$$u^*(t)=Bt^{\beta-1}-\int_0^1 G(t,s)D^{\alpha-1}u^*(s)\mathrm{d}s>0,\quad 0\leqslant t\leqslant 1.$$

If $B=C=0$ and $f(t,0,0)\neq 0$, $0\leqslant t\leqslant 1$, then the zero function is not a solution of problem (1.1).

## 4 Example

Consider the following boundary value problem

$$\begin{cases} D^{\frac52}u(t)+(k-\frac12)t^2u+\sin D^{\frac32}u(t)=0, & 0\leqslant t\leqslant 1,\\ u(0)=0,\quad u(1)=B,\quad D^{\frac32}u(0)=C.\end{cases} \tag{4.1}$$

Let $D^{\frac{3}{2}}u(t) = v(t)$, $R = 2\eta + d$, $d > 2$. If $(u, v) \in V_R$, then $\|u\| \leqslant R$, and $\|v\| \leqslant kR$. We have

$$|f(t,u,v)| = \left|\Big(k - \frac{1}{2}\Big)t^2u + \sin v(t)\right| \leqslant \Big(k - \frac{1}{2}\Big)(2\eta + d) + 1 \leqslant (2k-1)\eta + kd.$$

Therefore,

$$\max\{|f(t,u,v)| : t \in [0,1], |u| \leqslant 2\eta + d, |v| \leqslant k(2\eta + d)\} \leqslant (2k-1)\eta + kd.$$

Then by Theorem 3.1, the boundary value problem (4.1) has at least one solution $u^* \in C[0,1]$ satisfying $\|u^*\| \leqslant 2\eta + d$ and $\|D^{\frac{3}{2}}u^*\| \leqslant k(2\eta + d)$.

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