Ann. of Appl. Math.
33:3(2017), 221-238

THE APPLICATION OF RANDOM MATRICES IN MATHEMATICAL PHYSICS*

Boling Guo
(*Institute of Applied Physics and Computational Math., Beijing 100088, PR China*)
Fangfang Li†
(*School of Math. and Statistics, Zhoukou Normal University, Zhoukou 466001, Henan, PR China*)

Abstract

In this paper, we introduce the application of random matrices in mathematical physics including Riemann-Hilbert problem, nuclear physics, big data, image processing, compressed sensing and so on. We start with the Riemann-Hilbert problem and state the relation between the probability distribution of nontrivial zeros and the eigenvalues of the random matrices. Through the random matrices theory, we derive the distribution of Neutron width and probability density between energy levels. In addition, the application of random matrices in quantum chromo dynamics and two dimensional Einstein gravity equations is also present in this paper.

Keywords random matrices; Riemann Hypothesis; Riemann-Hilbert problem; nuclear physics

2000 Mathematics Subject Classification 15B52

1 Introduction

A random matrix, that is, every element in the matrix is a random variable, it was first reported in mathematical statistics in 1930, but it didn't get people's attention. The statistical asymptotic properties of its eigenvalues is seldom understood. It was not until 1950 that physicists discovered that the statistical properties of slow neutron resonances were related to random matrices in nuclear physics. In 1951, Wigner [1] pointed out that the local statistical properties of nuclear levels were related to the eigenvalues of random matrices. After that, it is closely related to the quantum chromodynamics, the two-dimensional quantum Einstein gravity equation, the electronic heat capacity of conventional superconducting nano-particles, the

*Manuscript received May 25, 2017
†Corresponding author. E-mail: lifang_868@126.com

magnetic susceptibility of conventional superconducting metal nano-particles and superconductivity. In 1962, mathematical physicist F.J. Dyson [2] pointed out an important conjecture: The two point correlation function for the random matrix and the two point correlation function for the zero point of Riemann Zeta function are equivalent. In 1973, H.L. Montgomery gave the mathematical proof of this. Computational mathematician Monte Caro performed a large number of calculations to verify that the distribution of a large number of zeros of $\zeta(s)$ is consistent with the Riemann Hypothesis.

As we known, Riemann proposed the famous Riemann Hypothesis in 1858: The Riemann Zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}\, s > 1,$$

where p labels primes and then by analytic continuation to the rest of the complex plane. It has a single simple pole at $s = 1$, zeros at $s = -2, -4, -6, \cdots$, and infinitely many zeros, called the non-trivial zeros, in $0 < \mathrm{Re}\, s < 1$. The Riemann Hypothesis states that all of the nontrivial zeros lie on the critical line of $\mathrm{Re}\, s = 1/2$. At the Second World Congress of Mathematicians, Hilbert presented twenty-three mathematical problems, one of which was Riemann Hypothesis. To this day, the proof of Riemann Hypothesis has become the most difficult and powerful yet unsolved problem at present, because many important mathematical results could be established follows from the proof of Riemann Hypothesis. Therefore, the random matrix becomes an important tool in proving the Riemann Hypothesis, the computation are used to verify that around 10^9 zeros of ζ function are all on $\mathrm{Re}\, s = 1/2$. In addition, the theory of random matrices also plays an important role in the numerical computation of large data and the robustness of perceptual compression.

2 Orthogonal Polynomial and Riemann-Hilbert Problem

Definition 2.1 Suppose that $w(x)$ is a nonnegative and integrable function on (a, b), where (a, b) is unbounded, then we ask the moments

$$\mu_n = \int_a^b x^n w(x) \mathrm{d}x, \quad n = 0, 1, 2, \cdots \tag{2.1}$$

are finite. If there exists a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ such that

$$\int_a^b P_m(x) P_n(x) w(x) \mathrm{d}x = K_n \delta_{mn}, \quad K_n \neq 0, \tag{2.2}$$

then $\{P_n(x)\}$ is called an orthogonal polynomial sequence with respect to the weight function $w(x)$ on (a, b).

2.1 The properties of orthogonal polynomial

Denote

$$\mathcal{L}[f] = \int_a^b f(x)w(x)\mathrm{d}x.$$

(1) Suppose that $\{P_n(x)\}$ is an orthogonal polynomial (OPS), for any n order polynomial, we have

$$\pi(x) = \sum_{k=0}^{n} c_k P_k(x), \quad c_k = \frac{\mathcal{L}[\pi(x)P_k(x)]}{\mathcal{L}[P_k^2(x)]}. \tag{2.3}$$

(2) Suppose that $\{P_n(x)\} \in$ OPS, and there exist $\alpha_n, \beta_n, \lambda_n$ such that

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \lambda_n P_{n-1}(x), \tag{2.4}$$

where $P_{-1}(x) = 0$,

$$\alpha_n = \frac{K_n}{K_{n+1}}, \quad \beta_n = \frac{\mathcal{L}[xP_n^2(x)]}{\mathcal{L}[P_n^2(x)]}, \quad \lambda_n = \frac{K_{n-1}}{K_n}\frac{\mathcal{L}[P_n^2(x)]}{\mathcal{L}[P_{n-1}^2(x)]},$$

K_n is the leading coefficient of $P_n(x)$, with $P_n(x) = K_n x^n + \cdots$, $K_n \neq 0$.

• monic OPS, $K_n = 1$,

$$\alpha_n = 1, \quad \beta_n = \frac{\mathcal{L}[xP_n^2(x)]}{\mathcal{L}[P_n^2(x)]}, \quad \lambda_n = \frac{\mathcal{L}[P_n^2]}{\mathcal{L}[P_{n-1}^2]},$$

$$xP_n = P_{n+1} + \beta_n P_n + \lambda_n p_{n-1}.$$

• normalized OPS,

$$\int_a^b w(x)P_n P_m(x)\mathrm{d}x = \delta_{mn},$$

then

$$xP_n = a_n P_{n+1} + \beta_n P_n + a_{n-1}p_{n-1}, \quad a_n = \frac{K_n}{K_{n+1}}, \quad \beta_n = \mathcal{L}[xP_n^2(x)].$$

(3) λ is zero of $P_n(x)$ if and only if λ is the eigenvalue of Jacobi matrix T_{n-1}.

n zeros of $P_n(x)$ are simple and real; the zeros of $P_n(x)$ and $P_{n+1}(x)$ are interlace, where

$$T_{n-1} = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ b_0 & a_1 & b_1 & \ddots & \vdots \\ 0 & b_1 & a_2 & b_2 & 0 \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & b_{n-2} & a_{n-1} \end{pmatrix}.$$

2.2 Important relationship of OPS and RHP

Theorem 2.1[3] *Suppose that $x^j\omega(x)\in H^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $j=0,1,\cdots$, for RHP, then the followings hold:*

(i) *Y is analytic in $\mathbb{C}\setminus\mathbb{R}$,*

(ii) $Y_+(z)=Y_-(z)\begin{pmatrix}1 & \omega(z)\\ 0 & 1\end{pmatrix}$, $z\in\mathbb{R}$,

(iii) $Y(z)=(I+O(z^{-1}))\begin{pmatrix}z^{-n} & 0\\ 0 & z^{-n}\end{pmatrix}$, $z\to\infty$, *the RHP has a unique solution*

$$Y=\begin{pmatrix}\pi_n(z) & \dfrac{1}{2\pi i}\displaystyle\int\frac{\pi_n(s)\omega(s)}{s-z}\mathrm{d}s\\ r_{n-1}\pi_{n-1}(z) & \dfrac{r_{n-1}}{2\pi i}\displaystyle\int\frac{\pi_{n-1}(s)\omega(s)}{s-z}\mathrm{d}s\end{pmatrix},$$

where

$$r_{n-1}=\frac{-2\pi i\Delta_{n-2}}{\Delta_{n-1}},\quad \pi_n(s)=z^n+a_{n,n-1}z^{n-1}+\cdots.$$

Proof The uniqueness follows from Liouwille theorem. (iii) follows from Plemelj formula and the properties of Beals-Coifman operator.

3 The Random Matrices Ensemble

The ensemble is a set of a large number of identical systems satisfying certain macroscopic constraints, these systems have certain statistical distribution in microscopic state.

3.1 Gauss ensemble

Suppose that x is a random variable which satisfies Gauss distribution

$$P(x)\mathrm{d}x=c\mathrm{e}^{-x^2}\mathrm{d}x,\quad x\in R,$$

where

$$c=\Big(\int_{-\infty}^{\infty}\mathrm{e}^{-x^2}\mathrm{d}x\Big)^{-1}=\frac{1}{\sqrt{\pi}}.$$

If the random variable x satisfies probability distribution

$$P(x)\mathrm{d}x=c\mathrm{e}^{-Q(x)}\mathrm{d}x,$$

where $x\to\pm\infty$, $Q(x)\to+\infty$ is sufficiently fast. For example, take $Q(x)=r_{2n}x^{2n}+\cdots+r_0$, $r_{2n}>0$.

The random variable x can be extended to an $n\times n$ random matrix M with probability distribution is

$$P(M)\mathrm{d}M=c\mathrm{e}^{-tr(Q(M))}\mathrm{d}M,$$

where $\mathrm{d}M$ is Lebesgue measure, $Q(x)\to+\infty$, $x\to\infty$, $Q(x)=M^2$, then it is Gauss ensemble.

3.2 Unitary ensemble

$$\mathcal{M} = \{M|\, M^* = M\}, \quad M = (M_{jk}), \quad M_{jk} = M_{jk}^R + M_{jk}^I,$$

and its probability distribution is given by

$$P(M)\mathrm{d}M = C_n \mathrm{e}^{-tr(Q(M))}\mathrm{d}M = C\mathrm{e}^{-tr(Q(M))}\prod_{i=1}^{n}\mathrm{d}M_{ii}\prod_{1\le j<k\le n}\mathrm{d}M_{jk}^R\mathrm{d}M_{jk}^I,$$

$$M = \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix}, \quad Q(M) = M^2,$$

and in U-transform

$$M \to \widetilde{M} = UMU^*, \quad UU^* = I,$$

it remains unchanged, namely $P(M)\mathrm{d}M = P(\widetilde{M})\mathrm{d}\widetilde{M}$.

Take $Q(M) = M^2$, U ensemble is Gauss U ensemble (GUE).

3.3 Orthogonal ensemble

$$\mathcal{M} = \{M = M^{\mathrm{T}}\}, \quad M = (M_{jk})_{n\times n},$$

where probability distribution is

$$P(M)\mathrm{d}M = c\mathrm{e}^{-tr(Q(M))}\mathrm{d}M = c\mathrm{e}^{-tr(Q(M))}\prod_{1\le j\le k\le n}\mathrm{d}M_{jk}.$$

In the following orthogonal transformation

$$M \to \widetilde{M} = UMU^{\mathrm{T}}, \quad UU^{\mathrm{T}} = I,$$

it remains unchanged, namely, $P(M)\mathrm{d}M = P(\widetilde{M})\mathrm{d}\widetilde{M}$.

If $Q(M) = M^2$, we call OE as GOE.

3.4 Sympletic ensemble

$$\mathcal{M} = \{M = M^* = JM^{\mathrm{T}}J^{\mathrm{T}}\}, \quad J = \begin{pmatrix} \sigma & & \\ & \ddots & \\ & & \sigma \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$M = (M_{jk})_{n\times n}, \quad M_{jk} = \begin{pmatrix} \alpha_{ij} + i\beta_{ij} & \gamma_{ik} + i\delta_{jk} \\ -(\gamma_{jk} - i\delta_{jk}) & \alpha_{jk} - i\beta_{jk} \end{pmatrix},$$

where probability distribution is

$$P(M)\mathrm{d}M = C\mathrm{e}^{-trQ(M)}\prod_{k=1}^{n}\mathrm{d}\alpha_{kk}\prod_{1\le j<k\le n}\mathrm{d}\alpha_{jk}\mathrm{d}\beta_{jk}\mathrm{d}\gamma_{jk}\mathrm{d}\delta_{jk}.$$

In U symplectic transformation

$$M \to \widetilde{M} = UMU^*, \quad UU^* = I, \quad UJU^{\mathrm{T}} = J,$$

it remains unchanged. $Q(M) = M^2$ is known as GSU.

4 The Eigenvalues of the Random Matrices

The probability of random matrices M satisfying A is

$$\int_A P(M)\mathrm{d}M = c\int_A \mathrm{e}^{trQ(M)}\mathrm{d}M.$$

For one of UE, OE and SE, M can be diagonalized,

$$\begin{cases} M = U\Lambda U^* & (UE, SE), \\ M = U\Lambda U^{\mathrm{T}} & (OE), \end{cases}$$

where Λ is a diagonal matrix composed of eigenvalues.

Take the transformation $M \to (\Lambda, U)$, then

$$C\int_A P(M)\mathrm{d}M = C\int_{f(A)} \mathrm{e}^{-tr(Q(M))}\Big|\det\Big(\frac{\partial M}{\partial(\Lambda U)}\Big)\Big|\mathrm{d}\Lambda\mathrm{d}U.$$

We can compute Jacobi $\big|\det(\frac{\partial M}{\partial(\Lambda U)})\big|$ as follows:

(i) Two dimension case

Suppose that $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, the eigenvalues are λ_1, λ_2, then there exists an orthogonal matrix

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

such that

$$M = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^{\mathrm{T}}.$$

Therefore,

$$\begin{cases} a = \lambda_1\cos^2\theta + \lambda_2\sin^2\theta, \\ b = (\lambda_1 - \lambda_2)\cos\theta\sin\theta, \\ c = \lambda_1\sin^2\theta + \lambda_2\cos^2\theta, \end{cases}$$

$$\begin{aligned} \Big|\det\Big(\frac{\partial M}{\partial(\Lambda U)}\Big)\Big| &= \Big|\det\Big(\frac{\partial(abc)}{\partial(\lambda_1\lambda_2, \theta)}\Big)\Big| \\ &= |\lambda_1 - \lambda_2|\det\begin{pmatrix} \cos^2\theta & \sin 2\theta & -\sin 2\theta \\ \frac{1}{2}\sin^2\theta & -\frac{1}{2}\sin 2\theta & \cos 2\theta \\ \cos^2\theta & \cos^2\theta & \sin 2\theta \end{pmatrix} \\ &= |\lambda_1 - \lambda_2| g_1(\theta). \end{aligned}$$

For the second order Hermite,

$$\Big|\det\Big(\frac{\partial M}{\partial(\Lambda U)}\Big)\Big| = (\lambda_1 - \lambda_2)^2 g_2(U).$$

For the forth order self-dual Hermite,

$$\left|\det\Big(\frac{\partial M}{\partial(\Lambda U)}\Big)\right| = (\lambda_1-\lambda_2)^4 g_3(U).$$

(ii) Arbitrary dimensions case

$$\left|\det\Big(\frac{\partial M}{\partial(\Lambda U)}\Big)\right| = \prod_{i<j}(\lambda_i-\lambda_j)^\beta g_3(U),$$

$\beta = 1, 2, 4$ correspond to OE, UE, SE respectively.

$$\begin{aligned} C\int_A P(M)\mathrm{d}M &= C\int_{f(A)} \mathrm{e}^{-trQ(\Lambda)}\prod(\lambda_i-\lambda_j)^\beta g(U)\mathrm{d}\Lambda\mathrm{d}U \\ &= \widetilde{C}\int_B \mathrm{e}^{-trQ(\Lambda)}\prod(\lambda_i-\lambda_j)^\beta \mathrm{d}\Lambda, \end{aligned}$$

$P(\Lambda) = \widetilde{C}\mathrm{e}^{-trQ(\Lambda)}\prod(\lambda_i-\lambda_j)^\beta$ is known as density function of eigenvalues.

5 The Correlation Kernel Function

The random matrix $Q(x) = x^2,\ Q(M) = M^2$, the probability density is

$$\begin{aligned} P(x_1,\cdots,x_n) &= \widetilde{C}_n \mathrm{e}^{-(x_1^2+\cdots+x_n^2)}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}^2 \\ &= \widetilde{C}\mathrm{e}^{-(x_1^2+\cdots+x_n^2)^2}\begin{vmatrix} \widehat{H}_0(x_1) & \cdots & \widehat{H}_0(x_n) \\ \widehat{H}_{n-1}(x_1) & \cdots & \widehat{H}_{n-1}(x_n)\end{vmatrix}^2 \\ &= \widetilde{C}\det(K(x_i,x_j))_{i,j=1}^n, \end{aligned}$$

where

$$K(x,y) = \exp\Big(-\frac{x^2+y^2}{2}\Big)\sum_{j=0}^{n-1}\widehat{H}_j(x)\widehat{H}_j(y)$$

is known as correlation kernel function.

$\widehat{H}_j$ is normalized Hermite OPS,

$$\int \widehat{H}_m(x)\widehat{H}_n(x)\mathrm{e}^{-x^2}\mathrm{d}x = \delta_{mn},\quad \widehat{H}_m = \frac{1}{2^{\frac{m}{2}}\pi^{\frac{1}{4}}\sqrt{m!}}H_m(x),$$

$H_m(x) = (-1)^m\mathrm{e}^{x^2}\dfrac{\mathrm{d}^m}{\mathrm{d}x^m}(\mathrm{e}^{-x^2}),\quad \widehat{H}_m = \dfrac{1}{2^{-\frac{n}{2}}\pi^{\frac{1}{4}}\sqrt{m!}}\pi_m(x) = k_m\pi_m(x)$, monic *Hermite*.

$$\int K_n(x,z)K_n(x,y)\mathrm{d}z = K_n(x,y) = C\mathrm{e}^{x_1^2+\cdots+x_n^2},$$

$$\int_{-\infty}^{\infty} K_n(x,x)\mathrm{d}x = n,$$

these imply the probability density

$$P(x_1,\cdots,x_n) = \frac{1}{n!}\det(K(x_i,x_j))_{i,j=1}^n.$$

Define $Fredholm$ determinant

$$\det(I-\lambda K_n) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_{(a,b)^j}\det(K_n(x_k,x_l))_{k,l=1}^j \mathrm{d}x_1\cdots\mathrm{d}x_j,$$

then the probability of (a,b) which has k eigenvalues is

$$P(x_{j1},\cdots,x_{jk}\in(a,b)) = (-1)^k\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\det(I-\lambda K_n)\Big|_{\lambda=1}.$$

5.1 The universality of correlation kernel

For GUE,

$$\begin{aligned}K_n(x,y) &= \exp\Big(-\frac{x^2+y^2}{2}\Big)\sum_{j=0}^{n-1}\widehat{H}_j(x)\widehat{H}_j(y)\\ &= \exp\Big(-\frac{x^2+y^2}{2}\Big)\frac{K_{n-1}}{K_n}\frac{\widehat{H}_n(x)\widehat{H}_{n-1}(y)-\widehat{H}_{n-1}(x)\widehat{H}_n(y)}{x-y}\\ &= \exp\Big(-\frac{x^2+y^2}{2}\Big)\frac{1}{2^n(n-1)!\sqrt{\pi}}\frac{H_n(x)H_{n-1}(y)-H_{n-1}(x)H_n(y)}{x-y},\end{aligned}$$

$$\exp\Big(-\frac{x^2}{2}\Big)H_n(x)=\frac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1)}\Big(\cos\Big(\sqrt{2n+1}x-\frac{n\pi}{2}\Big)+O(\frac{1}{\sqrt{n}})\Big),\ n \text{ is an even number},$$

$$\exp\Big(-\frac{x^2}{2}\Big)H_n(x)=\frac{\Gamma(n+2)}{\Gamma(\frac{n}{2}+\frac{3}{2})}\Big(\cos\Big(\sqrt{2n+1}x-\frac{n\pi}{2}\Big)+O(\frac{1}{\sqrt{n}})\Big),\ n \text{ is an odd number}.$$

Therefore,

$$\frac{1}{\sqrt{2n}}K_n\Big(\frac{x}{\sqrt{2n}},\frac{y}{\sqrt{2n}}\Big)\to\frac{\sin(x-y)}{\pi(x-y)},$$

$$\begin{aligned}&\frac{1}{\sqrt[n]{2n}}K_n\Big(\sqrt{2n}+\frac{x}{\sqrt[n]{2n}},\sqrt{2n}+\frac{y}{\sqrt[n]{2n}}\Big)\\ &\to K_{A_i}=\frac{A_i(x)A_i'(y)-A_i(y)A_i'(x)}{\pi(x-y)},\end{aligned}$$

where

$$A_i''(x)+xA_i(x)=0,\quad A_i(x)=\frac{1}{\pi}\int_0^{+\infty}\cos\Big(\frac{1}{3}t^2+xt\Big)\mathrm{d}t,$$

$$\sqrt{\frac{2}{n}}K_n(\sqrt{2n+1}x,\sqrt{2n+1}x)\to\begin{cases}\frac{2}{\pi}\sqrt{1-x^2}, & |x|\le 1,\\ 0, & |x|>1.\end{cases}$$

6 The Application of Random Matrices in Mathematics

6.1 The application in Riemann Hypothesis

In 1858, Riemann proposed the famous Riemann Conjecture or Riemann Hypothesis: We know that the Riemann Zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\Big(1-\frac{1}{p^s}\Big)^{-1}. \tag{6.1}$$

For $\mathrm{Re}\, s>1$, p labels primes, and then by analytic continuation to the rest of the complex plane. It exists simple pole ($s=1$); $s=-2,-4,-6$, there's an infinite number of trivial zeros,

$$\zeta(1-s)=2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s) \tag{6.2}$$

implies it exists nontrivial zero in $0<\mathrm{Re}\, s<1$. Riemann Hypothesis states that all of these non-trivial zeros lie on the the critical line $\mathrm{Re}\, s=1/2$, that is, $\zeta(\frac{1}{2}+it)=0$ has nontrivial zeros only when $t=t_n\in\mathbb{R}$ [4].

We assume the Riemann Hypothesis to be true. We would like to know what the probability distribution of t_n is and how it relates to the eigenvalues of the random matrices.

Suppose that the average density of nontrivial zeros increases with the $t\log t$ function.

The unfolded zeros

$$w_n=\frac{t_n}{2\pi}\log\frac{t_n}{2\pi} \tag{6.3}$$

satisfy

$$\lim_{w\to\infty}\frac{1}{w}\#\{w_n\in[0,w]\}=1, \tag{6.4}$$

that is, the mean of $w_{n+1}-w_n$ is 1. The question is how the presentation of zeros are distributed?

The connection between the pair correlation of the Riemann zeros, as measured by

$$F_\zeta(\alpha,\beta)=\lim_{w\to\infty}\frac{1}{w}\#\{w_n,w_m\in[0,w],\ \alpha\le w_n-w_m<\beta\}.$$

For an $N \times N$ random U matrix, $A \in U(N)$. Set the eigenvalue of A as $\mathrm{e}^{i\theta_n}$, $1 \le n \le N$, $\theta_n \in R$. The eigenphases θ_n have mean density $N/2\pi$.

Set

$$\phi_n = \theta_n \frac{N}{2\pi},$$

which has unit mean density (that is $\phi_n \in [0, N)$). Define

$$F(\alpha, \beta; A) = \frac{1}{N} \#\{\phi_n, \phi_m : \alpha \le \phi_n - \phi_m < \beta\}.$$

The key step is to give the average of $F(\alpha, \beta; A)$ over A, chosen uniformly with respect to Haar measure on $U(N)$. Let

$$F_U(\alpha, \beta; N) = \int_{U(N)} F(\alpha, \beta; A) \mathrm{d}A.$$

In 1963, Dyson proved that

$$F_U(\alpha, \beta) = \lim_{N \to \infty} F_U(\alpha, \beta; N)$$

exists, and takes the form

$$F_U(\alpha, \beta) = \int_\alpha^\beta \left[1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right] \mathrm{d}x,$$

where $\delta(x)$ is Dirac function. For the pair correlation function $F_\zeta(\alpha, \beta)$ of the Riemann zeros, ask

$$F_\zeta(\alpha, \beta) = F_U(\alpha, \beta)?$$

In 1973, Montgomery studied $F_\zeta(\alpha, \beta)$ of the Riemann zeros related to the two point correlation function $R_{2,\zeta}$

$$F_\zeta(\alpha, \beta) = \int_\alpha^\beta \big(R_{2,\zeta}(x) + \delta(x)\big) \mathrm{d}x,$$

where

$$\begin{aligned} R_{2,\zeta}(f, w) &= \frac{1}{w} \sum_{j \ne k, w_j, w_k \in w} f(w_j - w_k) \\ &= \int_{-\infty}^{\infty} f(x) \frac{1}{w} \sum_{j \ne k, w_j, w_k \in w} \delta(x - w_j + w_k) \mathrm{d}x. \end{aligned}$$

Montgomery obtained the following theorem:

Theorem 6.1[5] *The text function $f(x)$ satisfies*

$$\widehat{f}(\tau) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{2\pi i x \tau} \mathrm{d}\tau.$$

It has support set $(-1, 1)$, then the following limitation exists

$$\lim_{w \to \infty} R_{2,\zeta}(f, w) = \int_{-\infty}^{\infty} f(x) R_2(x) \mathrm{d}x,$$

where

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

Corollary 6.1[5] *$F_\zeta(\alpha, \beta)$ is defined as follows*

$$F_\zeta(\alpha, \beta) = \int_{\alpha}^{\beta} [R_{2,\zeta}(x) + \delta(x)] \mathrm{d}x,$$

then

$$R_{2,\zeta}(x) = R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

As before, define

$$F(\alpha, \beta; A, N) = \frac{1}{N} \#\{\phi_n, \phi_m : \alpha \leq \phi_n - \phi_m < \beta\},$$
$$F_U(\alpha, \beta, N) = \int_{U(N)} F(\alpha, \beta; A, N) \mathrm{d}A,$$

where $\mathrm{d}A$ is Haar measure. A is an $N \times N$ U matrix, $A \in U(N)$, the eigenvalues of A are $\mathrm{e}^{i\theta_n}$, $1 \leq n \leq N$; $\theta_n \in R$, $\phi_n = \theta_n \frac{N}{2\pi}$.

Theorem 6.2[2] *Limiting distribution*

$$F_U(\alpha, \beta) = \lim_{N \to \infty} F_U(\alpha, \beta, N)$$

exists, and has

$$F_U(\alpha, \beta) = \int_{\alpha}^{\beta} [R_{2,U}(x) + \delta(x)] \mathrm{d}x,$$

where $\delta(x)$ is Dirac function,

$$R_{2,U}(x) = R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

Therefore

$$F_\zeta(\alpha, \beta) = F_U(\alpha, \beta).$$

Figures 6.1-6.5 demonstrate this result.

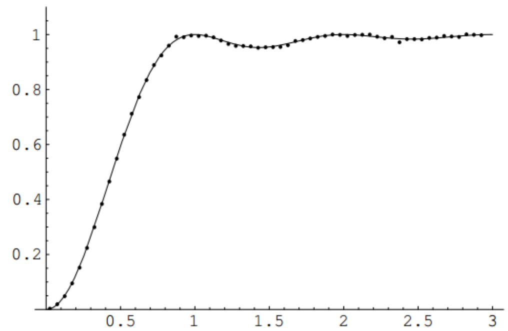

Figure 6.1

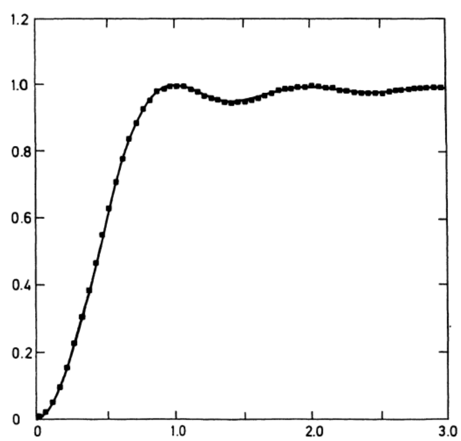

Figure 6.2

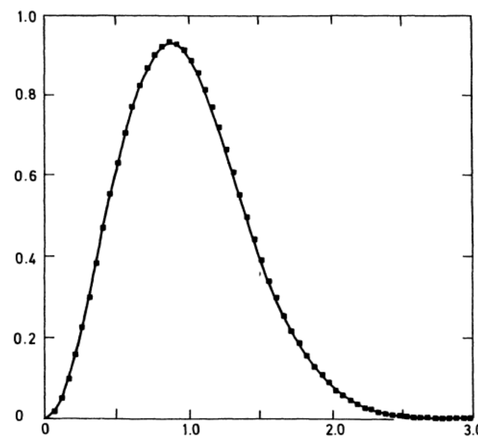

Figure 6.3

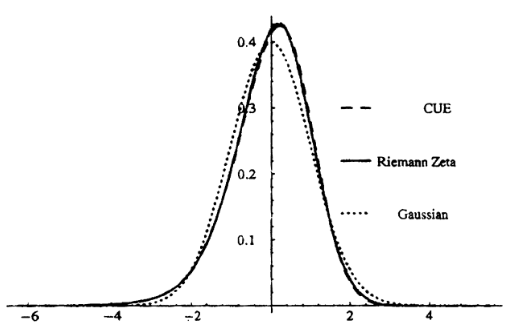

Figure 6.4

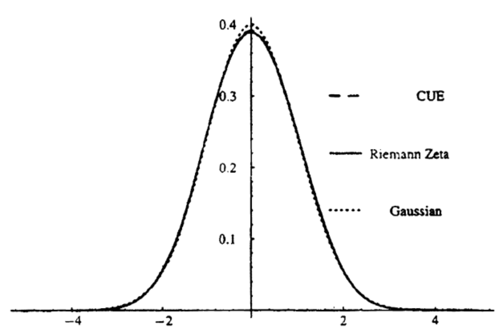

Figure 6.5

6.2 Application in big data, image processing, compressed sensing and so on

In the big data problems, for the huge matrix problem, level of current computer computing and storage are too low to solve algebraic equations. Random matrices such as Gauss distribution random variables can be chosen to solve them. In image processing, compression sensing and so on, one may reduce the vector of higher-dimensional vector to the lower average distribution or the random matrices of Gauss distribution to deal with. In dealing with large data approximation, the error satisfies the random matrices, and its convergence is proved. The robustness of the random matrices must be considered.

7 Application of Random Matrices in Physics

7.1 Random matrices in nuclear physics

As shown in Figure 7.1, it is a typical image of slow neutron resonance, with different peaks at different positions, with different heights and widths, and the positions of peaks are called atomic energy levels.

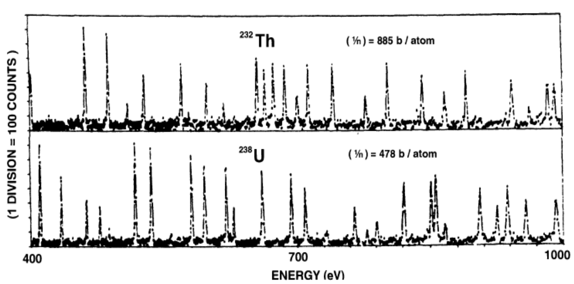

Figure 7.1

Physicists have collected a lot of resonant excitation data for nucleus, and whether all kinds of these energy levels only have statistical properties. The answer is that there is a process of slow change.

According to quantum mechanics, the energy levels of a system are described by the eigenvalues of the Hermitian operator, energy levels are generally composed of a large number of continuous and discrete energy levels. The eigenvalue structure of Hermitian is distributed in infinite dimensional Hilbert space. Because we are concerned with the discrete part of the energy level approximation, we can approximate Hilbert spaces as finite dimensional ones, select the basis function of the space, Hamiltonian is a finite dimensional matrix. We can solve eigenvalue problems

$$H\psi_i = E_i\psi_i. \tag{7.1}$$

How to determine the matrix H? We can take the principle of statistics so that each component of the matrices is a random variable, and it is assumed to have symmetric properties. Wigner first pointed out that the local statistical properties of energy levels are equal to the distribution of eigenvalues of random matrices, such as Gauss ensemble, orthogonal ensemble, sympletic ensemble, etc. In 1962, Dyson gave the theory that elements of matrices were Brown motion. Random matrices were also applied to glass crystals. There were random Ising models.

If two neighboring spins are σ_i, the interaction of σ_j is J_{ij}. If $J_{ij} = J$ is a fixed Ising model, it was calculated for Onsager. If J_{ij} is a random variable, it is a random Ising model.

7.2 Level density

The discontinuous number of excitation level is $\lambda(n)$,

$$n = \nu_1 + \nu_2 + \cdots + \nu_l, \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_l > 0, \tag{7.2}$$

when n is large, then

$$\lambda(n) \sim \exp\Big\{\Big(\frac{\theta\pi^2 n}{3}\Big)^{\frac{1}{2}}\Big\}, \tag{7.3}$$

where $\theta = 1$ or 2. And the level density is

$$\rho(E, j, \pi) \propto (2j+1)(E-\Delta)^{-\frac{5}{4}} \exp\Big(\frac{-j(j+1)}{2\sigma^2}\Big) \exp\{2\pi(E-\Delta)^{\frac{1}{2}}\}. \tag{7.4}$$

where E is excitation energy, j is span, π is parity, σ is variance.

7.3 The distribution of Neutron width

Its width probability distribution is

$$p(x) = \frac{\nu}{2}\Big[\Gamma\Big(\frac{\nu}{2}\Big)\Big]^{-1}\Big(\frac{\nu x}{2}\Big)^{\frac{\nu}{2}-1} \mathrm{e}^{-\frac{\nu x}{2}} = (2\pi x)^{-\frac{1}{2}}\mathrm{e}^{-\frac{x}{2}}. \tag{7.5}$$

The Gauss distribution corresponding to the amplitude of width of interval $\mathrm{d}x$ is

$$\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{1}{2}(\sqrt{x})^2\Big\}\,\mathrm{d}\sqrt{x}. \tag{7.6}$$

7.4 Probability density between energy levels

Let

$$S_i = E_{i+1} - E_i, \quad s_i = \frac{S_i}{D}, \tag{7.7}$$

and

$$p(s)\mathrm{d}s = \mathrm{e}^{-s}\mathrm{d}s \tag{7.8}$$

be Possion distribution as shown in Figures 7.2 and 7.3.

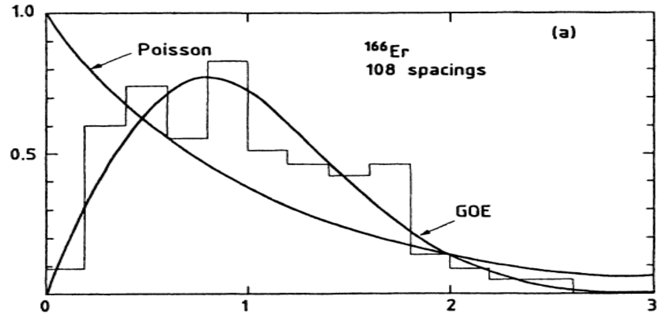

Figure 7.2

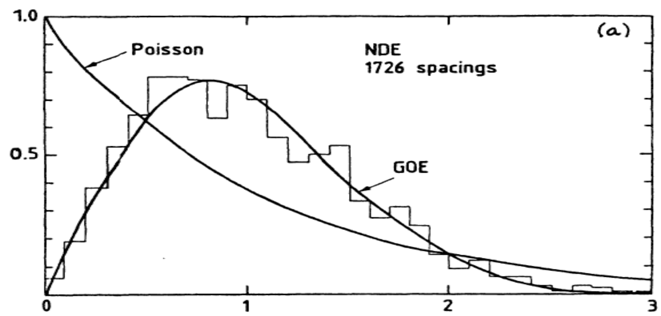

Figure 7.3

Wigner distribution

$$p\Big(\frac{S}{D}\Big)\mathrm{d}S = \lim_{m\to\infty}\prod_{r=0}^{m-1}\left(1-\frac{Sr}{m}\frac{S}{m}a\right)aS\mathrm{d}S = aS\mathrm{e}^{-aS^2/2}\mathrm{d}S, \tag{7.9}$$

a is a constant, $s = S/D$ satisfies

$$\int_0^\infty sp(s)\mathrm{d}s = 1, \tag{7.10}$$

as shown in Figure 7.4.

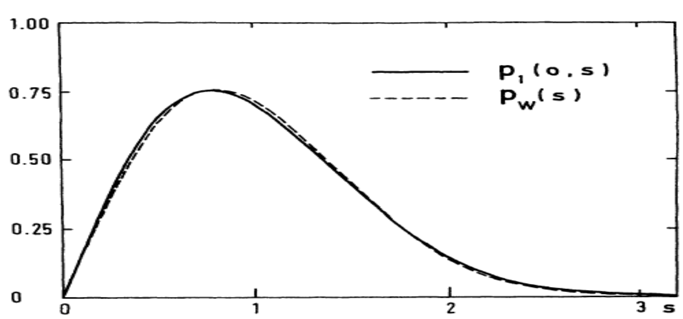

Figure 7.4

7.5 The application of quantum chromo dynamics and two-dimensional Einstein gravity equations

The so-called QCD (Quantum Chromo Dynamics) is quantum chromo dynamics, we know that quark is the smallest unit of matter. It has three colors (red, green and blue), six flavors, and it interacts with bosons, gluons and so on.

The QCD partition function can be expressed as

$$E^{QCD}=\sum_k \mathrm{e}^{-\beta E_k},$$

where it contains the eigenvalues of QCD Hamilton E_k, and the parameter β is related to the temperature of the system, and also can be expressed by the Euclidean integral of the non-Abelian gauge field A_μ,

$$E^{QCD}(M)=\int \mathrm{d}A_\mu \prod_{f=1}^{N_f} \det(D+m_f)\mathrm{e}^{-S^{YM}},$$

where S^{YM} is Yang-Mills action

$$S^{YM}=\int \mathrm{d}^4x\left[\frac{1}{4g^2}F_{\mu\nu}^{a2}-i\frac{\theta}{32\pi^2}F_{\mu\nu}^a\widetilde{F}_{\mu\nu}^a\right].$$

Field strength and its duality are given by

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+f_{abc}A_\mu^b A_\nu^c,\quad \widetilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta},$$

f_{abc} is the structural constant of the gauge group $SU(N_c)$. The gauge field is defined as $A_\mu=A_{\mu a}T^a/2$, where T^a is the generator of the normalized group. The parameter

$$\gamma\equiv\frac{1}{32\pi^2}\int \mathrm{d}^4x F_{\mu\nu}^a\widetilde{F}_{\mu\nu}^a$$

is topological invariant. The mass matrix M can be selected to be diagonal, $M=\mathrm{diag}(m_1,\cdots,m_N)$.

The Hermit Dirac operator D given by $D=r_\mu(\partial_\mu+iA_\mu)$, where r_μ is the Euclidean Dirac matrix which has an anti-commutation relation, $\{r_\mu,r_\nu\}=2\delta_{\mu\nu}$, and in the eigen representation, the r matrix is

$$r_k=\begin{pmatrix}0 & i\sigma_k\\ -i\sigma_k & 0\end{pmatrix},\quad r_4=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad r_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Dirac operator has the structure

$$D=\begin{pmatrix}0 & id\\ id^\dagger & 0\end{pmatrix}.$$

When integral measure is discrete space,

$$\mathrm{d}A_\mu^a = \prod_x \mathrm{d}A_\mu^a(x).$$

QCD partition function has many symmetries: axisymmetric, color symmetry, etc., including the color symmetry of boson, anti-U symmetry.

Because the intrinsic ensemble and the D operator are related to the Hermit matrix of the $N \times N$, its probability density is

$$p(c)\mathrm{d}c = N\det{}^{N_f}(D+M)\exp\Big(-\frac{N\beta_D}{4}TrC^\dagger C\Big)DC,$$

$$D = \begin{pmatrix} 0 & iC \\ iC^\dagger & 0 \end{pmatrix},$$

where $\beta_D = 1, 2, 4$. We can use orthogonal polynomials to solve.

We also find that the QCD partition function is the ζ function of integrable coefficients, and the Toda lattice equation and the Painlevé equation can be obtained, so it is related to the integrable systems.

Consider the 2D quantum gravity Einstein equation, and the Einstein gravitational action is

$$S_E = \Lambda \int_\Sigma \sqrt{g}\mathrm{d}^2\xi + N\int_\Sigma \sqrt{g}R\mathrm{d}^2\xi = \Lambda A(\Sigma) + N\chi(\Sigma),$$

where g is Riemann matrix, R is the curvature, $A(\Sigma)$ is the surface area, N is Newton's constant, $\chi(\Sigma)$ is the Euler characteristic function of the surface. Consider the Gauss average

$$\langle x^{2n}\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}} x^{2n}\mathrm{d}x = (2n-1)!! = \frac{(2n)!}{2^n n!}.$$

A stochastic Einstein equation can be obtained.

For any function f,

$$\begin{aligned} \langle f(M)\rangle &= \frac{1}{Z_0(N)}\int \mathrm{d}M\mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{2}} f(M), \\ f(M) &= \prod_{(ij)\in I} M_{ij}, \\ \mathrm{d}M &= \prod_i M_{ii}\prod_{i<j}\mathrm{dRe}(M_{ij})\mathrm{dIm}(M_{ij}). \end{aligned}$$

References

[1] E.P. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels, *Mathematical Proceedings of the Cambridge Philosophical Society*, **47**:4(1951),790-798.

[2] F.J. Dyson, A Brownian-Motion model for the eigenvalues of a random matrix, *Journal of Mathematical Physics*, **3**:6(1962),1191-1198.

[3] A.S. Fokas, A.R. Its, A.V. Kitaev, The isomonodromy approach to matric models in 2D quantum gravity, *Communications in Mathematical Physics*, **147**:2(1992),395-430.

[4] E.C. Titchmarsh, The Theory of the Riemann Zeta Function, Clarendon Press, Oxford, second edition, 1986.

[5] H.L. Montgomery, The pair correlation of zeros of the zeta function, *Proceedings of Symposia in Pure Mathematics*, **41**:1(1973),181-193.

[6] J.P. Keating, Random matrices and the Riemann zeta-function, *Highlights of mathematical physics* (London 2004), 153-163.

(*edited by Liangwei Huang*)