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A LYAPUNOV-TYPE INEQUALITY FOR A PERIODIC BOUNDARY VALUE PROBLEM OF A FRACTIONAL DIFFERENTIAL EQUATION*†

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Abstract

In this paper, we establish a Lyapunov-type inequality for fractional differential periodic boundary-value problems. As applications, a necessary condition is obtained to ensure the existence and uniqueness of nontrivial solutions to this problem.

Keywords fractional differential equation; periodic boundary condition; Lyapunov-type inequality; Mittag-Leffler functions

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1 Introduction

Derivatives and integrals of non-integer orders have a long history, and these operators have wide applications in theory and engineering technology, such as biophysics, bioengineering, finance, control theory, quantum mechanics, image and signal processing, viscoelasticity and sciences. For its important application, the existence and uniqueness of solutions to the fractional differential equations were investigated by many scholars. In [1] by applying the theory of Leray-Schauder degree, the existence of nontrivial solutions to the boundary value problems of fractional differential equations is considered under some conditions concerning the first eigenvalue corresponding to the relevant linear operator (see Theorem 2.1 of [1]). By using Banach contraction principle and Krasnoselskill fixed point theorem, the existence of solutions to the integral boundary value problems for the fractional differential equations was investigated in [2]. The Lyapunov inequality has been applied to very

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useful invarious problems related with differential equations, for example, see [3-5]. letting $q : [a, b] \to R$ be a continuous function, the Lyapunov inequality states that a necessary condition for the boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0 \end{cases}$$

to have nontrivial solutions is that

$$\int_a^b |q(s)| \mathrm{d}s > \frac{4}{b-a}.$$

Recently, the research of Lyapunov-type inequality for fraction boundary value problem has been increasingly concerned. The first work in this direction is due to Ferreira (see [6]), where the author derived a Lyapunov-type inequality for differential equations depending on the Riemann-Liouville fractional derivative, that is, for the boundary value problem

$$\begin{cases} {}_aD^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \le 2, \\ u(a) = u(b) = 0, \end{cases}$$

where ${}_aD^\alpha$ denotes the Riemann-Liouville fractional derivative of order α, and $q : [a, b] \to R$ is a real and continuous function. It was proved that the above problem has a nontrivial solution, then we have

$$\int_a^b |q(t)| \mathrm{d}t > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}.$$

Clearly, if let $\alpha = 2$ in the above inequality, then we obtain Lyapunov's standard inequality.

In [7], a Lyapunov-type inequality was obtain for the Caputo fractional boundary value problem

$$\begin{cases} {}_a^CD^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \le 2, \\ u(a) = u(b) = 0, \end{cases}$$

where ${}_a^CD^\alpha$ is the Caputo fractional derivative of order α, and $q : [a, b] \to R$ is a real and continuous function. In this work, Ferreira proved that if the above problem has a nontrivial solution, then

$$\int_a^b |q(t)| \mathrm{d}s > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

Similarly, if let $\alpha = 2$ in (6), then we obtain Lyapunov's classical inequality.

Meanwhile, in [8], Jleli and Samet considered the following fractional differential equation

$$ {}_a^C D^{\alpha} y(t) + q(t)y(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2, $$

with the two boundary conditions

$$ y(a) = 0 = y'(b) $$

or

$$ y'(a) = 0 = y(b). $$

They proved that if the above two boundary problem has a nontrivial solution, then

$$ \int_a^b (b-s)^{(\alpha-2)} |q(s)| \mathrm{d}s \ge \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)} $$

and

$$ \int_a^b (b-s)^{\alpha-1} |q(s)| \mathrm{d}s \ge \Gamma(\alpha). $$

In addition, in [9], Jleli and Samet considered the fractional differential equation under a Robin boundary condition

$$ \begin{cases} {}_a^C D^{\alpha} u(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2, \\ u(a) - u'(a) = u(b) + u'(b) = 0, \end{cases} $$

and proved that if the above problem has a nontrivial solution, then

$$ \int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)|q(s)| \mathrm{d}s \ge \frac{(b-a+2)\Gamma(\alpha)}{\max\{b-a+1, \frac{2-\alpha}{(\alpha-1)(b-a-1)}\}}. $$

Motivated by the above works, we consider in this paper a Riemann-Liouville fractional differential equation under periodic boundary conditions. More precisely, we consider the boundary value problem

$$ \begin{cases} D_{0^+}^{\alpha} u(t) + q(t)u(t) = 0, \quad t \in J := (0,1],\ 0 < \alpha < 1, \\ \lim\limits_{t \to 0^+} t^{1-\alpha} u(t) = u(1), \end{cases} \tag{1} $$

where $q : [a, b] \to R$ is a continuous function. We write (1) as an equivalent integral equation, then by using some properties of its Green function, we get a corresponding Lyapunov-type inequality, after that, we obtain a necessary condition for the periodic boundary condition has nontrivial continuous solution. Meanwhile, we get sufficient condition for periodic boundary conditions has an only trivial solution.

The organization of this paper is as follows. In Section 2, we introduce some preliminaries and recall some concepts. In Section 3, we devote to calculating the Green function and the correspond Lyapunov-type inequality and get a necessary condition for the periodic boundary value problem to have nontrivial solutions. Meanwhile, we get sufficient condition for periodic boundary conditions to have a unique trivial solution. In Section 4, an example is presented to illustrate the application of the obtained results.

2 Preliminaries

In this section, we introduce the definitions of the Riemann-Liouville fractional integral, the Riemann-Liouville derivative and Mittag-Leffler functions. For additional details, readers can refer to the references [10-13].

Definition 2.1 Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order α is defined by $({}_aI^0 f) \equiv f$ and

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad \alpha > 0,\ t \in [a, b],$$

where $\Gamma(\cdot)$ is Euler's gamma function, that is, $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ for $x > 0$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ is defined by $({}_a^C D^0 f) \equiv f$ and

$$(D_a^\alpha f)(t) = D^{\lceil\alpha\rceil} I_a^{\lceil\alpha\rceil-\alpha} f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \Big(\frac{\mathrm{d}}{\mathrm{d}x}^{\lceil\alpha\rceil}\Big) \int_a^x \frac{f(t)}{(x-t)^{\alpha-\lceil\alpha\rceil+1}}\mathrm{d}t,$$

where $\lceil\cdot\rceil$ is a ceiling function, that is, $\lceil x\rceil = \min\{z \in Z : z \geq x\}$.

Definition 2.3 Let $\alpha > 0,\ \beta > 0$, and the function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0,\ z \in C,$$

whenever the series converges is called the two-parameter Mittag-Leffler function with the parameters α, β.

Remark 2.1 It is evident that one-parameter Mittag-Leffler functions are defined in terms of their two-parameter counterparts via the relation $E_\alpha(z) = E_{\alpha,1}(z)$. For more details, we refer to [14].

Definition 2.4 $u = 0$ is called a trivial solution to problem (1). If u is a solution to problem (1), and u is not identically zero, then u is called a nontrivial solution.

3 Main Results

We begin by writing problem (1) in its equivalent integral form.

Lemma 3.1[13] *Let $f(x) = (x-a)^\beta$ for some $\beta > -1$ and $n > 0$, then*

$$D_a^n f(x) = D^{\lceil n\rceil} I_a^{\lceil n\rceil - n} f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\lceil n\rceil - n + \beta + 1)} D^{\lceil n\rceil}[(x-a)^{\lceil n\rceil - n + \beta}],$$

where $\lceil\cdot\rceil$ is a ceiling function, that is, $\lceil n\rceil = \min\{z \in Z : z \geq n\}$.

Lemma 3.2[15] *For any $x, y \in R$, when $x < 0 < y$, we have*

$$0 < E_{\alpha,\alpha}(x) < E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)} < E_{\alpha,\alpha}(y).$$

Let $C[0,1]$ be the set of all continuous functions defined on $[0,1]$. Define $\|u\| = \max\limits_{0\le t\le 1} |u(t)|$, for $u \in X$, then $(C[0,1], \|\cdot\|)$ is a Banach space.

For $0 < \alpha < 1,\ u \in C(0,1]$, define

$$t^{1-\alpha}u(t)|_{t=0} = \lim_{t\to 0^+} t^{1-\alpha}u(t), \quad C_{1-\alpha}[0,1] = \{u \in C(0,1] |\, t^{1-\alpha}u(t) \in C[0,1]\}.$$

Then

(1) $C_{1-\alpha}[0,1]$ endowed with the norm $\|u\|_\infty = \max\limits_{0\le t\le 1} |t^{1-\alpha}u(t)|$ is a Banach space.

(2) $C_{1-\alpha}[0,1] \subset C(0,1] \cap L^1[0,1]$, where $L^1[0,1]$ is the set of all Lebesgue integrable functions on $[0,1]$.

Lemma 3.3 *If $u \in C_{1-\alpha}[0,1]$ is a solution to the boundary value problem (1), then u satisfies the following integral equation*

$$u(t) = \int_0^1 G_{\lambda_0,\alpha}(t,s)(-\lambda_0 - q(s))u(s)\mathrm{d}s,$$

where $\lambda_0 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} < 0$ and

$$G_{\lambda_0,\alpha}(t,s) = \begin{cases} \dfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0 t^\alpha)E_{\alpha,\alpha}(\lambda_0(1-s)^\alpha)t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)} \\ +(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda_0(t-s))^\alpha, & 0 \le s \le t \le 1, \\ \dfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0 t^\alpha)E_{\alpha,\alpha}(\lambda_0(1-s)^\alpha)t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)}, & 0 \le t \le s \le 1. \end{cases}$$

Proof For $\lambda < 0$, in [15], J.J. Nieto verified that the following periodic boundary value problem of fractional differential equations

$$\begin{cases} D_{0^+}^\alpha u(t) - \lambda u(t) = f(t), \quad 0 < t < 1,\ 0 < \alpha < 1, \\ \lim\limits_{t\to 0^+} t^{1-\alpha}u(t) = u(1) \end{cases} \tag{2}$$

has solutions in the form

$$\begin{aligned} u(t) = {} & \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)t^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)f(s)\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)f(s)\mathrm{d}s. \end{aligned}$$

Periodic boundary value problems (1) is equivalent to the following problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) - \lambda_0 u(t) = (-\lambda_0 - q(t))u(t), & 0 < t < 1,\ 0 < \alpha < 1, \\ \lim\limits_{t\to 0^+} t^{1-\alpha}u(t) = u(1), \end{cases} \tag{3}$$

where $\lambda_0 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} < 0$. By Lemma 2.1 in [15], we can prove the conclusion.

Theorem 3.1 *If the boundary value problem* (1) *has a nontrivial solution in* $C_{1-\alpha}[0,1]$, *then*

$$\|q\| \geq \frac{\Gamma(2\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}{\Gamma(\alpha)} + \lambda_0,$$

where $\lambda_0 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)},\ \|q\| = \max\limits_{t\in[0,1]}|q(t)|$.

Proof Let $u \in C_{1-\alpha}[0,1]$ be a nontrivial solution to problem (1). According to Lemma 3.2, we have

$$\begin{aligned} |t^{1-\alpha}u(t)| &\leq \int_0^1 |G_{\lambda_0,\alpha}(t,s)| \cdot \left|\frac{t^{1-\alpha}}{s^{1-\alpha}}\right| \cdot |s^{1-\alpha}u(s)| \cdot |-\lambda_0 - q(s)|\,\mathrm{d}s \\ &\leq (|\lambda_0| + \|q\|) \int_0^1 |G_{\lambda_0,\alpha}(t,s)| \cdot \left|\frac{t^{1-\alpha}}{s^{1-\alpha}}\right| \cdot |s^{1-\alpha}u(s)|\mathrm{d}s, \end{aligned}$$

then

$$\|u\|_\infty \leq \| u \|_\infty (|\lambda_0| + \|q\|) \int_0^1 |G_{\lambda_0,\alpha}(t,s)| \cdot \left|\frac{t^{1-\alpha}}{s^{1-\alpha}}\right|\mathrm{d}s,$$

that is

$$(|\lambda_0| + \|q\|) \int_0^1 |G_{\lambda_0,\alpha}(t,s)| \cdot \left|\frac{t^{1-\alpha}}{s^{1-\alpha}}\right|\mathrm{d}s \geq 1.$$

(i) If $0 \leq t \leq s \leq 1$, according to Lemma 3.2, we have

$$(|\lambda_0| + \|q\|) \int_0^1 \frac{(1-s)^{\alpha-1}s^{\alpha-1}}{\Gamma(\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}\mathrm{d}s \geq 1,$$

$$\frac{(|\lambda_0| + \|q\|)B(\alpha,\alpha)}{\Gamma(\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|} \geq 1 \quad\Rightarrow\quad \frac{(|\lambda_0| + \|q\|)\Gamma(\alpha)}{\Gamma(2\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|} \geq 1,$$

that is,

$$|\lambda_0| + \|q\| \geq \frac{\Gamma(2\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}{\Gamma(\alpha)},$$

where $B(\alpha,\alpha)$ is a Euler's Beta function. that is let $\alpha > 0,\ \beta > 0$, then

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(\alpha+\beta)}.$$

(ii) If $0 \le s \le t \le 1$, we have

$$(|\lambda_0| + \|q\|) \int_0^1 |G_{\lambda_0,\alpha}(t,s)| \cdot \Big|\frac{t^{1-\alpha}}{s^{1-\alpha}}\Big| \mathrm{d}s \ge 1,$$

that is

$$(|\lambda_0| + \|q\|) \int_0^t \Big[\frac{(1-s)^{\alpha-1}s^{\alpha-1}}{\Gamma(\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|} + \Big|\frac{t^{1-\alpha}(t-s)^{\alpha-1}s^{\alpha-1}}{\Gamma(\alpha)}\Big|\Big] \mathrm{d}s \ge 1$$

or equivalent

$$(|\lambda_0| + \|q\|)\Big[\frac{\Gamma(\alpha)}{\Gamma(2\alpha)|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|} + \int_0^t \Big|\frac{t^{1-\alpha}(t-s)^{\alpha-1}s^{\alpha-1}}{\Gamma(\alpha)}\Big|\Big] \mathrm{d}s \ge 1. \qquad (4)$$

Let $u = \frac{s}{t}$, then

$$\begin{aligned}\int_0^t t^{1-\alpha}(t-s)^{\alpha-1}s^{\alpha-1}\mathrm{d}s &= \int_0^t (t-s)^{\alpha-1}\Big(\frac{s}{t}\Big)^{\alpha-1}\mathrm{d}s = \int_0^t \Big(1-\frac{s}{t}\Big)^{\alpha-1}\Big(\frac{s}{t}\Big)^{\alpha-1)} t^{\alpha-1}\mathrm{d}s \\ &= \int_0^1 t^{\alpha}(1-u)^{\alpha-1}u^{\alpha-1}\mathrm{d}u = t^{\alpha}B(\alpha,\alpha).\end{aligned}$$

This leads to

$$(|\lambda_0| + \|q\|)\Big[\frac{\Gamma(\alpha)}{\Gamma(2\alpha)|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|} + \frac{t^{\alpha}B(\alpha,\alpha)}{\Gamma(\alpha)}\Big] \ge 1.$$

Because $0 < t \le 1,\ 0 < \alpha \le 1$,

$$(|\lambda_0| + \|q\|)\Big[\frac{\Gamma(\alpha)}{\Gamma(2\alpha)|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|} + \frac{B(\alpha,\alpha)}{\Gamma(\alpha)}\Big] \ge 1,$$

that is

$$(|\lambda_0| + \|q\|)\Big[\frac{\Gamma(\alpha)(1+|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|)}{\Gamma(2\alpha)|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}\Big] \ge 1,$$

or equivalent

$$|\lambda_0| + \|q\| \ge \frac{\Gamma(2\alpha)|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}{\Gamma(\alpha)(1+|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|)}.$$

Therefore, combing (i) with (ii) completes the proof.

Corollary 3.1 *If*

$$\|q\| < \frac{\Gamma(2\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}{\Gamma(\alpha)} + \lambda_0,$$

where $\lambda_0 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}$, *then* (1) *has a unique trivial solution in* $C_{1-\alpha}[0,1]$.

Corollary 3.2 *If*

$$\|q\| < \frac{\Gamma(2\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}{\Gamma(\alpha)} + \lambda_0,$$

where $\lambda_0 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}$, *the problem*

$$\begin{cases} D_{0^+}^{\alpha}u(t) + q(t)u(t) = f(t), \quad t \in J := (0,1],\ 0<\alpha<1, \\ \lim\limits_{t\to 0^+} t^{1-\alpha}u(t) = u(1) \end{cases}$$

has a unique solution in $C_{1-\alpha}[0,1]$ *for each* $f \in C[0,1]$.

4 Example

Example 4.1 Consider the following problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) + \frac{1}{3}t^{\alpha}u(t) = 0, \quad 0<t\le 1,\ 0<\alpha<1, \\ \lim\limits_{t\to 0^+} t^{1-\alpha}u(t) = u(1). \end{cases} \tag{5}$$

We assert that problem (5) has a unique trivial solution in $C_{1-\alpha}[0,1]$.

In fact, from Corollary 3.1, we only need to prove that

$$H(\lambda_0,\alpha) = \frac{\Gamma(2\alpha)\,|1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_0)|}{\Gamma(\alpha)} + \lambda_0 - \|q\| > 0. \tag{6}$$

In [16], C. Zeng and Y. Chen established a global Pade approximation of the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ with $x \in [0,+\infty)$. In [16] Table 1, when $x \in [0,+\infty)$, the parameters $0<\alpha=\beta<1$, the global Pade approximation of the function $E_{\alpha,\alpha}(-x)$ is

$$E_{\alpha,\alpha}(-x) \approx \frac{\frac{1}{\Gamma(\alpha)}}{1 + \frac{2\Gamma(1-\alpha)^2}{\Gamma(1+\alpha)\Gamma(1-2\alpha)}x + \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}x^2}. \tag{7}$$

From (6) and (7), since $\lambda_0 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} < 0$,

$$H(\lambda_0,\alpha) \approx \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\left|1 - \frac{1}{1 - \frac{2\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}}\right| + \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - \|q\|.$$

Noting that

$$\|q\| = \max_{0<t\le 1}\left|\frac{1}{3}t^{\alpha}\right| = \frac{1}{3}, \quad \alpha = \frac{3}{4},$$

according to the Gamma function table, we obtain $\Gamma(-0.5) \approx -3.65$, $\Gamma(0.25) \approx 3.63$, $\Gamma(0,5) \approx 1.78$, $\Gamma(0.75) \approx 1.23$.

In consequence,

$$H(\lambda_0,\alpha)\approx\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\left|1-\frac{1}{1-\frac{2\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}}\right|+\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}-\|q\|$$
$$\approx\frac{1.5}{0.75}\left|1-\frac{1}{3.34}\right|-\frac{1.23}{3.63}-\frac{1}{3}\approx 0.66-\frac{1}{3}\approx 0.33>0.$$

From corollary 3.1, problem (5) has a unique trivial solution.

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