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A CLASS OF SPECTRALLY ARBITRARY RAY PATTERNS *

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Abstract

An $n \times n$ ray pattern A is said to be spectrally arbitrary if for every monic nth degree polynomial $f(x)$ with coefficients from $\mathbb{C}$, there is a complex matrix in the ray pattern class of A such that its characteristic polynomial is $f(x)$. In this paper, a family ray patterns is proved to be spectrally arbitrary by using Nilpotent-Jacobian method.

Keywords ray pattern; Nilpotent-Jacobian method; spectrally arbitrary
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1 Introduction

A *ray pattern* $A = (a_{jk})$ of order n is a matrix with entries $a_{jk} \in \{e^{i\theta} | \, 0 \le \theta < 2\pi\} \cup \{0\}$, where $i^2 = -1$. Its ray pattern class is

$$Q_R(A) = \{B = (b_{jk}) \in M_n(\mathbb{C}) | \, b_{jk} = r_{jk} a_{jk}, \, r_{jk} \in \mathbb{R}^+, \, 1 \le j, k \le n\}.$$

It is easy to see that ray patterns are a generalization of the sign patterns.

A ray pattern A is said to be *spectrally arbitrary* if for any monic nth degree polynomial $f(x)$ with coefficients from $\mathbb{C}$, there is a complex matrix $B \in Q_R(A)$ such that the characteristic polynomial of B is $f(x)$.

Spectrally arbitrary problem is a basic subject in combinatorial matrix theory and a hot topic for some international scholars. The problem of the spectrally arbitrary sign patterns was introduced in [2]. J.H. Drew et al. developed the Nilpotent-Jacobian method to show that a sign pattern is spectrally arbitrary in [2]. Work on spectrally arbitrary sign patterns has continued in several articles including [1,3,4]. J.J. Mcdonald and J. Stuart in [6] extended the Nilpotent-Jacobian method from sign patterns to the ray patterns. Y.Z. Mei and Y.B. Gao in [7] showed that the minimum number of nonzeros in an $n \times n$ irreducible spectrally arbitrary ray pattern is $3n - 1$.

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Though the general method — Nilpotent-Jacobian method — to prove the spectrally arbitrary property has been developed, the proof procedure is not very easy. Let $A_{n,m} = (a_{jk})$ be an $n \times n$ complex square matrix as follows

$$A_{n,m}=\begin{array}{cc} & \begin{array}{cccccccccccc} 1 & & \cdots & & m & & & \cdots & & & & n \end{array} \\ \begin{array}{c} 1 \\ \\ \vdots \\ \\ m \\ \\ \\ \\ \vdots \\ \\ \\ n \end{array} & \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & e^{i\theta} & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 \\ 1 & -i & -i & -i & \cdots & \cdots & \cdots & \cdots & \cdots & -i & -i & -i \end{pmatrix} \end{array} (0 \le \theta < 2\pi),$$

where $n,\ m,\ j$ and k are positive integers; $2 \le m \le n-2$, $1 \le j,k \le n$; and the (m,m) entry is $e^{i\theta}$.

In [6], the ray pattern $A_{n,2}$ was proved to be spectrally arbitrary. In [8], the ray pattern $A_{n,3}$ was proved to be spectrally arbitrary. In [5,9], several families ray patterns were proved to be spectrally arbitrary.

In this paper, we show that for $n \ge 8$ if $\theta \in \Big(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\Big)$, then the ray pattern $A_{n,4}$ is spectrally arbitrary.

2 The Extended Nilpotent-Jacobian Method

A square matrix A is called to be *nilpotent* if there exists a positive integer k such that $A^k = 0$ but $A^{k-1} \ne 0$. A ray pattern B is said to be *potentially nilpotent* if there is a complex matrix $A \in Q_R(B)$ with characteristic polynomial $g(x) = x^n$. If the ray pattern A is spectrally arbitrary, then A is potentially nilpotent affirmatively.

In [6], the extended Nilpotent-Jacobian method can be summarized as follows:

(1) Find a nilpotent matrix in the given ray pattern class.

(2) Change $2n$ of the positive coefficients (denoted $r_1, r_2, \cdots, r_{2n}$) of the $e^{i\theta_{jk}}$ in this nilpotent matrix to variables $t_1, t_2, \cdots, t_{2n}$.

(3) Express the characteristic polynomial of the resulting matrix as:

$$x^n + \sum_{k=1}^{n}(f_k(t_1, t_2, \cdots, t_{2n}) + i g_k(t_1, t_2, \cdots, t_{2n}))x^{n-k}.$$

(4) Compute the determinant of the Jacobi matrix $J = \frac{\partial(f_1,g_1,\cdots,f_n,g_n)}{\partial(t_1,t_2,\cdots,t_{2n})}$.

(5) If the determinant of J, evaluated at $(t_1,t_2,\cdots,t_{2n}) = (r_1,r_2,\cdots,r_{2n})$ is nonzero, then the given ray pattern and all of its superpatterns are spectrally arbitrary, where a ray pattern $P=(p_{jk})$ is a *superpattern* of a ray pattern $A=(a_{jk})$ if $p_{jk}=a_{jk}$ whenever $a_{jk}\neq 0$.

3 Main Results

Let n be an integer with $n\geq 8$. For convenience, we restrict θ to $0\leq\theta\leq\frac{\pi}{2}$. Let $p=q+i\sqrt{1-q^2}$ and $q=\cos\theta$. We consider the following $n\times n$ complex matrix

$$B=\begin{pmatrix} -a_1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ -a_2 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ -a_3 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\ a_4 & 0 & 0 & p & 1 & 0 & \cdots & \cdots & 0 & 0\\ a_5 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0\\ a_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0\\ -a_{n-2} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & 0\\ -a_{n-1} & b_n & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1\\ a_n & -ib_{n-1} & -ib_{n-2} & \cdots & \cdots & \cdots & \cdots & -ib_3 & -ib_2 & -ib_1 \end{pmatrix}. \tag{3.1}$$

If $a_j>0$ and $b_j>0$ for all $1\leq j\leq n$, then $B\in Q_R(A_{n,4})$. Denote

$$|\lambda I-B|=\lambda^n+\sum_{k=1}^{n}(f_k+ig_k)\lambda^{n-k}=\lambda^n+\sum_{k=1}^{n}\alpha_k\lambda^{n-k}, \tag{3.2}$$

where

$$f_k=f_k(a_1,a_2,\cdots,a_n;b_1,b_2,\cdots,b_n),\quad g_k=g_k(a_1,a_2,\cdots,a_n;b_1,b_2,\cdots,b_n)$$

are the real and imaginary parts of the coefficient α_k of λ^{n-k}, for $k=1,2,\cdots,n$, respectively.

Lemma 1 $|\lambda I-B|=\lambda^n+\sum\limits_{k=1}^{n}(f_k+ig_k)\lambda^{n-k}$, *where* f_k *and* g_k *are defined as above, then*

$$\begin{cases} f_1=a_1-q,\\ f_2=b_1\sqrt{1-q^2}-a_1q+a_2,\\ f_3=b_2\sqrt{1-q^2}+a_1b_1\sqrt{1-q^2}-a_2q+a_3,\\ f_4=b_3\sqrt{1-q^2}+a_1b_2\sqrt{1-q^2}+a_2b_1\sqrt{1-q^2}-a_3q-a_4,\\ f_j=(b_{j-1}+a_1b_{j-2}+a_2b_{j-3}+a_3b_{j-4})\sqrt{1-q^2}-a_j\quad(5\leq j\leq n-3),\\ f_{n-2}=-b_n+(a_3b_{n-6}+a_2b_{n-5}+a_1b_{n-4})\sqrt{1-q^2}+a_{n-2},\\ f_{n-1}=-a_1b_n+(a_3b_{n-5}+a_2b_{n-4})\sqrt{1-q^2}+a_{n-1},\\ f_n=-a_n+a_3b_{n-4}\sqrt{1-q^2}, \end{cases}$$

and

$$
\begin{cases}
g_1 = b_1 - \sqrt{1-q^2},\\
g_2 = b_2 + a_1b_1 - b_1q - a_1\sqrt{1-q^2},\\
g_3 = b_3 + a_1b_2 - b_2q - a_1b_1q + a_2b_1 - a_2\sqrt{1-q^2},\\
g_4 = b_4 + a_1b_3 - b_3q - a_1b_2q + a_3b_1 + a_2b_2 - a_2b_1q - a_3\sqrt{1-q^2},\\
g_j = b_j + a_1b_{j-1} + a_2b_{j-2} + a_3b_{j-3} - b_{j-1}q - a_1b_{j-2}q\\
\qquad -\sum\limits_{k=4}^{j-1} a_kb_{j-k} - a_2b_{j-3}q - a_3b_{j-4}q \quad (5 \le j \le n-3),\\
g_{n-2} = b_{n-2} + a_1b_{n-3} - a_1b_{n-4}q + a_2b_{n-4} + a_3b_{n-5}\\
\qquad -\sum\limits_{k=4}^{n-3} a_kb_{n-2-k} - a_2b_{n-5}q - a_3b_{n-6}q,\\
g_{n-1} = b_{n-1} - b_1b_n + a_1b_{n-2} + a_2b_{n-3} + a_3b_{n-4}\\
\qquad -\sum\limits_{k=4}^{n-3} a_kb_{n-1-k} - a_2b_{n-4}q - a_3b_{n-5}q + a_{n-2}b_1,\\
g_n = -a_1b_1b_n + a_1b_{n-1} + a_2b_{n-2} + a_3b_{n-3}\\
\qquad -\sum\limits_{k=4}^{n-3} a_kb_{n-k} - a_3b_{n-4}q + a_{n-2}b_2 + a_{n-1}b_1.
\end{cases}
$$

Proof For $2 \le t \le n-4$, let

$$
\Delta_t = \begin{vmatrix}
\lambda & -1 & 0 & \cdots & 0\\
0 & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & 0\\
0 & \cdots & 0 & \lambda & -1\\
ib_t & ib_{t-1} & \cdots & ib_2 & \lambda + ib_1
\end{vmatrix},
$$

then

$$
|\lambda I - B| = (\lambda + a_1)\begin{vmatrix}
\lambda & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\
0 & \lambda & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0\\
0 & 0 & \lambda - p & -1 & 0 & \cdots & \cdots & 0 & 0\\
0 & 0 & 0 & \lambda & -1 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & \lambda & -1 & 0 & 0\\
0 & 0 & \cdots & \cdots & 0 & 0 & \lambda & -1 & 0\\
-b_n & 0 & \cdots & \cdots & \cdots & 0 & 0 & \lambda & -1\\
ib_{n-1} & ib_{n-2} & \cdots & \cdots & \cdots & \cdots & ib_3 & ib_2 & \lambda + ib_1
\end{vmatrix}
$$

$$
\begin{aligned}
&+a_2\Gamma + a_3\Omega - \sum_{k=4}^{n-3} a_k\Delta_{n-k} + a_{n-2}\Delta_2 + a_{n-1}(\lambda + ib_1) - a_n \\
= {}& \lambda^2\Gamma + a_1\Gamma\lambda - b_n(\lambda + ib_1)\lambda + ib_{n-1}\lambda - a_1 b_n(\lambda + ib_1) + a_1 ib_{n-1} \\
&+a_2\Gamma + a_3\Omega - \sum_{k=4}^{n-3} a_k\Delta_{n-k} + a_{n-2}\Delta_2 + a_{n-1}(\lambda + ib_1) - a_n,
\end{aligned}
$$

where

$$
\begin{aligned}
&\Delta_t = \lambda^t + i\sum_{j=1}^{t} b_j\lambda^{t-j} \quad (2 \le t \le n-4), \\
&\Omega = \lambda^{n-3} + (b_1 i - p)\lambda^{n-4} + i\sum_{j=2}^{n-3}(b_j - b_{j-1}p)\lambda^{n-3-j}, \\
&\Gamma = \lambda^{n-2} + (b_1 i - p)\lambda^{n-3} + i\sum_{j=2}^{n-3}(b_j - b_{j-1}p)\lambda^{n-2-j} + ib_{n-2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{cases}
\alpha_1 = ib_1 + a_1 - p, \\
\alpha_2 = ib_2 + ib_1(a_1 - p) - a_1 p + a_2, \\
\alpha_3 = ib_3 + ib_2(a_1 - p) - ia_1 b_1 p + ia_2 b_1 - a_2 p + a_3, \\
\alpha_4 = ib_4 + ib_3(a_1 - p) - ia_1 b_2 p + ia_3 b_1 + ia_2 b_2 - a_3 p - ia_2 b_1 p - a_4, \\
\alpha_j = ib_j + ib_{j-1}(a_1 - p) - \Big(a_j + i\sum\limits_{k=4}^{j-1} a_k b_{j-k} - ia_2 b_{j-2} - ia_3 b_{j-3}\Big) \\
\qquad -ip(a_1 b_{j-2} + a_2 b_{j-3} + a_3 b_{j-4}) \quad (5 \le j \le n-3), \\
\alpha_{n-2} = ib_{n-2} - b_n - ia_1 b_{n-4} p + ia_1 b_{n-3} + ia_2 b_{n-4} + ia_3 b_{n-5} \\
\qquad -i\sum\limits_{k=4}^{n-3} a_k b_{n-2-k} - ip(a_2 b_{n-5} + a_3 b_{n-6}) + a_{n-2}, \\
\alpha_{n-1} = ib_{n-1} - a_1 b_n - ib_1 b_n + ia_1 b_{n-2} + ia_2 b_{n-3} + ia_3 b_{n-4} \\
\qquad -i\sum\limits_{k=4}^{n-3} a_k b_{n-1-k} - ip(a_2 b_{n-4} + a_3 b_{n-5}) + ia_{n-2} b_1 + a_{n-1}, \\
\alpha_n = -ia_1 b_1 b_n + ia_1 b_{n-1} + ia_2 b_{n-2} + ia_3 b_{n-3} - i\sum\limits_{k=4}^{n-3} a_k b_{n-k} \\
\qquad -ia_3 b_{n-4} p + ia_{n-2} b_2 + ia_{n-1} b_1 - a_n.
\end{cases}
$$

Then the lemma holds.

Lemma 2 *For $n \ge 8$, if $\theta \in \Big(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\Big)$, then the ray pattern $A_{n,4}$ is potentially nilpotent.*

Proof Let B have form (3.1), $q = \cos\theta$. In (3.2), assume that $f_k = 0$ and $g_k = 0$ for $1 \le k \le n$, by Lemma 1, we have

$$\begin{cases} a_1 = q, \\ a_2 = a_1 q - b_1\sqrt{1-q^2}, \\ a_3 = a_2 q - a_1 b_1\sqrt{1-q^2} - b_2\sqrt{1-q^2}, \\ a_4 = -a_3 q + (a_1 b_2 + a_2 b_1 + b_3)\sqrt{1-q^2}, \\ a_j = b_{j-1}\sqrt{1-q^2} + a_1 b_{j-2}\sqrt{1-q^2} \\ \qquad +a_3 b_{j-4}\sqrt{1-q^2} + a_2 b_{j-3}\sqrt{1-q^2} \quad (5 \le j \le n-3), \\ a_{n-2} = b_n - (a_3 b_{n-6} + a_2 b_{n-5} + a_1 b_{n-4})\sqrt{1-q^2}, \\ a_{n-1} = a_1 b_n - (a_3 b_{n-5} + a_2 b_{n-4})\sqrt{1-q^2}, \\ a_n = a_3 b_{n-4}\sqrt{1-q^2}, \end{cases}$$

and

$$\begin{cases} b_1 = \sqrt{1-q^2}, \\ b_2 = a_1\sqrt{1-q^2} - a_1 b_1 + b_1 q, \\ b_3 = a_2\sqrt{1-q^2} - a_1 b_2 + a_1 b_1 q - a_2 b_1 + b_2 q, \\ b_4 = a_3\sqrt{1-q^2} - a_1 b_3 + a_1 b_2 q - a_3 b_1 - a_2 b_2 + a_2 b_1 q + b_3 q, \\ b_j = -a_1 b_{j-1} + b_{j-1} q + a_1 b_{j-2} q + \sum\limits_{k=4}^{j-1} a_k b_{j-k} - a_3 b_{j-3} \\ \qquad -a_2 b_{j-2} + a_2 b_{j-3} q + a_3 b_{j-4} q \quad (5 \le j \le n-3), \\ b_{n-2} = -a_1 b_{n-3} + a_1 b_{n-4} q - a_2 b_{n-4} - a_3 b_{n-5} \\ \qquad + \sum\limits_{k=4}^{n-3} a_k b_{n-2-k} + a_2 b_{n-5} q + a_3 b_{n-6} q, \\ b_{n-1} = b_1 b_n - a_1 b_{n-2} - a_2 b_{n-3} - a_3 b_{n-4} - a_{n-2} b_1 \\ \qquad + \sum\limits_{k=4}^{n-3} a_k b_{n-1-k} + a_2 b_{n-4} q + a_3 b_{n-5} q, \\ b_n = \Big(a_1 b_{n-1} + a_2 b_{n-2} + a_3 b_{n-3} - \sum\limits_{k=4}^{n-3} a_k b_{n-k} - a_3 b_{n-4} q + a_{n-2} b_2 + a_{n-1} b_1\Big)(a_1 b_1)^{-1}. \end{cases}$$

So we only need to show that for $1 \le j \le n$ if $\theta \in \Big(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\Big)$, then $a_j > 0$ and $b_j > 0$, thus $B \in Q_R(A_{n,4})$ is nilpotent. The following proofs are limited to $\theta \in \Big(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\Big)$.

Firstly, for $1 \le j \le 4$, it is easy to see that

$$\begin{cases} a_1 = q > 0, \\ a_2 = 2q^2 - 1 > 0, \\ a_3 = 4q^3 - 3q > 0, \end{cases}$$

and

$$\begin{cases} a_4 = -8q^4 + 8q^2 - 1 > 0, \\ b_1 = \sqrt{1-q^2} > 0, \\ b_2 = q\sqrt{1-q^2} > 0, \\ b_3 = q^2\sqrt{1-q^2} > 0, \\ b_4 = q^3\sqrt{1-q^2} > 0. \end{cases}$$

Secondly, we consider a_j for $5 \le j \le n-3$ and b_j for $5 \le j \le n-3$. We proceed by induction on j. For $j = 5$, we have

$$\begin{aligned} &a_5 = (b_4 + a_1b_3 + a_2b_2 + a_3b_1)\sqrt{1-q^2} = 4q(2q^2-1)(1-q^2) > 0, \\ &b_5 = a_1b_3q + a_4b_1 - a_3b_2 - a_2b_3 + a_2b_2q + a_3b_1q = (-7q^4 + 8q^2 - 1)\sqrt{1-q^2} > 0. \end{aligned}$$

Suppose that $a_j > 0$ and $b_j > 0$ hold for any $j < l$ where $5 \le l \le n-3$, then

$$\begin{aligned} a_l &= b_{l-1}\sqrt{1-q^2} + a_1b_{l-2}\sqrt{1-q^2} + a_2b_{l-3}\sqrt{1-q^2} + a_3b_{l-4}\sqrt{1-q^2} > 0, \\ b_l &= -a_1b_{l-1} + b_{l-1}q + a_1b_{l-2}q + \sum_{k=4}^{l-1} a_kb_{l-k} - a_3b_{l-3} - a_2b_{l-2} + a_2b_{l-3}q + a_3b_{l-4}q \\ &= (1-q^2)b_{l-2} + \sum_{k=4}^{l-1} a_kb_{l-k} + 2q(1-q^2)b_{l-3} + a_3b_{l-4}q > 0. \end{aligned}$$

Therefore by induction, $a_j > 0$ and $b_j > 0$ for $5 \le j \le n-3$.

Thirdly, it is obvious that $a_n > 0$. Thus we only have to consider the following equations

$$a_{n-2} = b_n - (a_3b_{n-6} + a_2b_{n-5} + a_1b_{n-4})\sqrt{1-q^2}, \tag{3.3}$$

$$a_{n-1} = a_1b_n - (a_3b_{n-5} + a_2b_{n-4})\sqrt{1-q^2}, \tag{3.4}$$

$$\begin{aligned} b_{n-2} = &-a_1b_{n-3} + a_1b_{n-4}q - a_2b_{n-4} - a_3b_{n-5} \\ &+ \sum_{k=4}^{n-3} a_kb_{n-2-k} + a_3b_{n-6}q + a_2b_{n-5}q, \end{aligned} \tag{3.5}$$

$$\begin{aligned} b_{n-1} = &b_1b_n - a_1b_{n-2} - a_2b_{n-3} - a_3b_{n-4} - a_{n-2}b_1 \\ &+ \sum_{k=4}^{n-3} a_kb_{n-1-k} + a_3b_{n-5}q + a_2b_{n-4}q, \end{aligned} \tag{3.6}$$

$$b_n = \Big(a_1b_{n-1} + a_2b_{n-2} + a_3b_{n-3} - \sum_{k=4}^{n-3} a_kb_{n-k} - a_3b_{n-4}q + a_{n-2}b_2 + a_{n-1}b_1\Big)(a_1b_1)^{-1}. \tag{3.7}$$

For the formula (3.5), by

$$b_{n-3} = a_1b_{n-5}q - a_2b_{n-5} - a_3b_{n-6} + \sum_{k=4}^{n-4} a_kb_{n-3-k} + a_3b_{n-7}q + a_2b_{n-6}q,$$

we have

$$b_{n-2}=(1-q^2)b_{n-4}+q(1-q^2)b_{n-5}+(6q^4-5q^2)b_{n-6}+a_{n-3}b_1 \\ +\sum_{k=4}^{n-4}a_kb_{n-2-k}-a_1\sum_{k=4}^{n-4}a_kb_{n-3-k}-q^2(4q^3-3q)b_{n-7}.$$

Since

$$a_{n-3}=(a_3b_{n-7}+a_2b_{n-6}+a_1b_{n-5}+b_{n-4})\sqrt{1-q^2},$$

we obtain

$$b_{n-2}=2(1-q^2)b_{n-4}+2q(1-q^2)b_{n-5}+(4q^4-2q^2-1)b_{n-6} \\ +(4q^3-3q)(1-2q^2)b_{n-7}+\sum_{k=4}^{n-4}a_k(b_{n-2-k}-qb_{n-3-k}).$$

By

$$b_{n-4}=a_1b_{n-6}q-a_2b_{n-6}-a_3b_{n-7}+\sum_{k=4}^{n-5}a_kb_{n-4-k}+a_3b_{n-8}q+a_2b_{n-7}q,$$

it follows that

$$b_{n-2}=2q(1-q^2)b_{n-5}+(6q^4-6q^2+1)b_{n-6}+(-4q^5+2q^3+q)b_{n-7} \\ +2q(1-q^2)(4q^3-3q)b_{n-8}+\sum_{k=4}^{n-4}a_k(b_{n-2-k}-qb_{n-3-k})+2(1-q^2)\sum_{k=4}^{n-5}a_kb_{n-4-k}.$$

Now we only need to show that $b_{n-2-k}-qb_{n-3-k}\geq 0$ for $4\leq k\leq n-4$. It is obvious that $b_j-qb_{j-1}=0$ for $2\leq j\leq 4$. For $j=5$, we can obtain

$$b_5-qb_4=(-8q^4+8q^2-1)\sqrt{1-q^2}>0.$$

For $6\leq j\leq n-3$, we have

$$\begin{aligned} a_j-qa_{j-1}&=(b_{j-1}+a_1b_{j-2}+a_2b_{j-3}+a_3b_{j-4})\sqrt{1-q^2} \\ &\quad -q(b_{j-2}+a_1b_{j-3}+a_2b_{j-4}+a_3b_{j-5})\sqrt{1-q^2} \\ &=[b_{j-1}-(1-q^2)b_{j-3}-2q(1-q^2)b_{j-4}-qa_3b_{j-5}]\sqrt{1-q^2} \\ &=\sum_{k=4}^{j-2}a_kb_{j-1-k}\sqrt{1-q^2}>0. \end{aligned}$$

Thus, for $6\leq j\leq n-3$, we have

$$\begin{aligned}b_j - qb_{j-1} = {} & a_1b_{j-2}q + \sum_{k=4}^{j-1} a_kb_{j-k} - a_3b_{j-3} - a_2b_{j-2} + a_2b_{j-3}q + a_3b_{j-4}q - a_1b_{j-3}q^2 \\ & - \sum_{k=4}^{j-2} a_kb_{j-1-k}q + a_3b_{j-4}q + a_2b_{j-3}q - a_2b_{j-4}q^2 - a_3b_{j-5}q^2 \\ = {} & (1-q^2)b_{j-2} + q(1-q^2)b_{j-3} + (-2q^4+3q^2-1)b_{j-4} \\ & + (-4q^5+7q^3-3q)b_{j-5} + \sum_{k=6}^{j-1} b_{j-k}(a_k - qa_{k-1}) > 0.\end{aligned}$$

Then $b_{n-2} > 0$.

For the formula (3.7), by (3.4) and (3.6), we have

$$\begin{aligned}a_1b_1b_n = {} & (1-q^2)b_{n-2} + 2q(1-q^2)b_{n-3} + (4q^4-2q^2-1)b_{n-4} \\ & + (4q^3-3q)(1-2q^2)b_{n-5} + \sum_{k=4}^{n-3} a_k(b_{n-k} - qb_{n-1-k}),\end{aligned}$$

then by (3.5) we can obtain

$$\begin{aligned}a_1b_1b_n = {} & q(1-q^2)b_{n-3} + (5q^4-4q^2)b_{n-4} + (-6q^5+6q^3-q)b_{n-5} \\ & + (4q^3-3q)(1-q^2)qb_{n-6} + \sum_{k=4}^{n-3} a_k(b_{n-k} - qb_{n-1-k}) \\ & + (1-q^2)\sum_{k=4}^{n-4} a_kb_{n-2-k} + (1-q^2)a_{n-3}b_1.\end{aligned}$$

Note that

$$a_{n-3} = (a_3b_{n-7} + a_2b_{n-6} + a_1b_{n-5} + b_{n-4})\sqrt{1-q^2},$$

then

$$\begin{aligned}a_1b_1b_n = {} & q(1-q^2)b_{n-3} + (6q^4-6q^2+1)b_{n-4} + (-5q^5+4q^3)b_{n-5} \\ & + (1-q^2)(2q^4-1)b_{n-6} + (4q^3-3q)(1-q^2)^2b_{n-7} \\ & + \sum_{k=4}^{n-3} a_k(b_{n-k} - qb_{n-1-k}) + (1-q^2)\sum_{k=4}^{n-4} a_kb_{n-2-k} > 0. \qquad (3.8)\end{aligned}$$

Mutiply both sides of the formula (3.3) by a_1b_1, then

$$a_1b_1a_{n-2} = a_1b_1b_n - a_1b_1(a_3b_{n-6} + a_2b_{n-5} + a_1b_{n-4})\sqrt{1-q^2}.$$

By (3.8), we have

$$\begin{aligned}
a_1b_1a_{n-2} &= q(1-q^2)b_{n-3} + (7q^4 - 7q^2 + 1)b_{n-4} + (3q^5 + q^3 + q)b_{n-5} \\
&\quad + (2q^2-1)(1-q^2)^2 b_{n-6} + (4q^3 - 3q)(1-q^2)^2 b_{n-7} \\
&\quad + \sum_{k=4}^{n-3} a_k(b_{n-k} - qb_{n-1-k}) + (1-q^2)\sum_{k=4}^{n-4} a_k b_{n-2-k} \\
&= q(1-q^2)b_{n-3} + q^2(1-q^2)b_{n-4} + (11q^5 - 7q^3 + 2q)b_{n-5} \\
&\quad + (2q^2-1)(1-q^2)^2 b_{n-6} + (4q^3 - 3q)(1-q^2)^2 b_{n-7} \\
&\quad + \sum_{k=5}^{n-3} a_k(b_{n-k} - qb_{n-1-k}) + (1-q^2)\sum_{k=4}^{n-4} a_k b_{n-2-k} > 0.
\end{aligned}$$

For the formula (3.4), by (3.3), we have

$$b_n = (a_3b_{n-6} + a_2b_{n-5} + a_1b_{n-4})\sqrt{1-q^2} + a_{n-2}. \tag{3.9}$$

By (3.9),

$$a_{n-1} = a_1a_{n-2} + a_1a_3b_{n-6}\sqrt{1-q^2} + 2q(1-q^2)\sqrt{1-q^2}b_{n-5} + (1-q^2)\sqrt{1-q^2}b_{n-4} > 0.$$

For the formula (3.6), by (3.5) and (3.9), we have

$$\begin{aligned}
b_{n-1} &= (1-q^2)b_{n-3} + 2q(1-q^2)b_{n-4} + (4q^4 - 2q^2 - 1)b_{n-5} \\
&\quad + (4q^3 - 3q)(1-2q^2)b_{n-6} + \sum_{k=4}^{n-3} a_k(b_{n-1-k} - qb_{n-2-k}).
\end{aligned}$$

Since

$$b_{n-3} = a_1b_{n-5}q - a_2b_{n-5} - a_3b_{n-6} + \sum_{k=4}^{n-4} a_k b_{n-3-k} + a_3b_{n-7}q + a_2b_{n-6}q,$$

we have

$$\begin{aligned}
b_{n-1} &= 2q(1-q^2)b_{n-4} + (5q^4 - 4q^2)b_{n-5} + (-6q^5 + 6q^3 - q)b_{n-6} + (1-q^2)a_3qb_{n-7} \\
&\quad + \sum_{k=4}^{n-3} a_k(b_{n-1-k} - qb_{n-2-k}) + (1-q^2)\sum_{k=4}^{n-5} a_k b_{n-3-k} + (1-q^2)a_{n-4}b_1.
\end{aligned}$$

By $a_{n-4} = (a_3b_{n-8} + a_2b_{n-7} + a_1b_{n-6} + b_{n-5})\sqrt{1-q^2}$, we can obtain

$$\begin{aligned}
b_{n-1} &= 2q(1-q^2)b_{n-4} + (6q^4 - 6q^2 + 1)b_{n-5} + (-5q^5 + 4q^3)b_{n-6} \\
&\quad + (1-q^2)(2q^4 - 1)b_{n-7} + (4q^3 - 3q)(1-q^2)^2 b_{n-8} \\
&\quad + \sum_{k=4}^{n-3} a_k(b_{n-1-k} - qb_{n-2-k}) + (1-q^2)\sum_{k=4}^{n-5} a_k b_{n-3-k} > 0.
\end{aligned}$$

From the above statement, we can verify that for $1 \le j \le n$, if $\theta \in \left(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\right)$, then $a_j > 0$ and $b_j > 0$. Consequently, $B \in Q_R(A_{n,4})$ is nilpotent, which completes the proof.

Lemma 3 *If* $\theta\in\left(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\right)$, *then* $\det J=\det\frac{\partial(f_1,g_1,\cdots,f_n,g_n)}{\partial(a_1,b_1,\cdots,a_n,b_n)}\neq 0$.

Proof Let the Jacobian matrix $J = \frac{\partial(f_1,g_1,\cdots,f_n,g_n)}{\partial(a_1,b_1,\cdots,a_n,b_n)}$. By Lemma 1, we assume the $2n \times 2n$ matrix J has the following block form

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where J_{11} is a $(2n-6)\times(2n-6)$ principal submatrix of J; J_{12} is a $(2n-6)\times 6$ submatrix formed by deleting the 1th, the 2th, $\cdots$, the $(2n-6)$th column of the matrix J; J_{21} is a $6\times(2n-6)$ submatrix formed by deleting the 1th, the 2th, $\cdots$, the $(2n-6)$th row of the matrix J; J_{22} is a 6×6 submatrix formed by deleting the 1th, the 2th, $\cdots$, the $(2n-6)$th column and the 1th, the 2th, $\cdots$, the $(2n-6)$th row of the matrix J.

It is easy to see that for $1 \le j \le 3$, $\frac{\partial f_j}{\partial a_j} = 1$ and $\frac{\partial g_j}{\partial b_j} = 1$; for $4 \le j \le n-3$, $\frac{\partial f_j}{\partial a_j} = -1$ and $\frac{\partial g_j}{\partial b_j} = 1$. Thus the matrix J_{11} is a lower triangular matrix with a diagonal entries of n copies of 1s and $n-6$ copies of -1s. It is easy to see that J_{12} is a zero matrix. Furthermore, the matrix

$$J_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -a_1 \\ b_1 & a_1 & 0 & 1 & 0 & -b_1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ b_2 & a_2 & b_1 & a_1 & 0 & -a_1b_1 \end{pmatrix},$$

then

$$\det J=\det J_{11}\det J_{22}=(-1)^{n-6}\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -a_1 \\ b_1 & a_1 & 0 & 1 & 0 & -b_1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ b_2 & a_2 & b_1 & a_1 & 0 & -a_1b_1 \end{vmatrix}=(-1)^{n-7}q\sqrt{1-q^2}\neq 0.$$

Therefore the lemma holds.

By the Nilpotent-Jacobian method on ray patterns and Lemmas 1-3, we have the following theorem.

Theorem 1 *For* $n \ge 8$, *if* $\theta \in \left(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\right)$, *then the ray pattern* $A_{n,4}$ *is spectrally arbitrary.*

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