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SUMUDU TRANSFORM IN BICOMPLEX SPACE AND ITS APPLICATIONS*

Ritu Agarwal†, Mahesh Puri Goswami
(*Dept. of Math., Malaviya National Institute of Technology, Jaipur-302017, India*)
Ravi P. Agarwal
(*Dept. of Math., Texas A&M University - Kingsville 700 University Blvd. Kingsville, TX 78363-8202, USA*)

Abstract

In this paper, we define Sumudu transform with convergence conditions in bicomplex space. Also, we derive some of its basic properties and its inverse. Applications of bicomplex Sumudu transform are illustrated to find the solution of differential equation of bicomplex-valued functions and find the solution for Cartesian transverse electric magnetic (TEM) waves in homogeneous space.

Keywords Sumudu transform; bicomplex numbers; bicomplex functions and bicomplex Laplace transform

2000 Mathematics Subject Classification 30G35; 42B10

1 Introduction

In literature, various integral transforms have been widely used in physics and engineering mathematics. In the sequence of these integral transforms, Watugala [26] defined Sumudu transform and applied it to find the solution of ordinary differential equations in control engineering problems.

Over the set of functions

$$\mathcal{A} = \left\{ f(t) : \text{ there exists an } M,\ \tau_j > 0,\ |f(t)| < M\mathrm{e}^{\frac{|t|}{\tau_j}},\ \text{if } t \in (-1)^j[0,\infty),\ j = 1,2 \right\}, \tag{1}$$

the Sumudu transform is defined by the formula

$$\mathcal{S}[f(t); s] = \frac{1}{s}\int_0^\infty \mathrm{e}^{-\frac{t}{s}} f(t)\mathrm{d}t, \quad s \in (\tau_1, \tau_2). \tag{2}$$

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†Corresponding author. E-mail: ragarwal.maths@mnit.ac.in

Sumudu transform has scale and unit preserving properties. In [11], Belgacem et al. discussed fundamental properties of Sumudu transform and showed that Sumudu transform is a theoretical dual of Laplace transform. Also, it is used to solve an integral production-depreciation problem. In [10], Belgacem and Karaballi generalized Sumudu differentiations, integrations and convolution theorems existing in the previous literatures. Also they generalized Sumudu shifting theorems and introduced recurrence formulas of the transform.

In [27], Zhang developed an algorithm based on Sumudu transform which can be implemented in computer algebra systems like Maple and used to solve differential equations. In [17], Hussian and Belgacem obtained the solution of Maxwell's differential equations for transient excitation functions propagating in a lossy conducting medium by using Sumudu transform in time domain.

In [9], Belgacem found the electric field solutions of Maxwell's equations, pertaining to transient electromagnetic planner, (TEMP), waves propagating in lossy media, through Sumudu transform. In [13], Eltayeb et al. discussed the Sumudu transform on a space of distributions. In [19, 20], Kilicman and Gadain produced some properties and relationship between double Laplace and double Sumudu transform and also, used the double Sumudu transform to solve wave equation in one dimension having singularity at initial conditions.

In [18], Kilicman et al. discussed the existence of double Sumudu transform with convergence conditions and applied it to find the solution of linear ordinary differential equations with constant coefficients. In [6], Al-Omari and Belgacem investigated certain class of quaternions and Sumudu transform. Motivated by the work of Al-Omari et al., we make efforts to extend the Sumudu transform to bicomplex variable.

2 Bicomplex Number

In 1892, Segre Corrado [25] defined bicomplex numbers as

$$C_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 |\ x_0, x_1, x_2, x_3 \in C_0\},$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2z_2 |\ z_1, z_2 \in C_1\},$$

where i_1 and i_2 are imaginary units such that $i_1^2 = i_2^2 = -1$, $i_1i_2 = i_2i_1 = j$, $j^2 = 1$ and C_0, C_1 and C_2 are sets of real numbers, complex numbers and bicomplex numbers, respectively. The set of bicomplex numbers is a commutative ring with unit and zero divisors. Hence, contrary to quaternions, bicomplex numbers are commutative with some non-invertible elements situated on the null cone.

In 1928 and 1932, Futagawa Michiji originated the concept of holomorphic functions of a bicomplex variable in many papers [14, 15]. In 1934, Dragoni [12] gave some basic results in the theory of bicomplex holomorphic functions while Price [22] and Rönn [24] developed the bicomplex algebra and function theory.

In recent developments, authors have done efforts to extend Polygamma function [16], inverse Laplace transform, it's convolution theorem [2], Stieltjes transform [1], Tauberian Theorem of Laplace-Stieltjes transform [3], Bochner Theorem of Fourier-Stieltjes transform [4] and Mellin transform and its application [5] in the bicomplex variable from their complex counterpart. In their procedure, the idempotent representation of bicomplex numbers plays a vital role.

Idempotent Representation: Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, that is,

$$\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,$$

where $e_1 = \frac{1+j}{2}$, $e_2 = \frac{1-j}{2}$; $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$.

This representation of a bicomplex number is known as *Idempotent Representation* of ξ. The coefficients $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called the *Idempotent Components* of the bicomplex number $\xi = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called *Idempotent Basis*.

With the help of idempotent representation, we define projection mappings $P_1 : C_2 \to A_1 \subseteq C_1$, $P_2 : C_2 \to A_2 \subseteq C_1$ as follows:

$$P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1,$$
$$P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2,$$

for any $z_1 + i_2 z_2 \in C_2$.

In the following theorem, Price discussed the convergence of bicomplex function with respect to it's idempotent complex component functions. This theorem is useful in proving our results.

Theorem 2.1[22] *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq C_2$ if and only if $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$ under projection mappings $P_1 : D \to D_1 \subseteq C_1$ and $P_2 : D \to D_2 \subseteq C_1$ are convergent in domains D_1 and D_2, respectively.*

In this article, we obtain bicomplex Sumudu transform, its inverse and some of their properties. In Section 3, we derive the bicomplex Sumudu transform with convergence conditions by using the bicomplex Laplace transform. In Section 4, we discuss some basic properties of bicomplex Sumudu transform. In Section 5, we derive the inversion theorem for bicomplex Sumudu transform. In Section 6, we find applications of bicomplex Sumudu transform in solving differential equation of bicomplex-valued function and last section presents the conclusion of the paper.

3 Bicomplex Sumudu Transform

Let $f(t)$ be bicomplex-valued piecewise continuous function of exponential order K. Then bicomplex Laplace transform (Kumar and Kumar [21]) of $f(t)$ is

$$L[f(t);\xi]=\int_0^\infty \mathrm{e}^{-\xi t}f(t)\mathrm{d}t,\quad \xi\in D, \tag{3}$$

where

$$D=\{\xi=s_1e_1+s_2e_2\in C_2:\mathrm{Re}(s_1)>K \text{ and } \mathrm{Re}(s_2)>K\} \tag{4}$$

or, equivalently

$$D=\{\xi\in C_2:\mathrm{Re}(\xi)>K+|\mathrm{Im}_j(\xi)|\}, \tag{5}$$

$\mathrm{Im}_j(\xi)$ denotes the imaginary part w.r.t. j. In (3) if we replace ξ by $\frac{1}{\xi}$ and multiply the integral obtained by $\frac{1}{\xi}$, we get

$$\frac{1}{\xi}L\Big[f(t);\frac{1}{\xi}\Big]=\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}f(t)\mathrm{d}t=\mathcal{S}[f(t);\xi]=\bar{f}(\xi),\quad \xi\in\Omega, \tag{6}$$

where $\mathcal{S}[\cdot]$ denotes the Sumudu transform of f and

$$\Omega=\Big\{\xi=s_1e_1+s_2e_2\in C_2:\mathrm{Re}\Big(\frac{1}{s_1}\Big)>K,\ \mathrm{Re}\Big(\frac{1}{s_2}\Big)>K \text{ and } \xi\notin\mathcal{O}_2\Big\}, \tag{7}$$

or equivalently,

$$\Omega=\Big\{\xi\in C_2:\mathrm{Re}\Big(\frac{1}{\xi}\Big)>K+\Big|\mathrm{Im}_j\Big(\frac{1}{\xi}\Big)\Big| \text{ and } \xi\notin\mathcal{O}_2\Big\}, \tag{8}$$

where $\mathcal{O}_2=\{z_1+i_2z_2\in C_2:z_1^2+z_2^2=0\}$.

Further, we show that $\|\bar{f}(\xi)\|<\infty$. Now, for $\xi=s_1e_1+s_2e_2$, $s_1=x_1+i_1y_1$ and $s_2=x_2+i_1y_2$, we obtain

$$\begin{aligned}\|\bar{f}(\xi)\|&=\Big\|\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}f(t)\mathrm{d}t\Big\|\\&\le\frac{1}{\|\xi\|}\int_0^\infty \|\mathrm{e}^{-\frac{1}{\xi}t}\|\,\|f(t)\|\,\mathrm{d}t,\quad[\xi\neq 0 \text{ i.e. } \xi\notin\mathcal{O}_2]\\&\le\frac{1}{\|\xi\|}\int_0^\infty \|\mathrm{e}^{-\frac{1}{(s_1e_1+s_2e_2)}t}\|M\mathrm{e}^{Kt}\mathrm{d}t\\&=\frac{M}{\|\xi\|}\int_0^\infty \|\mathrm{e}^{-\frac{1}{s_1}t}e_1+\mathrm{e}^{-\frac{1}{s_2}t}e_2\|\mathrm{e}^{Kt}\mathrm{d}t,\end{aligned}$$

because of $\|s_1e_1+s_2e_2\|=\Big(\frac{|s_1|^2+|s_2|^2}{2}\Big)^{\frac{1}{2}}$, which is the bicomplex norm defined by Rochon and Shapiro [23, p.85].

$$\begin{aligned}
&\frac{M}{\|\xi\|}\int_0^\infty \frac{1}{\sqrt{2}}\left(\left|\mathrm{e}^{-\frac{1}{s_1}t}\right|^2+\left|\mathrm{e}^{-\frac{1}{s_2}t}\right|^2\right)^{\frac{1}{2}}\mathrm{e}^{Kt}\mathrm{d}t\\
&=\frac{M}{\sqrt{2}\|\xi\|}\int_0^\infty \left(\mathrm{e}^{-\frac{2x_1}{x_1^2+y_1^2}t}+\mathrm{e}^{-\frac{2x_2}{x_2^2+y_2^2}t}\right)^{\frac{1}{2}}\mathrm{e}^{Kt}\mathrm{d}t\\
&\le\frac{M}{\sqrt{2}\|\xi\|}\Big[\int_0^\infty \mathrm{e}^{-\frac{x_1}{x_1^2+y_1^2}t}\mathrm{e}^{Kt}\mathrm{d}t+\int_0^\infty \mathrm{e}^{-\frac{x_2}{x_2^2+y_2^2}t}\mathrm{e}^{Kt}\mathrm{d}t\Big]\\
&=\frac{M}{\sqrt{2}\|\xi\|}\Big[\int_0^\infty \mathrm{e}^{-\left(\frac{x_1}{x_1^2+y_1^2}-K\right)t}\mathrm{d}t+\int_0^\infty \mathrm{e}^{-\left(\frac{x_2}{x_2^2+y_2^2}-K\right)t}\mathrm{d}t\Big]\\
&=\frac{M}{\sqrt{2}\|\xi\|}\left(\frac{1}{\frac{x_1}{x_1^2+y_1^2}-K}+\frac{1}{\frac{x_2}{x_2^2+y_2^2}-K}\right),
\end{aligned}$$

by Minkowski's inequality $(|x|^2+|y|^2)^{\frac{1}{2}}\le|x|+|y|$, for any $x,y\in C_0$. Then the requirement $\|\bar{f}(\xi)\|<\infty$ is satisfied only if $\frac{x_1}{x_1^2+y_1^2}>K$ that is $\mathrm{Re}\big(\frac{1}{s_1}\big)>K$ and $\frac{x_2}{x_2^2+y_2^2}>K$ that is $\mathrm{Re}\big(\frac{1}{s_2}\big)>K$. Therefore, $\bar{f}(\xi)$ is analytic and convergent in the strip Ω, defined by (8).

Thus, we can summarize the above discussion to define the bicomplex Sumudu transform as follows:

Definition 3.1 Let $f(t)$ be bicomplex-valued piecewise continuous function of exponential order K. Then the bicomplex Sumudu transform of $f(t)$ is defined as

$$\mathcal{S}[f(t);\xi]=\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}f(t)\mathrm{d}t=\bar{f}(\xi),\quad \xi\in\Omega, \tag{9}$$

where Ω is defined as

$$\Omega=\Big\{\xi\in C_2:\mathrm{Re}\Big(\frac{1}{\xi}\Big)>K+\Big|\mathrm{Im}_j\Big(\frac{1}{\xi}\Big)\Big|\text{ and }\xi\notin\mathcal{O}_2\Big\}. \tag{10}$$

4 Properties of Bicomplex Sumudu Transform

In this section, we discuss some properties of bicomplex Sumudu transform that is linearity property and change of scale property as follows:

Theorem 4.1(Linearity Property) *Let $\bar{f}(\xi)=\mathcal{S}[f(t);\xi]$ and $\bar{g}(\xi)=\mathcal{S}[g(t);\xi]$ be bicomplex Sumudu transforms of bicomplex-valued functions $f(t)$ and $g(t)$ of exponential orders K_1 and K_2, respectively. Then*

$$\mathcal{S}[c_1f(t)+c_2g(t)]=c_1\bar{f}(\xi)+c_2\bar{g}(\xi),\quad \xi\in\Omega, \tag{11}$$

where Ω is defined in (10) *and* $K=\max\{K_1,K_2\}$.

Proof By applying the definition of bicomplex Sumudu transform, we get

$$\begin{aligned}\mathcal{S}[c_1f(t)+c_2g(t)] &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}[c_1f(t)+c_2g(t)]\mathrm{d}t\\ &= \frac{c_1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}f(t)\mathrm{d}t+\frac{c_2}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}g(t)\mathrm{d}t\\ &= c_1\bar{f}(\xi)+c_2\bar{g}(\xi).\end{aligned}$$

Theorem 4.2(Change of Scale Property) *Let $\bar{f}(\xi)=\mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

$$\mathcal{S}[f(at);\xi]=\bar{f}(a\xi),\quad a>0,\ \xi\in\Omega, \tag{12}$$

where Ω is defined in (10).

Proof By applying the definition of bicomplex Sumudu transform, we get

$$\begin{aligned}\mathcal{S}[f(at);\xi] &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}f(at)\mathrm{d}t,\quad [\text{put } at=u]\\ &= \frac{1}{a\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{a\xi}u}f(u)\mathrm{d}u\\ &= \bar{f}(a\xi).\end{aligned}$$

Theorem 4.3 *Let $\bar{f}(\xi)=\mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

(i) $\mathcal{S}[f'(t);\xi]=\frac{1}{\xi}[\bar{f}(\xi)-f(0)],\ \xi\in\Omega,$

(ii) $\mathcal{S}[f''(t);\xi]=\frac{1}{\xi^2}[\bar{f}(\xi)-f(0)-\xi f'(0)],\ \xi\in\Omega,$

(iii) $\mathcal{S}[f^{(n)}(t);\xi]=\frac{1}{\xi^n}\Big[\bar{f}(\xi)-f(0)-\sum\limits_{k=1}^{n-1}\xi^k f^{(k)}(0)\Big],\ \xi\in\Omega,$

where Ω is defined in (10).

Proof (i) By applying the definition of bicomplex Sumudu transform, we get

$$\begin{aligned}\mathcal{S}[f'(t);\xi] &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{1}{\xi}t}f'(t)\mathrm{d}t\\ &= \Big(\frac{1}{s_1}\int_0^\infty \mathrm{e}^{-\frac{1}{s_1}t}f_1'(t)\mathrm{d}t\Big)e_1+\Big(\frac{1}{s_2}\int_0^\infty \mathrm{e}^{-\frac{1}{s_2}t}f_2'(t)\mathrm{d}t\Big)e_2.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\mathcal{S}[f'(t);\xi] &= \Big(-\frac{1}{s_1}f_1(0)+\frac{1}{s_1}\bar{f}_1(s_1)\Big)e_1+\Big(-\frac{1}{s_2}f_2(0)+\frac{1}{s_2}\bar{f}_2(s_2)\Big)e_2\\ &= \frac{1}{s_1e_1+s_2e_2}\big[\bar{f}_1(s_1)e_1+\bar{f}_2(s_2)e_2-f_1(0)e_1-f_2(0)e_2\big]\\ &= \frac{1}{\xi}[\bar{f}(\xi)-f(0)].\end{aligned} \tag{13}$$

(ii) If we replace $f'(t) = g(t)$, using (13) then

$$\mathcal{S}[f''(t);\xi] = \mathcal{S}[g'(t);\xi] = \frac{1}{\xi}\big[\bar{g}(\xi) - g(0)\big] = \frac{1}{\xi}\Big[\frac{1}{\xi}\big(\bar{f}(\xi) - f(0)\big) - f'(0)\Big]$$
$$= \frac{1}{\xi^2}\big[\bar{f}(\xi) - f(0) - \xi f'(0)\big].$$

(iii) By (13) the result (iii) is true for $n = 1$. Now, assume the result is true for $n = m$, that is

$$\mathcal{S}[f^{(m)}(t);\xi] = \frac{1}{\xi^m}\left[\bar{f}(\xi) - f(0) - \sum_{k=1}^{m-1} \xi^k f^{(k)}(0)\right]. \tag{14}$$

Now, for $n = m + 1$, using (14),

$$\mathcal{S}[f^{(m+1)}(t);\xi] = \frac{1}{\xi}\big[\mathcal{S}[f^{(m)}(t);\xi] - f^{(m)}(0)\big]$$
$$= \frac{1}{\xi^{m+1}}\left[\bar{f}(\xi) - f(0) - \sum_{k=1}^{m} \xi^k f^{(k)}(0)\right].$$

Hence, by the principle of mathematical induction, the result is true for all $n \in \mathbb{N}$.

Theorem 4.4 *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

$$\mathcal{S}[tf(t);\xi] = \xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\bar{f}(\xi) + \xi\bar{f}(\xi), \quad \xi \in \Omega, \tag{15}$$

where Ω is defined in (10).

Proof Since

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{f}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty \frac{1}{\xi}\mathrm{e}^{-\frac{t}{\xi}} f(t)\mathrm{d}t$$
$$= \Big(\frac{\mathrm{d}}{\mathrm{d}s_1}\int_0^\infty \frac{1}{s_1}\mathrm{e}^{-\frac{t}{s_1}} f_1(t)\mathrm{d}t\Big)e_1 + \Big(\frac{\mathrm{d}}{\mathrm{d}s_2}\int_0^\infty \frac{1}{s_2}\mathrm{e}^{-\frac{t}{s_2}} f_2(t)\mathrm{d}t\Big)e_2,$$

using Leibniz rule for integration for complex functions, we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{f}(\xi) = \Big(\int_0^\infty \frac{\partial}{\partial s_1}\frac{1}{s_1}\mathrm{e}^{-\frac{t}{s_1}} f_1(t)\mathrm{d}t\Big)e_1 + \Big(\int_0^\infty \frac{\partial}{\partial s_2}\frac{1}{s_2}\mathrm{e}^{-\frac{t}{s_2}} f_2(t)\mathrm{d}t\Big)e_2$$
$$= \Big(\int_0^\infty \frac{1}{s_1^3}\mathrm{e}^{-\frac{t}{s_1}} tf_1(t)\mathrm{d}t - \int_0^\infty \frac{1}{s_1^2}\mathrm{e}^{-\frac{t}{s_1}} f_1(t)\mathrm{d}t\Big)e_1$$
$$+\Big(\int_0^\infty \frac{1}{s_2^3}\mathrm{e}^{-\frac{t}{s_2}} tf_2(t)\mathrm{d}t - \int_0^\infty \frac{1}{s_2^2}\mathrm{e}^{-\frac{t}{s_2}} f_2(t)\mathrm{d}t\Big)e_2$$

$$
\begin{aligned}
&= \Big(\frac{1}{s_1^2}\mathcal{S}[tf_1(t);s_1] - \frac{1}{s_1}\mathcal{S}[f_1(t);s_1]\Big)e_1 + \Big(\frac{1}{s_2^2}\mathcal{S}[tf_2(t);s_2] - \frac{1}{s_2}\mathcal{S}[f_2(t);s_2]\Big)e_2 \\
&= \frac{1}{(s_1e_1+s_2e_2)^2}\mathcal{S}\big[t\big(f_1(t)e_1+f_2(t)e_2\big);s_1e_1+s_2e_2\big] \\
&\quad -\frac{1}{s_1e_1+s_2e_2}\mathcal{S}\big[\big(f_1(t)e_1+f_2(t)e_2\big);s_1e_1+s_2e_2\big] \\
&= \frac{1}{\xi^2}\mathcal{S}[tf(t);\xi] - \frac{1}{\xi}\mathcal{S}[f(t);\xi].
\end{aligned}
$$

Therefore,

$$\mathcal{S}[tf(t);\xi] = \xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{f}(\xi) + \xi\bar{f}(\xi).$$

Theorem 4.5 *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

$$\mathcal{S}\Big[\int_0^t f(u)\mathrm{d}u;\xi\Big] = \xi\bar{f}(\xi), \quad \xi\in\Omega, \tag{16}$$

where Ω is defined in (10).

Proof From the definition of bicomplex Sumudu transform, we have

$$\mathcal{S}\Big[\int_0^t f(u)\mathrm{d}u;\xi\Big] = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{t}{\xi}}\int_0^t f(u)\mathrm{d}u\mathrm{d}t.$$

By changing the order of integration, we get

$$\mathcal{S}\Big[\int_0^t f(u)\mathrm{d}u;\xi\Big] = \frac{1}{\xi}\int_0^\infty f(u)\mathrm{d}u\int_u^\infty \mathrm{e}^{-\frac{t}{\xi}}\mathrm{d}t = \frac{1}{\xi}\int_0^\infty \xi\mathrm{e}^{-\frac{u}{\xi}}f(u)\mathrm{d}u = \xi\bar{f}(\xi).$$

Theorem 4.6 *Let $f(t)$ be the bicomplex-valued function of period $T>0$, then*

$$\mathcal{S}[f(t);\xi] = \frac{\frac{1}{\xi}\int_0^T \mathrm{e}^{-\frac{t}{\xi}}f(t)\mathrm{d}t}{1-\mathrm{e}^{-\frac{T}{\xi}}}, \quad \xi\in\Omega, \tag{17}$$

where Ω is defined in (10).

Proof By the definition, we have

$$\mathcal{S}[f(t);\xi] = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{t}{\xi}}f(t)\mathrm{d}t = \frac{1}{\xi}\sum_{n=0}^\infty\int_{nT}^{(n+1)T}\mathrm{e}^{-\frac{t}{\xi}}f(t)\mathrm{d}t. \tag{18}$$

Taking $t=\tau+nT$ in the n^{th} integral, (18) becomes

$$\begin{aligned}\mathcal{S}[f(t);\xi] &= \frac{1}{\xi}\sum_{n=0}^{\infty}\mathrm{e}^{-\frac{nT}{\xi}}\int_0^T \mathrm{e}^{-\frac{\tau}{\xi}}f(\tau+nT)\mathrm{d}\tau\\ &= \frac{1}{\xi}\sum_{n=0}^{\infty}\mathrm{e}^{-\frac{nT}{\xi}}\int_0^T \mathrm{e}^{-\frac{\tau}{\xi}}f(\tau)\mathrm{d}\tau, \quad \big[\text{because } f(t+nT)=f(t)\big]\\ &= \frac{1}{\xi\big(1-\mathrm{e}^{-\frac{T}{\xi}}\big)}\int_0^T \mathrm{e}^{-\frac{t}{\xi}}f(t)\mathrm{d}t.\end{aligned}$$

4.1 Convolution

The way of combining two signals is known as convolution. It is such a widespread and useful formula that it has its own shorthand notation, $*$. For any two signals x and y, there will be another signal z obtained by convolving x with y as

$$z(t) = x * y = \int_0^t x(s)y(t-s)\mathrm{d}s.$$

We derive here the convolution theorem for bicomplex Sumudu transform as follows:

Theorem 4.7 *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ and $\bar{g}(\xi) = \mathcal{S}[g(t);\xi]$ be the bicomplex Sumudu transforms of bicomplex-valued function $f(t)$ and $g(t)$ of exponential orders K_1 and K_2 respectively, then*

$$\mathcal{S}[(f*g)(t);\xi] = \xi\bar{f}(\xi)\bar{g}(\xi), \quad \xi\in\Omega, \tag{19}$$

where $K = \max\{K_1, K_2\}$ and Ω is defined in (10).

Proof

$$\mathcal{S}[(f*g)(t);\xi] = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{t}{\xi}}(f*g)(t)\mathrm{d}t = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{t}{\xi}}\int_0^t f(u)g(t-u)\mathrm{d}u\mathrm{d}t.$$

Changing the order of integration, by putting $t-u=z$, we get

$$\begin{aligned}\mathcal{S}[(f*g)(t);\xi] &= \frac{1}{\xi}\int_0^\infty f(u)\mathrm{d}u\int_u^\infty \mathrm{e}^{-\frac{t}{\xi}}g(t-u)\mathrm{d}t\\ &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{u}{\xi}}f(u)\mathrm{d}u\int_0^\infty \mathrm{e}^{-\frac{z}{\xi}}g(z)\mathrm{d}z = \xi\bar{f}(\xi)\bar{g}(\xi).\end{aligned}$$

5 Inverse Bicomplex Sumudu Transform

In this section, we discuss the inversion theorem for Sumudu transform in bicomplex space as follows:

Theorem 5.1 *Let $\bar{f}(\xi)$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$ of exponential order K, analytic in Ω, then*

$$f(t) = \frac{1}{2\pi i_1}\int_\Gamma \mathrm{e}^{\frac{t}{\xi}}\xi\bar{f}(\xi)\mathrm{d}\xi = \sum \mathrm{Res}\,\big[\mathrm{e}^{\frac{t}{\xi}}\xi\bar{f}(\xi)\big], \tag{20}$$

where $\Gamma = \big(\Gamma_1, \Gamma_2\big)$ *is piecewise continuous differentiable closed contour in Bicomplex space and* Ω *is defined in* (10).

Proof Since we know that bicomplex Laplace transform [21] of bicomplex-valued function $f(t)$ is

$$L\Big[f(t); \frac{1}{\xi}\Big] = F\Big(\frac{1}{\xi}\Big) = \int_0^\infty \mathrm{e}^{-\frac{t}{\xi}} f(t)\mathrm{d}t, \quad \xi \in \Omega,$$

which deduces

$$\frac{1}{\xi}F\Big(\frac{1}{\xi}\Big) = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-\frac{t}{\xi}} f(t)\mathrm{d}t = \bar{f}(\xi),$$

then

$$F\Big(\frac{1}{\xi}\Big) = \xi\bar{f}(\xi). \tag{21}$$

Using (21), inverse bicomplex Laplace transform [8, Definition 3.1] of (21) is

$$f(t) = \frac{1}{2\pi i_1}\int_\Gamma \mathrm{e}^{\frac{t}{\xi}} F\Big(\frac{1}{\xi}\Big)\mathrm{d}\xi = \frac{1}{2\pi i_1}\int_\Gamma \mathrm{e}^{\frac{t}{\xi}}\xi\bar{f}(\xi)\mathrm{d}\xi.$$

Hence the result (20) holds.

6 Applications

In this section, we find the solution of an application of differential equation of bicomplex-valued functions. Also, we find the solution for Cartesian transverse electric magnetic (TEM) waves in homogeneous space using bicomplex Sumudu transform.

(a) Consider the general differential equation of order n of bicomplex-valued function $y(t)$

$$a_n\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_0 y = f(t) \tag{22}$$

with initial conditions $y(0), y'(0), \cdots, y^{n-1}(0)$ being given and finite, a_0, $a_1, \cdots, a_n$ being bicomplex constants and $f(t)$ being bicomplex-valued function. Taking the bicomplex Sumudu transform of (22), we get

$$a_n\frac{1}{\xi^n}\Big[\bar{y}(\xi)-y(0)-\sum_{k=1}^{n-1}\xi^k y^{(k)}(0)\Big]+a_{n-1}\frac{1}{\xi^{n-1}}\Big[\bar{y}(\xi)-y(0)-\sum_{k=1}^{n-2}\xi^k y^{(k)}(0)\Big]+\cdots+a_0\bar{y}(\xi)=\bar{f}(\xi),$$

which deduces

$$a_n\Big[\bar{y}(\xi)-y(0)-\sum_{k=1}^{n-1}\xi^k y^{(k)}(0)\Big]+a_{n-1}\xi\Big[\bar{y}(\xi)-y(0)-\sum_{k=1}^{n-2}\xi^k y^{(k)}(0)\Big]+\cdots+a_0\xi^n = \xi^n\bar{f}(\xi).$$

Therefore,

$$
\begin{aligned}
\bar{y}(\xi) =& \frac{y(0)\big[a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_1\xi^{n-1}\big]}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \\
&+ \frac{y'(0)\xi\big[a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_2\xi^{n-2}\big]}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \\
&+ \cdots + \frac{y^{(n-1)}(0)a_n\xi^{n-1}}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \\
&+ \frac{\xi^n \bar{f}(\xi)}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n}.
\end{aligned} \tag{23}
$$

Taking the inverse bicomplex Sumudu transform of (23), we get the solution of differential equation (22) as

$$
y(t) = \frac{1}{2\pi i_1} \int_\Gamma \mathrm{e}^{\frac{t}{\xi}} \xi \bar{y}(\xi) \mathrm{d}\xi, \tag{24}
$$

where $\Gamma = \big(\Gamma_1, \Gamma_2\big)$ is piecewise continuous differentiable closed contour in Bicomplex space, and Ω and $\bar{y}(\xi)$ are defined in (10) and (23), respectively.

In particular, consider a differential equation

$$
\frac{\mathrm{d}y}{\mathrm{d}t} + y = \mathrm{e}^{at}, \quad a \in C_2 \tag{25}
$$

with the initial condition $y(0) = 0$, where $y(t)$ is a bicomplex-valued function.

Taking the bicomplex Sumudu transform of (25), we get

$$
\begin{aligned}
&\frac{\bar{y}(\xi) - y(0)}{\xi} + \bar{y}(\xi) = \sum_{n=0}^{\infty} a^n \xi^n, \\
&\bar{y}(\xi) = \frac{\xi}{\xi + 1} \sum_{n=0}^{\infty} a^n \xi^n, \\
&\bar{y}(\xi) = \xi\big(1 - \xi + \xi^2 - \xi^3 + \cdots\big) \sum_{n=0}^{\infty} a^n \xi^n.
\end{aligned}
$$

Taking the inverse bicomplex Sumudu transform and using Theorem 4.7, we get

$$
y(t) = \int_0^t \mathrm{e}^{-u} \mathrm{e}^{a(t-u)} \mathrm{d}u = \frac{1}{1+a}\big(\mathrm{e}^{at} - \mathrm{e}^{-t}\big),
$$

which is the required solution of the given differential equation (25).

(b) To apply this purely mathematical concept in Electromagnetic, Maxwell's equations (in a source-free domain) are first written in a form involving the wave number k and the medium intrinsic impedance η, rather than the medium permittivity and permeability Bicomplex Maxwell's equation is described in Anastassiu H.T. et al. [7], that is

$$\nabla \times \mathbf{E} = -i_1 k\eta \mathbf{H}, \tag{26}$$

$$\nabla \times \mathbf{H} = i_1 \frac{k}{\eta}\mathbf{E}, \tag{27}$$

for the time convention $\mathrm{e}^{i_1 wt}$. Vector fields $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic fields, respectively. The bicomplex vector field $\mathbf{F}$ is defined:

$$\mathbf{F} \equiv \frac{1}{\sqrt{\eta}}\mathbf{E} + i_2\sqrt{\eta}\mathbf{H} \tag{28}$$

with the implication that each directional component of $\mathbf{F}$ is a scalar bicomplex function, combining the corresponding field directional components. Multiplying (27) with i_2 and adding the result to (26), after some manipulation, the bicomplex Maxwell's equation is derived, that is

$$\nabla \times \mathbf{F} = i_2 i_1 k\mathbf{F}. \tag{29}$$

Assuming a TEM to be z wave, that is, a vanishing z-component, and after introducing $Q_y = i_2 F_y$, (29) is reduced to the following system of bicomplex differential equations

$$\frac{\mathrm{d}Q_y}{\mathrm{d}z} = i_1 k F_x, \tag{30}$$

$$\frac{\mathrm{d}F_x}{\mathrm{d}z} = i_1 k Q_y, \tag{31}$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0. \tag{32}$$

Differentiating (30) and (31) and using respectively (31) and (30) therein, we get

$$\frac{\mathrm{d}^2 F_x}{\mathrm{d}z^2} + k^2 F_x = 0, \tag{33}$$

$$\frac{\mathrm{d}^2 Q_y}{\mathrm{d}z^2} + k^2 Q_y = 0. \tag{34}$$

For the solution, taking the bicomplex Sumudu transform of (33), we get

$$\frac{1}{\xi^2}\left[\bar{F}_x(\xi) - F_x(0) - \xi F_x'(0)\right] + k^2 \bar{F}_x(\xi) = 0,$$

which implies

$$\bar{F}_x(\xi) = \frac{F_x(0) + \xi F_x'(0)}{\xi^2 k^2 + 1}. \tag{35}$$

Taking the inverse bicomplex Sumudu transform of (35), we get

$$\begin{aligned} F_x(z) &= \frac{1}{2\pi i_1}\int_\Omega \mathrm{e}^{\frac{z}{\xi}}\xi\bar{F}_x(\xi)\mathrm{d}\xi = \frac{1}{2\pi i_1}\int_\Omega \mathrm{e}^{\frac{z}{\xi}}\xi\frac{F_x(0)+\xi F_x'(0)}{\xi^2k^2+1}\mathrm{d}\xi \\ &= \lim_{\xi\to\frac{i_1}{k}}\Big(\xi-\frac{i_1}{k}\Big)\xi\mathrm{e}^{\frac{z}{\xi}}\frac{F_x(0)+\xi F_x'(0)}{\xi^2k^2+1} + \lim_{\xi\to-\frac{i_1}{k}}\Big(\xi+\frac{i_1}{k}\Big)\xi\mathrm{e}^{\frac{z}{\xi}}\frac{F_x(0)+\xi F_x'(0)}{\xi^2k^2+1} \\ &= \frac{kF_x(0)+i_1F_x'(0)}{k^3}\mathrm{e}^{-i_1kz} + \frac{kF_x(0)-i_1F_x'(0)}{k^3}\mathrm{e}^{i_1kz} \\ &= R\mathrm{e}^{-i_1kz}+K\mathrm{e}^{i_1kz}. \end{aligned} \tag{36}$$

Similarly,

$$F_y(z) = -i_2Q_y(z) = -i_2\big[L\mathrm{e}^{-i_1kz}+S\mathrm{e}^{i_1kz}\big]. \tag{37}$$

Therefore,

$$\mathbf{F}\equiv\frac{1}{\sqrt{\eta}}\mathbf{E}+i_2\sqrt{\eta}\mathbf{H} = \big[R\mathrm{e}^{-i_1kz}+K\mathrm{e}^{i_1kz}\big]\hat{x} - i_2\big[L\mathrm{e}^{-i_1kz}+S\mathrm{e}^{i_1kz}\big]\hat{y}, \tag{38}$$

where $\hat{x}$ and $\hat{y}$ are the fundamental position unit vectors in the direction of X-axis and Y-axis respectively and

$$\begin{aligned} R &= \frac{kF_x(0)+i_1F_x'(0)}{k^3} = R_1+i_2R_2, \quad K = \frac{kF_x(0)-i_1F_x'(0)}{k^3} = K_1+i_2K_2, \\ L &= \frac{kQ_y(0)+i_1Q_y'(0)}{k^3} = L_1+i_2L_2, \quad S = \frac{kQ_y(0)-i_1Q_y'(0)}{k^3} = S_1+i_2S_2. \end{aligned}$$

Equation (32) implies that L, R, S and K are bicomplex constants. Since (38) satisfies bicomplex Maxwell's equation (29)

$$L=-R \quad \text{and} \quad S=K.$$

Hence (38) becomes

$$\mathbf{F}\equiv\frac{1}{\sqrt{\eta}}\mathbf{E}+i_2\sqrt{\eta}\mathbf{H} = \big[R\mathrm{e}^{-i_1kz}+K\mathrm{e}^{i_1kz}\big]\hat{x} - i_2\big[-R\mathrm{e}^{-i_1kz}+K\mathrm{e}^{i_1kz}\big]\hat{y}. \tag{39}$$

Extracting the bi-real and bi-imaginary parts of the solutions (39) yields the electric and magnetic fields components

$$\mathbf{E} = \sqrt{\eta}\big[R_1\mathrm{e}^{-i_1kz}+K_1\mathrm{e}^{i_1kz}\big]\hat{x} + \sqrt{\eta}\big[-R_2\mathrm{e}^{-i_1kz}+K_2\mathrm{e}^{i_1kz}\big]\hat{y}, \tag{40}$$

$$\mathbf{H} = \frac{1}{\sqrt{\eta}}\big[R_2\mathrm{e}^{-i_1kz}+K_2\mathrm{e}^{i_1kz}\big]\hat{x} - \frac{1}{\sqrt{\eta}}\big[-R_1\mathrm{e}^{-i_1kz}+K_1\mathrm{e}^{i_1kz}\big]\hat{y}, \tag{41}$$

where R_1, R_2, K_1 and K_2 are arbitrary complex constants, and (40) and (41) is the solution of Maxwell's equations (26) and (27).

7 Conclusion

Bicomplex numbers play a vital role in providing large class of frequency domain. In this paper, we derive bicomplex Sumudu transform with convergence conditions with its some basic properties which are useful in finding the solution of differential equations involving bicomplex-valued functions. Bicomplex analysis has great advantage that it separates the electric and magnetic fields as complex components.

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