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# A CANONICAL CONSTRUCTION OF $H^m$-NONCONFORMING TRIANGULAR FINITE ELEMENTS*†

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**Abstract**

We design a family of 2D $H^m$-nonconforming finite elements using the full $P_{2m-3}$ degree polynomial space, for solving $2m$th elliptic partial differential equations. The consistent error is estimated and the optimal order of convergence is proved. Numerical tests on the new elements for solving tri-harmonic, 4-harmonic, 5-harmonic and 6-harmonic equations are presented, to verify the theory.

**Keywords** nonconforming finite element; minimum element; high order partial differential equation

**2000 Mathematics Subject Classification** 65N30; 73C02

## 1 Introduction

For solving $2m$th order elliptic partial differential equations, the finite element spaces are designed as either a subspace of $H^m$ Sobolev space, or not a subspace. In the first case, the finite element is called a conforming element. In the latter case, the finite element is called a non-conforming element. But some continuity is still required for non-conforming finite elements. The Courant triangle, the space of continuous piecewise linear functions, is an $H^1$ conforming finite element, solving second order elliptic equations. The Crouzeix-Raviart triangle, the space of piecewise linear functions continuous at mid-edge points of each triangle, is a $P_1$ $H^1$-nonconforming finite element. The possible minimum polynomial degree is $m$ for an $H^m$ conforming

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and non-conforming finite element. This is because an $m$th order derivative of polynomial degree $m-1$ or less would be zero. Wang and Xu constructed a family of $P_m$ nonconforming finite elements for $2m$th-order elliptic partial differential equations in $R^n$ for any $n \geq m$, on simplicial grids [18]. Such minimum finite elements are very simple compared with the standard conforming elements. For example, in 3D, for $m = 2, 3, 4$ the polynomial degrees of the $H^2$, $H^3$ and $H^4$ elements are 9, 17 and 25, respectively, cf. [1, 2, 20], while those of Wang-Xu's elements are 2, 3 and 4 only, respectively. However, there is a limit that the space dimension $n$ must be no less than the Sobolev space index $m$. For example, Wang and Xu constructed a $P_3$ $H^3$-nonconforming element in 3D [18], but not in 2D.

On rectangular grids, the problem of constructing $H^m$ conforming elements is relatively simple. Hu, Huang and Zhang constructed an $n$-D $C^1$-$Q_2$ element on rectangular grids [10]. Here $Q_k$ means the space of polynomials of separated degree $k$ or less. Then, the element is extended to a whole family of $C^{k-1}$-$Q_k$ elements, i.e., $H^k$-conforming $Q_k$ elements for any space dimension $n$, in [11]. That is, the minimum polynomial degree $k$ $(= m)$ is achieved in constructing $H^m$-conforming finite elements, on rectangular grids for any space dimension $n$. There is no limit of Wang-Xu [18] that $n \geq m$.

It is a challenge to remove the limit $n \geq m$ in the Wang-Xu's work [18], by constructing the minimum degree non-conforming $H^m$ finite elements for the space dimension $n < m$. First, in 2D, we need to construct $H^m$ non-conforming finite elements of polynomial degree $m$ on triangular grids, $m > 2$. This is not possible on general grids. In [12] Hu-Zhang constructed an $H^3$ non-conforming finite element of cubic polynomials, but on the Hsieh-Clough-Tocher macro-triangle grids, following the idea in the construction of $H^m$ conforming elements on macro rectangular grids in [10,11]. In [19], Wu-Xu enriched the $P_3$ polynomial space by 3 $P_4$ bubble functions to obtain a working $H^3$ non-conforming element in 2D. In fact, they extended this technique to $n$ space dimension [19] so that $H^{n+1}$ non-conforming elements in $n$ space dimension is constructed by $P_{n+1}$ polynomials enriched by $n$ $P_{n+2}$ face-bubble functions. In this work, we use the full $P_{2m-3}$ polynomial space for $m \geq 4$ to construct 2D $H^m$ non-conforming elements. For $m = 3 > n = 2$, we have the $P_4$ non-conforming finite element. That is, the new element is of full $P_4$ space, two more degrees of freedom locally than Wu-Xu's element [19].

## 2 Definition of Nonconforming Elements

Let a 2D polygonal domain be triangulated by a quasi-uniform triangular grid of size $h$, $\mathcal{T}_h$. Let $\mathcal{E}_h$ denote the set of edges of $\mathcal{T}_h$, and $\mathcal{E}_h(\Omega)$ denote the set of internal edges. Given $e = K_1 \cap K_2$, the jump and average of a piecewise function $v$ across it are defined as, respectively,

$$[v] := (v|_{K_1})|_e - (v|_{K_2})|_e \quad \text{and} \quad \{v\} := \frac{(v|_{K_1})|_e + (v|_{K_2})|_e}{2}.$$

For any boundary edge $e \subset \partial K$, let

$$[v] := \{v\} := (v|_K)|_e.$$

On each element $K$ of grid $\mathcal{T}_h$, we denote the polynomial space of degree $k$ by $P_k(K)$. For defining an $H^m$-nonconforming element, we need the weak continuity

$$⨍_e [\nabla^{m-1} v] \mathrm{d}s = 0, \tag{2.1}$$

for any function $v$ in the nonconforming finite element space and any internal edge $e$ of $\mathcal{T}_h$. In this paper, $\nabla^m$ is the $m$-th Hessian tensor. For example, $\nabla^1 u = \nabla u$ is the vector gradient, $\nabla^2 u = (\partial_i \partial_j u)$ is the 2-Hessian matrix. A sufficient condition for (2.1) is up to additional possible degrees of freedom for the uni-solvency to take the following degrees of freedom on each element $K$:

- The values of $\nabla^\ell v$, $\ell = 0, \cdots, m-2$, at its three vertices;
- the integral means of $\frac{\partial^{m-1} v}{\partial \mathbf{n}^{m-1}}$ over its three edges.

On one hand, such a set of conditions imposes $3 + 3\frac{m(m-1)}{2}$ degrees of freedom, which requires a minimal degree of the polynomials, say $d(m)$. Note that $d(1) = 1$, $d(2) = 2$, and $d(3) = 4$. On the other hand, on any edge $e$ of element $K$, the restriction of the function $v_h$ is a polynomial with respect to the arc length. This set of conditions in fact imposes all the values of $\frac{\partial^\ell v_h|_e}{\partial \mathbf{t}^\ell}$, $\ell = 0, \cdots, m-2$, at the two endpoints of edge $e$, which determine uniquely a polynomial with respect to the arch length of degree $\leq 2m-3$. It is elementary to show that

$$d(m) \geq 2m - 3 \quad \text{when } m \leq 3$$

and

$$d(m) < 2m - 3 \quad \text{otherwise.}$$

This indicates that the minimal degree of the polynomials should be

$$d(m) = \begin{cases} 1, & m = 1, \\ 2, & m = 2, \\ 4, & m = 3, \\ 2m-3, & m > 3. \end{cases}$$

Therefore, we denote the finite element space on one element $K$ by

$$V_m(K) := \begin{cases} P_1(K), & m=1, \\ P_2(K), & m=2, \\ P_4(K), & m=3, \\ P_{2m-3}(K), & m>3. \end{cases}$$

For $m = 1$, it recovers the celebrated Crouzeix–Raviart element which uses $P_1(K)$ as the shape function space on element $K$ [5]. For $m = 2$ it becomes the simplest nonconforming element for fourth order elliptic problems, namely the Morley element which uses $P_2(K)$ as the shape function space on element $K$ [5]. For $m = 3$, it implies that the recent elements from Wu and Xu [19] and [12] are the simplest $H^3$ nonconforming elements in $2D$ which can not be essentially improved. Here, we propose a new set of degrees of freedom for $P_4(K)$ which yields somehow a new $H^3$ nonconforming element. In the sequel, we propose a set of degrees of freedom for the spaces $V_3(K) = P_4(K)$ and $V_m(K) = P_{2m-3}(K)$ with $m \geq 4$. We define the finite element space by the following 6 cases.

<u>Case 1</u> For $m = 3$, on each element $K \in \mathcal{T}_h$, the degrees of freedom for $P_4(K)$ are as follows,

$$v(\mathbf{x}_i),\ \nabla v(\mathbf{x}_i),\ v(\mathbf{m}_i),\ \text{and}\ \fint_{e_i} \frac{\partial^2 v}{\partial \mathbf{n}_i^2} \mathrm{d}s, \tag{2.2}$$

where $\mathbf{x}_i$ are three vertexes of $K$, $e_i$ are three edges of $K$, and $\mathbf{m}_i$ are three mid-points of the edges $e_i$ of $K$, respectively, (see Figure 1). Here $\mathbf{n}_i$ is the unit normal vector to an edge $e_i$. The degrees of freedom of $V_3(K)$ are plotted in Figure 2. Note that there are $3 \times (1 + 2) + 3 \times (1 + 1) = 15$ dofs, which is the dimension of the 2D $P_4$ polynomial space. The new element is continuous, that is, an $H^1$ conforming element.

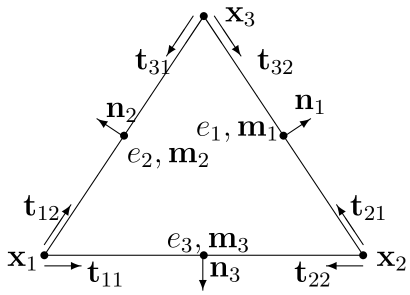


Figure 1: Vertex, edge, mid-point, unit normal vector, unit tangent vectors, of a triangle $K$.

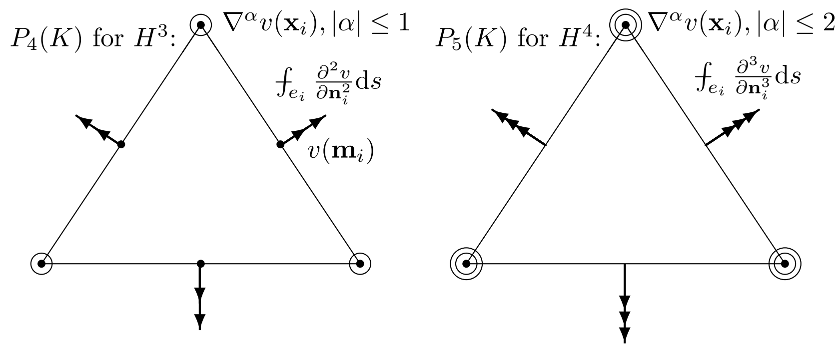


Figure 2: The degrees of freedom for $P_4(K)$ and $P_5(K)$, defined in (2.2) and (2.3).

Case 2 For $m = 4$, on each element $K \in \mathcal{T}_h$, the degrees of freedom for $P_5(K)$ are as follows

$$v(\mathbf{x}_i),\ \nabla v(\mathbf{x}_i),\ \nabla^2 v(\mathbf{x}_i),\ \text{and}\ ⨍_{e_i} \frac{\partial^3 v}{\partial \mathbf{n}_i^3} \mathrm{d}s, \tag{2.3}$$

where $\mathbf{x}_i$ are three vertexes of $K$, $e_i$ are three edges of $K$, and $\mathbf{n}_i$ is the unit normal vector to an edge $e_i$ (see Figure 1). The degrees of freedom of $P_5(K)$ are plotted in Figure 2. We note that the number of linear functionals is $3\times(1+2+3)+3\times(1+1) = 21$ dofs, which is the dimension of the 2D $P_5$ polynomial space. Note that this element is the same as the famous Argyris element [2], except the first order normal derivative of the Argyris element which is replaced by the integral mean of the third order normal derivative. However the element is only an $H^1$ conforming element, not an $H^2$ conforming element.

Case 3 For $m = 5$, the $H^5$ non-conforming element is made by $P_{2m-3} = P_7$ polynomials, which is defined by the following degrees of freedom:

$$\nabla^\alpha v(\mathbf{x}_i),\ |\alpha| \le 3,\ \frac{\partial v(\mathbf{m}_i)}{\partial \mathbf{n}_i},\ ⨍_{e_i} \frac{\partial^4 v}{\partial \mathbf{n}_i^4} \mathrm{d}s, \tag{2.4}$$

where $\mathbf{x}_i$ are three vertices of $K$, $e_i$ are three edges of $K$, $\mathbf{n}_i$ is the unit normal vector to an edge $e_i$ and $\mathbf{m}_i$ is its mid-point. Here the multi-index $\alpha = (i_1, i_2)$ defines the order of mixed derivatives of $v$ at a vertex $\mathbf{x}_i$. Let us count the number of linear functionals,

$$3 \times (1+2+3+4) + 3 \times (1+1) = 30 + 6 = 36 = \dim P_7.$$

In particular, this element is also an $H^2$-conforming element, that is, a $C^1$ element.

Case 4 For $m = 3k+3$, $k = 1, 2, \cdots$, the $H^m$ non-conforming finite element consists of all $P_{2m-3} = P_{6k+3}$ polynomials, whose degrees of freedom are as follows:

$$
\begin{aligned}
&\nabla^\alpha v(\mathbf{x}_i),\ |\alpha|\le 3k+1,\ \fint_{e_i}\frac{\partial^{3k+2}v}{\partial\mathbf{n}_i^{3k+2}}\mathrm{d}s,\\
&\frac{\partial v(\mathbf{m}_i)}{\partial\mathbf{n}_i},\ \frac{\partial^2 v(\mathbf{m}_{i,j,2})}{\partial\mathbf{n}_i^2},\ \cdots,\ \frac{\partial^{m_0}v(\mathbf{m}_{i,j,m_0})}{\partial\mathbf{n}_i^{m_0}},\\
&\frac{\partial^{m_0+1}v(\mathbf{m}_{i,j,m_1})}{\partial\mathbf{n}_i^{m_0+1}},\ \frac{\partial^{m_0+2}v(\mathbf{m}_{i,j,m_1-3})}{\partial\mathbf{n}_i^{m_0+2}},\ \cdots,\ \frac{\partial^{m_0+m_2}v(\mathbf{m}_{i,j,7})}{\partial\mathbf{n}_i^{m_0+m_2}},\\
&\frac{\partial^{4k}v(\mathbf{x}_1)}{\partial\mathbf{t}_2^{2k}\partial\mathbf{t}_3^{2k}},\ \frac{\partial^{4k}v(\mathbf{x}_2)}{\partial\mathbf{t}_3^{2k}\partial\mathbf{t}_1^{2k}},\ \frac{\partial^{4k}v(\mathbf{x}_3)}{\partial\mathbf{t}_1^{2k}\partial\mathbf{t}_2^{2k}},\ \frac{\partial^{4k+1}v(\mathbf{x}_1)}{\partial\mathbf{t}_2^{2k+1}\partial\mathbf{t}_3^{2k}},\ \frac{\partial^{4k+1}v(\mathbf{x}_2)}{\partial\mathbf{t}_3^{2k+1}\partial\mathbf{t}_1^{2k}},\ \frac{\partial^{4k+1}v(\mathbf{x}_3)}{\partial\mathbf{t}_1^{2k+1}\partial\mathbf{t}_2^{2k}},\\
&v\Big(\frac{\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3}{3}\Big),
\end{aligned}
\tag{2.5}
$$

where $\mathbf{t}_i$ is the unit tangent vector in the direction of edge $\mathbf{x}_{i+1}\mathbf{x}_{i+2}$, and $\mathbf{m}_{i,j,l}$, $1\le j\le l$, are $l$ uniformly distributed internal points on edge $e_i$. But when $k=1$, the six tangential derivatives in (2.5) are replaced by 6 internal values. Here $m_0 = [(3k+2)/2]$, which is the integer part of the number, if $k\ge 2$ or else $m_0=0$, namely,

$$
m_0=\begin{cases} 3\ell+1, & \text{if } k=2\ell,\ \ell\ge 1,\\ 3\ell+2, & \text{if } k=2\ell+1,\ \ell\ge 1,\\ 0, & \text{otherwise,}\end{cases}
$$

$$
m_1=\begin{cases} 3l-2, & \text{if } k=2\ell,\ \ell\ge 1,\\ 3l+1, & \text{if } k=2\ell+1,\ \ell\ge 1,\\ 0, & \text{otherwise,}\end{cases}
$$

and

$$
m_2=\begin{cases} l-2, & \text{if } k=2\ell,\ \ell\ge 1,\\ l-1, & \text{if } k=2\ell+1,\ \ell\ge 1,\\ 0, & \text{otherwise.}\end{cases}
$$

That is, we first fill the missing dofs on each edge to make function $v$ $C^1$, $C^2$, and so on until $C^{m_0}$ (if $k\ge 2$), which implies that we add one 1st normal derivative, two 2nd normal derivatives, and so on until $m_0$ $m_0$-th normal derivatives. The maximum level of added full normal derivatives is $m_0$. After that, we can add some high order normal derivatives, due to the constraint of adding higher order normal derivatives on the two other edges. So the number of higher normal derivatives is reduced by 3 each level, until reaching 7. By this time, the number of undefined dofs is exactly 7. Consequently, in this case, we always have 7 internal degrees of freedom (independent of dofs on neighboring triangles), which are imposed by six tangential derivatives at three vertices and one value at the center. We depict the dofs of the element when $k=1$ in Figure 3.

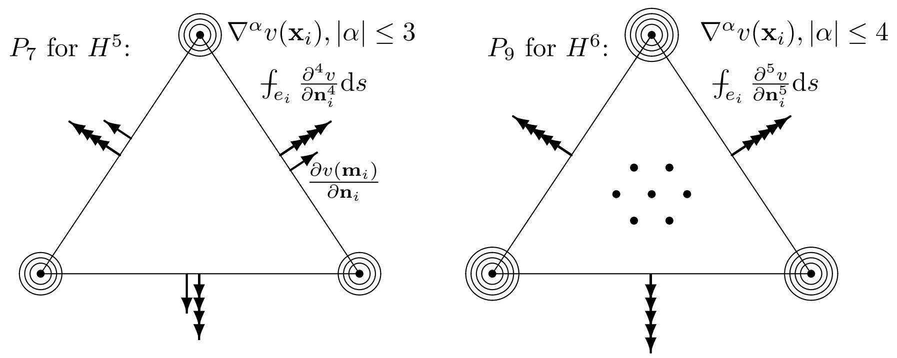


Figure 3: The degrees of freedom for $P_7$ and $P_9$ defined in (2.4) and (2.5) respectively.

<u>Case 5</u> For $m = 3k + 4$, $k = 1, 2, \cdots$, the $H^m$ non-conforming finite element consists of all $P_{2m-3} = P_{6k+5}$ polynomials, whose degrees of freedom are as follows:

$$\begin{aligned}
&\nabla^\alpha v(\mathbf{x}_i),\ |\alpha| \le 3k+2,\ ⨍_{e_i} \frac{\partial^{3k+3} v}{\partial \mathbf{n}_i^{3k+3}} \mathrm{d}s,\\
&\frac{\partial v(\mathbf{m}_i)}{\partial \mathbf{n}_i},\ \frac{\partial^2 v(\mathbf{m}_{i,j,2})}{\partial \mathbf{n}_i^2},\ \cdots,\ \frac{\partial^{m_0} v(\mathbf{m}_{i,j,m_0})}{\partial \mathbf{n}_i^{m_0}},\\
&\frac{\partial^{m_0+1} v(\mathbf{m}_{i,j,m_1})}{\partial \mathbf{n}_i^{m_0+1}},\ \frac{\partial^{m_0+2} v(\mathbf{m}_{i,j,m_1-3})}{\partial \mathbf{n}_i^{m_0+2}},\ \cdots,\ \frac{\partial^{m_0+m_2} v(\mathbf{m}_{i,j,6})}{\partial \mathbf{n}_i^{m_0+m_2}},\\
&\frac{\partial^{2(m_0+m_2+1)} v(\mathbf{x}_l)}{\partial \mathbf{n}_i^{m_0+m_2+1} \partial \mathbf{n}_j^{m_0+m_2+1}} \text{ with } (i,j,l) \text{ permutations of } (1,2,3),
\end{aligned} \tag{2.6}$$

where $\mathbf{m}_{i,j,l}$, $1 \le j \le l$, are $l$ uniformly distributed internal points on edge $e_i$. Here $m_0 = [(3k+2)/2]$, the integer part of number, namely,

$$m_0 = \begin{cases} 3\ell+1, & \text{if } k = 2\ell,\ \ell \ge 1,\\ 3\ell+2, & \text{if } k = 2\ell+1,\ \ell \ge 1,\\ 0, & \text{otherwise,} \end{cases}$$
$$m_1 = \begin{cases} 3l, & \text{if } k = 2\ell,\ \ell \ge 1,\\ 3l+3, & \text{if } k = 2\ell+1,\ \ell \ge 1,\\ 0, & \text{otherwise,} \end{cases}$$

and

$$m_2 = \begin{cases} l-1, & \text{if } k = 2\ell,\ \ell \ge 1,\\ l, & \text{if } k = 2\ell+1,\ \ell \ge 1,\\ 0, & \text{otherwise.} \end{cases}$$

That is, we first fill the missing dofs on each edge to make function $v$ be $C^1$, $C^2$, and so on until $C^{m_0}$, namely, we add one 1st normal derivative, two 2nd normal derivatives, and so on until $m_0$ $m_0$-th normal derivatives. After that, we can only add some high order normal derivatives, due to the constraint of adding higher order normal derivatives on the two other edges. So the number of higher normal derivatives is reduced by 3 each level, until reaching 6 on each edge. By this time, the number of undefined dofs is exactly 3. That is, in this case, we always have 3 internal degrees of freedom (independent of dofs on neighboring triangles), which can be determined by three higher order derivatives at three vertices $\frac{\partial^{2(m_0+m_2+1)}v(\mathbf{x}_l)}{\partial\mathbf{n}_i^{m_0+m_2+1}\partial\mathbf{n}_j^{m_0+m_2+1}}$ with $(i,j,l)$ permutations of $(1,2,3)$.

We depict the dofs of the element when $k=1$, that is, $P_{11}(K)$, in Figure 4.

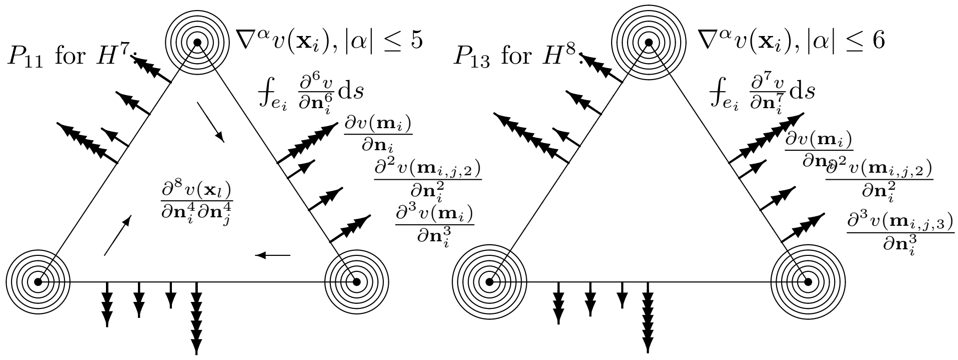


Figure 4: The degrees of freedom for $P_{11}(K)$ and $P_{13}(K)$ defined in (2.6) and (2.7).

Case 6 For $m=3k+5$, $k=1,2,\cdots$, the $H^m$ non-conforming finite element consists of all $P_{2m-3}=P_{6k+7}$ polynomials, whose degrees of freedom are as follows:

$$\begin{aligned}&\nabla^\alpha v(\mathbf{x}_i),\ |\alpha|\le 3k+3,\ \fint_{e_i}\frac{\partial^{3k+4}v}{\partial\mathbf{n}_i^{3k+4}}\mathrm{d}s,\\ &\frac{\partial v(\mathbf{m}_i)}{\partial\mathbf{n}_i},\ \frac{\partial^2 v(\mathbf{m}_{i,j,2})}{\partial\mathbf{n}_i^2},\ \cdots,\ \frac{\partial^{m_0}v(\mathbf{m}_{i,j,m_0})}{\partial\mathbf{n}_i^{m_0}},\\ &\frac{\partial^{m_0+1}v(\mathbf{m}_{i,j,m_1})}{\partial\mathbf{n}_i^{m_0+1}},\ \frac{\partial^{m_0+2}v(\mathbf{m}_{i,j,m_1-3})}{\partial\mathbf{n}_i^{m_0+2}},\ \cdots,\ \frac{\partial^{m_0+m_2}v(\mathbf{m}_{i,j,5})}{\partial\mathbf{n}_i^{m_0+m_2}},\end{aligned}\tag{2.7}$$

where $\mathbf{m}_{i,j,l}$, $1\le j\le l$, are $l$ uniformly distributed internal points on edge $e_i$. Here $m_0=[(3k+3)/2]$, which is the integer part of number, namely,

$$m_0=\begin{cases}3\ell+1, & \text{if } k=2\ell,\ \ell\ge 1,\\ 3\ell+3, & \text{if } k=2\ell+1,\ \ell\ge 1,\\ 0, & \text{otherwise},\end{cases}$$
$$m_1=\begin{cases}3l+2, & \text{if } k=2\ell,\ \ell\ge 1,\\ 3l+2, & \text{if } k=2\ell+1,\ \ell\ge 1,\\ 0, & \text{otherwise},\end{cases}$$

and

$$m_2 = \begin{cases} l, & \text{if } k = 2\ell,\ \ell \geq 1, \\ l, & \text{if } k = 2\ell + 1,\ \ell \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

That is, we first fill the missing dofs on each edge to make function $v$ $C^1$, $C^2$, and so on until $C^{m_0}$. After that, we can only add some high order normal derivatives, due to the constraint of adding higher order normal derivatives on the two other edges. So the number of higher normal derivatives is reduced by 3 each level, until reaching 5 on each edge. By this time, the number of undefined dofs is exactly 0. We depict the dofs of the element when $k = 1$, that is, $P_{13}(K)$, in Figure 4.

The global $H^m$ non-conforming finite element space is defined by

$$\begin{aligned} V_m(\mathcal{T}_h) := \{v \in L^2(\Omega) |\ v|_K \in V_m(K) \text{ for any } K \in \mathcal{T}_h, \text{the} \\ \text{inter-element dofs (on neighboring elements)} \\ \text{have same values, the boundary dofs take value } 0\}, \end{aligned} \tag{2.8}$$

where $V_m(K)$ are defined in (2.2)-(2.7).

For the $m$-harmonic equations:

$$\begin{aligned} (-\Delta)^m u_m = f \quad & \text{in } \Omega, \\ \frac{\partial^\ell u_m}{\partial \mathbf{n}^\ell} = 0 \qquad & \text{on } \partial\Omega,\ \ell = 0, 1, \cdots, m-1, \end{aligned} \tag{2.9}$$

the finite element approximation problem is: Find $u_{m,h} \in V_m(\mathcal{T}_h)$ such that

$$(\nabla_h^m u_{m,h}, \nabla_h^m v) = (f, v) \quad \text{for any } v \in V_m(\mathcal{T}_h), \tag{2.10}$$

where $\nabla_h^m$ is the discrete $m$-th Hessian tensor which is defined elementwise.

# 3 Well-posedness of Non-conforming Element

**Lemma 3.1** *The finite element functions in the space $V_m(K)$ are uniquely defined by the specified degrees of freedom.*

**Proof** The proof for $V_1(K)$ and $V_2(K)$ can be found in [18]. We also skip the proof for $V_3(K)$, $V_4(K)$ and $V_5(K)$ since it is similar to those of the high order cases. So we have three cases, $m = 3k+5, 3k+4$, and $m = 3k+3$ for $H^m$ nonconforming elements. For each case, we will show the square linear system of equations, with homogeneous right-hand side, has a unique solution $v = 0$. This ensures the existence and the uniqueness of the finite element functions.

For the first case, $m = 3k+5$, cf. degrees of freedom in (2.7), by the vertex and edge (low-order normal derivatives) degrees of freedom, we have

$$v = Bp_1, \quad \text{where } B = \lambda_1^r \lambda_2^r \lambda_3^r \text{ and } p_1 = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3, \tag{3.1}$$

where $r = m_0 + m_2 + 1$ with $m_0$ and $m_2$ being defined in (2.7), and $\lambda_i$ being barycenter coordinates of the triangle. We shall prove that the parameters $a_i$ are zero. It follows from the definition of the barycenter coordinates that

$$\begin{aligned}\nabla v &= r\Big(\frac{\mathbf{n}_1}{h_1}\lambda_1^{r-1}\lambda_2^r\lambda_3^r + \frac{\mathbf{n}_2}{h_2}\lambda_1^r\lambda_2^{r-1}\lambda_3^r + \frac{\mathbf{n}_3}{h_3}\lambda_1^r\lambda_2^r\lambda_3^{r-1}\Big)\\ &\quad\times (a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3) + \lambda_1^r\lambda_2^r\lambda_3^r\Big(a_1\frac{\mathbf{n}_1}{h_1} + a_2\frac{\mathbf{n}_2}{h_2} + a_3\frac{\mathbf{n}_3}{h_3}\Big),\\ \frac{\partial v}{\partial \mathbf{n}_1} &= \Big(\frac{r}{h_1}\lambda_1^{r-1}\lambda_2^r\lambda_3^r + \frac{rc_{12}}{h_2}\lambda_1^r\lambda_2^{r-1}\lambda_3^r + \frac{rc_{13}}{h_3}\lambda_1^r\lambda_2^r\lambda_3^{r-1}\Big)\\ &\quad\times (a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3) + \lambda_1^r\lambda_2^r\lambda_3^r\Big(a_1\frac{1}{h_1} + a_2\frac{c_{12}}{h_2} + a_3\frac{c_{13}}{h_3}\Big),\end{aligned}$$

where $h_i$ is the height of the triangle from vertex $v_i$ to the opposite edge $e_i$, and

$$c_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j.$$

Note that the total degree of polynomial is $2m - 3 = 2(3k + 5) - 3 = 6k + 7$. We compute the $m - 1$(st) normal derivative of $v$ on edge $e_1$, where $m - 1 = r + l$, $r = 2k + 2$ is an even number and $l = k + 2$. When restricted on the edge $e_1$, any term containing $\lambda_1$ would vanish. Therefore, the $r$-th normal derivative must be on the term $\lambda_1^r$, and the rest $l$-th normal derivative is on the other terms.

$$\begin{aligned}\frac{\partial^{r+l}v}{\partial\mathbf{n}_1^{r+l}}\Big|_{e_1} &= p_1\frac{\partial^{r+l}B}{\partial\mathbf{n}_1^{r+l}}\Big|_{e_1} + (r+l)\frac{\partial^{r+l-1}B}{\partial\mathbf{n}_1^{r+l-1}}\frac{\partial p_1}{\partial\mathbf{n}_1}\Big|_{e_1}\\ &= p_1\binom{r+l}{r}\frac{\partial^r\lambda_1^r}{\partial\mathbf{n}_1^r}\frac{\partial^l(\lambda_2\lambda_3)^r}{\partial\mathbf{n}_1^l}\Big|_{e_1} + (r+l)\binom{r+l-1}{r}\frac{\partial^r\lambda_1^r}{\partial\mathbf{n}_1^r}\frac{\partial^{l-1}(\lambda_2\lambda_3)^r}{\partial\mathbf{n}_1^{l-1}}\frac{\partial p_1}{\partial\mathbf{n}_1}\Big|_{e_1}.\end{aligned}$$

This leads to

$$\begin{aligned}\frac{\partial^{r+l}v}{\partial\mathbf{n}_1^{r+l}}\Big|_{e_1} &= \frac{r!}{h_1^r}(a_2\lambda_2 + a_3\lambda_3)\binom{r+l}{r}\\ &\quad\times\sum_{i=0}^{l}\binom{l}{i}\frac{r!}{(r-i)!}\lambda_2^{r-i}\Big(\frac{c_{12}}{h_2}\Big)^i\frac{r!}{(r-l+i)!}\lambda_3^{r-l+i}\Big(\frac{c_{13}}{h_3}\Big)^{l-i}\Big|_{e_1}\\ &\quad+(r+l)\Big(a_1\frac{1}{h_1} + a_2\frac{c_{12}}{h_2} + a_3\frac{c_{13}}{h_3}\Big)\frac{r!}{h_1^r}\binom{r+l-1}{r}\\ &\quad\times\sum_{i=0}^{l-1}\binom{l-1}{i}\frac{r!}{(r-i)!}\lambda_2^{r-i}\Big(\frac{c_{12}}{h_2}\Big)^i\frac{r!}{(r-l+i+1)!}\lambda_3^{r-l+i+1}\Big(\frac{c_{13}}{h_3}\Big)^{l-i-1}\Big|_{e_1}.\end{aligned}$$

Note that

$$\int_0^1 x^m(1-x)^n\mathrm{d}x = \frac{m!n!}{(m+n+1)!}. \qquad (3.2)$$

Therefore,

$$
\begin{aligned}
0 &= \int_{e_1} \frac{\partial^{r+l} v}{\partial \mathbf{n}_1^{r+l}} \mathrm{d}s \\
&= \frac{r!}{h_1^r} \binom{r+1}{r} \sum_{i=0}^{l} \binom{l}{i} \frac{r!}{(r-i)!} \frac{r!}{(r-l+i)!} \Big(\frac{c_{12}}{h_2}\Big)^i \Big(\frac{c_{13}}{h_3}\Big)^{l-i} \\
&\quad \times \Big(a_2 \frac{s_1 (r-i+1)!(r-l+i)!}{(2r-l+2)!} + a_3 \frac{s_1 (r-i)!(r-l+i+1)!}{(2r-l+2)!}\Big) \\
&\quad + \frac{r!}{h_1^r} \binom{r+1}{r} l \Big(a_1 \frac{1}{h_1} + a_2 \frac{c_{12}}{h_2} + a_3 \frac{c_{13}}{h_3}\Big) \\
&\quad \times \sum_{i=0}^{l-1} \binom{l-1}{i} \frac{r!}{(r-i)!} \frac{r!}{(r-l+i+1)!} \Big(\frac{c_{12}}{h_2}\Big)^i \Big(\frac{c_{13}}{h_3}\Big)^{l-i-1} \frac{s_1 (r-i)!(r-l+i+1)!}{(2r-l+2)!},
\end{aligned}
$$

where $s_1$ is the length of the edge $e_1$. This yields

$$
\begin{aligned}
0 = &\sum_{i=0}^{l} \binom{l}{i} \Big(\frac{c_{12}}{h_2}\Big)^i \Big(\frac{c_{13}}{h_3}\Big)^{l-i} \big(a_2(r-i+1) + a_3(r-l+i+1)\big) \\
&+ l \Big(a_1 \frac{1}{h_1} + a_2 \frac{c_{12}}{h_2} + a_3 \frac{c_{13}}{h_3}\Big) \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big)^{l-1}.
\end{aligned}
$$

Further, by the equation

$$
\frac{\mathrm{d}}{\mathrm{d}x} x(x+c)^l = \sum_{i=0}^{l} \binom{l}{i} (i+1) x^i c^{l-i},
$$

we get

$$
\begin{aligned}
0 &= \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big)^{l-1} \Big[a_2(r-l+1)\Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big) + a_2 l \frac{c_{13}}{h_3} + a_3(r-l+1)\Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big) \\
&\quad + a_3 l \frac{c_{12}}{h_2} + l\Big(a_1 \frac{1}{h_1} + a_2 \frac{c_{12}}{h_2} + a_3 \frac{c_{13}}{h_3}\Big)\Big] \\
&= \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big)^{l-1} \Big\{\frac{a_1}{h_1} l + a_2(r+1)\Big[\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big] + a_3(r+1)\Big[\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big]\Big\}.
\end{aligned}
$$

We derive the equation

$$
\frac{a_1}{h_1} l + a_2(r+1)\Big[\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big] + a_3(r+1)\Big[\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big] = 0.
$$

Multiplying the equation by the twice area of triangle $K$, noting that $|K| = s_i h_i/2$, $i = 1, 2, 3$, it follows

$$
l a_1 s_1 + (r+1) a_2 (c_{12} s_2 + c_{13} s_3) + (r+1) a_3 (c_{12} s_2 + c_{13} s_3) = 0.
$$

Noting that

$$
c_{12} s_2 + c_{13} s_3 = \mathbf{n}_1 \cdot s_2 \mathbf{n}_2 + \mathbf{n}_1 \cdot s_3 \mathbf{n}_3 = -s_2 \cos(\theta_{12}) - s_3 \cos(\theta_{13}) = -s_1,
$$

where $\theta_{ij}$ is the angle between edge $e_i$ and $e_j$, we get

$$la_1 - (r+1)a_2 - (r+1)a_3 = 0.$$

Symmetrically,

$$la_2 - (r+1)a_1 - (r+1)a_3 = 0, \quad la_3 - (r+1)a_1 - (r+1)a_2 = 0.$$

Adding the three equations, we obtain

$$(l - 2(r+1))(a_1 + a_2 + a_3) = -(3k+4)(a_1 + a_2 + a_3) = 0,$$

that it follows

$$a_1 + a_2 + a_3 = 0.$$

Combing this equation with the first equation above, we obtain

$$(r+1+l)a_1 = 0, \quad a_1 = 0.$$

Thus $a_i = 0$, $p_1 = 0$, and the unique solution $v = 0$.

We study next the second case $m = 3k+4$, cf. (2.6). In this case, we have a $P_2$ internal polynomial in $v$ after factoring out the boundary factors. That is, when $v$ satisfies the homogeneous vertex and low-order normal derivative conditions,

$$v = Bp_2, \tag{3.3}$$

where $B = \lambda_1^r\lambda_2^r\lambda_3^r$ and $p_2 = a_1\lambda_2\lambda_3 + a_2\lambda_3\lambda_1 + a_3\lambda_1\lambda_2 + a_4\lambda_1^2 + a_5\lambda_2^2 + a_6\lambda_3^2$. Again, here $r = m_0 + m_2 + 1$ with $m_0$ and $m_2$ being defined in (2.6). Next, we show that $a_i = 0$, $i = 1, \cdots, 6$. By the condition

$$0 = \frac{\partial^{2r} v(\mathbf{x}_3)}{\partial \mathbf{n}_1^r \partial \mathbf{n}_2^r} = \frac{r!}{h_1^r}\frac{r!}{h_2^r}\sum_{i=0}^{r}\binom{r}{i}^2 c_{12}^{2(r-i)}(\lambda_3^r p_2)(\mathbf{x}_3) + 0 = \frac{r!}{h_1^r}\frac{r!}{h_2^r}\sum_{i=0}^{r}\binom{r}{i}^2 c_{12}^{2(r-i)} a_6,$$

where the rest terms contain at least one factor of $\lambda_1$ or $\lambda_2$, we get $a_6 = 0$. Symmetrically, we derive

$$v = \lambda_1^r\lambda_2^r\lambda_3^r(a_1\lambda_2\lambda_3 + a_2\lambda_3\lambda_1 + a_3\lambda_1\lambda_2).$$

Consider the $m-1$(st) normal derivative on an edge, $m - 1 = r + 1 + l$, $r = 2k+1$ and $l = k+1$,

$$\begin{aligned}\left.\frac{\partial^{r+l+1} v}{\partial \mathbf{n}_1^{r+1+l}}\right|_{e_1} = \left.\frac{\partial^{3l} v}{\partial \mathbf{n}_1^{3l}}\right|_{e_1} &= a_3\binom{3l}{l}\frac{\partial^{2l}\lambda_1^{2l}}{\partial \mathbf{n}_1^{2l}}\left.\frac{\partial^{l}(\lambda_2^{2l}\lambda_3^{2l-1})}{\partial \mathbf{n}_1^{l}}\right|_{e_1} + a_2\binom{3l}{l}\frac{\partial^{2l}\lambda_1^{2l}}{\partial \mathbf{n}_1^{2l}}\left.\frac{\partial^{l}(\lambda_2^{2l-1}\lambda_3^{2l})}{\partial \mathbf{n}_1^{l}}\right|_{e_1} \\ &\quad + a_1\binom{3l}{l+1}\frac{\partial^{2l-1}\lambda_1^{2l-1}}{\partial \mathbf{n}_1^{2l-1}}\left.\frac{\partial^{l+1}(\lambda_2^{2l}\lambda_3^{2l})}{\partial \mathbf{n}_1^{l+1}}\right|_{e_1}.\end{aligned}$$

This leads to

$$\frac{\partial^{r+l+1}v}{\partial \mathbf{n}_1^{r+1+l}}\Big|_{e_1} = a_3\binom{3l}{l}\frac{(2l)!}{h_1^{2l}}\sum_{i=0}^{l}\binom{l}{i}\frac{(2l)!c_{12}^i}{(2l-i)!h_2^i}\lambda_2^{2l-i}\frac{(2l-1)!c_{13}^{l-i}}{(l+i-1)!h_3^{l-i}}\lambda_3^{l+i-1}\Big|_{e_1}$$
$$+a_2\binom{3l}{l}\frac{(2l)!}{h_1^{2l}}\sum_{i=0}^{l}\binom{l}{i}\frac{(2l)!c_{13}^i}{(2l-i)!h_3^i}\lambda_3^{2l-i}\frac{(2l-1)!c_{12}^{l-i}}{(l+i-1)!h_2^{l-i}}\lambda_2^{l+i-1}\Big|_{e_1}$$
$$+a_1\binom{3l}{l+1}\frac{(2l-1)!}{h_1^{2l-1}}\sum_{i=0}^{l+1}\binom{l+1}{i}\frac{(2l)!c_{12}^i}{(2l-i)!h_2^i}\lambda_2^{2l-i}\frac{(2l)!c_{13}^{l+1-i}}{(l+i-1)!h_3^{l+1-i}}\lambda_3^{l+i-1}\Big|_{e_1}.$$

So, by the Euler formula (3.2),

$$0 = \int_{e_1}\frac{\partial^{r+l+1}v}{\partial \mathbf{n}_1^{r+1+l}}\mathrm{d}s$$
$$= \binom{3l}{l}\frac{(2l)!}{h_1^{2l}}s_1\Bigg[a_3\sum_{i=0}^{l}\binom{l}{i}\frac{(2l)!c_{12}^i(2l-1)!c_{13}^{l-i}}{(3l)!h_2^ih_3^{l-i}} + a_2\sum_{i=0}^{l}\binom{l}{i}\frac{(2l)!c_{13}^i(2l-1)!c_{12}^{l-i}}{(3l)!h_3^ih_2^{l-i}}$$
$$+a_1\frac{h_1}{l+1}\sum_{i=0}^{l+1}\binom{l+1}{i}\frac{(2l)!c_{12}^i(2l)!c_{13}^{l+1-i}}{(3l)!h_2^ih_3^{l+1-i}}\Bigg].$$

Consequently

$$0 = \binom{3l}{l}\frac{((2l)!)^2(2l-1)!}{(3l)!h_1^{2l}}s_1\Big[a_3\Big(\frac{c_{12}}{h_2}+\frac{c_{13}}{h_3}\Big)^l+a_2\Big(\frac{c_{12}}{h_2}+\frac{c_{13}}{h_3}\Big)^l+a_1\frac{2lh_1}{l+1}\Big(\frac{c_{12}}{h_2}+\frac{c_{13}}{h_3}\Big)^{l+1}\Big].$$

Noting that $\frac{c_{12}}{h_2}+\frac{c_{13}}{h_3} = -\frac{s_1}{2|K|}$,

$$0 = a_3 + a_2 - a_1\frac{2lh_1}{l+1}\frac{s_1}{2|K|} = a_3 + a_2 - a_1\frac{2l}{l+1}.$$

Symmetrically, we get two other equations. Adding these three equations, we obtain

$$\frac{2}{l+1}(a_1+a_2+a_3) = 0.$$

Subtracting this equation from the above equation, we derive

$$a_1 = 0, \quad \text{and symmetrically, } a_2 = a_3 = 0.$$

So $v \equiv 0$.

For the third case, $m = 3k+3$, cf. (2.5). Similar to the previous two cases, instead of a $P_1$ or a $P_2$ internal polynomial, after setting low-order boundary/inter-element degrees of freedom to zero, we have a $P_3$ internal polynomial that

$$v = \lambda_1^r\lambda_2^r\lambda_3^r p_3, \tag{3.4}$$

where $r = m_0+m_2+1$ with $m_0$ and $m_2$ being defined in (2.5), and $p_3$ being a degree 3 polynomial in $\lambda_1$, $\lambda_2$, and $\lambda_3$,

$$p_3 = \sum_{i+j+k=3} a_{ijk}\lambda_1^i\lambda_2^j\lambda_3^j.$$

Here we use the standard homogeneous polynomial basis. Now we apply an internal degree of freedom, a high-order tangential derivative, to get (with notations $s_i = \|\mathbf{x}_i - \mathbf{x}_j\|$, $\mathbf{t}_k = (\mathbf{x}_i - \mathbf{x}_j)/s_i$, $i = 1, 2, 3$, $j = \mathrm{mod}(i, 3) + 1$, $k = \mathrm{mod}(j, 3) + 1$)

$$\frac{\partial^{2r} v(\mathbf{x}_1)}{\partial \mathbf{t}_2^r \partial \mathbf{t}_3^r} = \frac{r!r!}{(-s_3)^r s_2^r} a_{300} = 0, \quad \text{which implied} \quad a_{300} = 0,$$

because $\partial_{\mathbf{t}_2}\lambda_2^r = 0$, $\partial^i_{\mathbf{t}_2^i}\lambda_3^r(\mathbf{x}_1) = 0$ if $i < r$. Repeating this calculation for the other two $2r$-order partial derivatives, we get $a_{030} = a_{003} = 0$. For the $2r + 1$ order tangential derivative, we have

$$\frac{\partial^{2r+1} v(\mathbf{x}_1)}{\partial \mathbf{t}_2^{r+1} \partial \mathbf{t}_3^r} = \frac{(r+1)!r!}{(-s_3)^r s_2^{r+1}} a_{201} = 0, \quad \text{which implied} \quad a_{201} = 0.$$

Computing the other two $2(r + 1)$-order partial derivatives, we get $a_{120} = a_{012} = 0$. Therefore, we have only four non-zero terms,

$$v = \lambda_1^r\lambda_2^r\lambda_3^r(a_{102}\lambda_1\lambda_3^2 + a_{210}\lambda_1^2\lambda_2 + a_{021}\lambda_2^2\lambda_3 + a_{111}\lambda_1\lambda_2\lambda_3).$$

Consider the $m - 1$(st) normal derivative on an edge, $m - 1 = r + l$, $r = 2k$ and $l = k + 3$,

$$\begin{aligned}
&\frac{\partial^{r+l} v}{\partial \mathbf{n}_1^{r+l}}\Big|_{e_1} \\
&= a_{102}\binom{r+1}{l-1}\frac{\partial^{r+1}\lambda_1^{r+1}}{\partial \mathbf{n}_1^{r+1}}\frac{\partial^{l-1}(\lambda_2^r\lambda_3^{r+2})}{\partial \mathbf{n}_1^{l-1}}\Big|_{e_1} + a_{210}\binom{r+1}{l-2}\frac{\partial^{r+2}\lambda_1^{r+2}}{\partial \mathbf{n}_1^{r+2}}\frac{\partial^{l-2}(\lambda_2^{r+1}\lambda_3^{r})}{\partial \mathbf{n}_1^{l-2}}\Big|_{e_1} \\
&\quad + a_{021}\binom{r+1}{l}\frac{\partial^{r}\lambda_1^{r}}{\partial \mathbf{n}_1^{r}}\frac{\partial^{l}(\lambda_2^{r+2}\lambda_3^{r+1})}{\partial \mathbf{n}_1^{l}}\Big|_{e_1} + a_{111}\binom{r+1}{l-1}\frac{\partial^{r+1}\lambda_1^{r+1}}{\partial \mathbf{n}_1^{r+1}}\frac{\partial^{l-1}(\lambda_2^{r+1}\lambda_3^{r+1})}{\partial \mathbf{n}_1^{l-1}}\Big|_{e_1} \\
&= a_{102}\binom{r+1}{l-1}\frac{(r+1)!}{h_1^{r+1}}\sum_{i=0}^{l-1}\binom{l-1}{i}\frac{r!\lambda_2^{r-i}c_{12}^i}{(r-i)!h_2^i}\frac{(r+2)!\lambda_3^{r-l+3+i}c_{13}^{l-1-i}}{(r-l+3+i)!h_3^{l-1-i}}\Big|_{e_1} \\
&\quad + a_{210}\binom{r+1}{l-2}\frac{(r+2)!}{h_1^{r+2}}\sum_{i=0}^{l-2}\binom{l-2}{i}\frac{(r+1)!\lambda_2^{r+1-i}c_{12}^i}{(r+1-i)!h_2^i}\frac{r!\lambda_3^{r-l+2+i}c_{13}^{l-2-i}}{(r-l+2+i)!h_3^{l-2-i}}\Big|_{e_1} \\
&\quad + a_{021}\binom{r+1}{l}\frac{r!}{h_1^{r}}\sum_{i=0}^{l}\binom{l}{i}\frac{(r+2)!\lambda_2^{r+2-i}c_{12}^i}{(r+2-i)!h_2^i}\frac{(r+1)!\lambda_3^{r-l+1+i}c_{13}^{l-i}}{(r-l+1+i)!h_3^{l-i}}\Big|_{e_1} \\
&\quad + a_{111}\binom{r+1}{l-1}\frac{(r+1)!}{h_1^{r+1}}\sum_{i=0}^{l-1}\binom{l-1}{i}\frac{(r+1)!\lambda_2^{r+1-i}c_{12}^i}{(r+1-i)!h_2^i}\frac{(r+1)!\lambda_3^{r-l+2+i}c_{13}^{l-1-i}}{(r-l+2+i)!h_3^{l-1-i}}\Big|_{e_1}.
\end{aligned}$$

By the Euler formula (3.2),

$$
\begin{aligned}
0 =&\int_{e_1} \frac{\partial^{r+l} v}{\partial \mathbf{n}_1^{r+l}} \mathrm{d}s = s_1 a_{102} \binom{r+1}{l-1} \frac{(r+1)!}{h_1^{r+1}} \sum_{i=0}^{l-1} \binom{l-1}{i} \frac{r! c_{12}^i}{h_2^i} \frac{(r+2)! c_{13}^{l-1-i}}{(2r-l+4)! h_3^{l-1-i}} \\
&+ s_1 a_{210} \binom{r+1}{l-2} \frac{(r+2)!}{h_1^{r+2}} \sum_{i=0}^{l-2} \binom{l-2}{i} \frac{(r+1)! c_{12}^i}{h_2^i} \frac{r! c_{13}^{l-2-i}}{(2r-l+4)! h_3^{l-2-i}} \\
&+ s_1 a_{021} \binom{r+1}{l} \frac{r!}{h_1^r} \sum_{i=0}^{l} \binom{l}{i} \frac{(r+2)! c_{12}^i}{h_2^i} \frac{(r+1)! c_{13}^{l-i}}{(2r-l+4)! h_3^{l-i}} \\
&+ s_1 a_{111} \binom{r+1}{l-1} \frac{(r+1)!}{h_1^{r+1}} \sum_{i=0}^{l-1} \binom{l-1}{i} \frac{(r+1)! c_{12}^i}{h_2^i} \frac{(r+1)! c_{13}^{l-1-i}}{(2r-l+4)!} \bigg|_{e_1}.
\end{aligned}
$$

That is

$$
\begin{aligned}
0 = &\, s_1 \binom{r+1}{l-1} \frac{(r!)^3}{h_1^r (2r-l+4)!} \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big)^{l-2} \\
&\times \Big[ a_{102} \frac{(r+1)^2 (r+2)}{h_1} \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big) + a_{210} \frac{(l-1)(r+1)^2}{h_1^2} \\
&+ a_{021} \frac{(r+1)^3 (r+2)}{l} \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big)^2 + a_{111} \frac{(r+1)^3}{h_1} \Big(\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3}\Big) \Big].
\end{aligned}
$$

By $\frac{c_{12}}{h_2} + \frac{c_{13}}{h_3} = -\frac{s_1}{2|K|}$,

$$
-a_{102}(r+2) + a_{210}(l-1) + a_{021} \frac{(r+1)(r+2)}{l} - a_{111}(r+1) = 0.
$$

Symmetrically, we get two other equations,

$$
-a_{210}(r+2) + a_{021}(l-1) + a_{102} \frac{(r+1)(r+2)}{l} - a_{111}(r+1) = 0, \quad (3.5)
$$

$$
-a_{021}(r+2) + a_{102}(l-1) + a_{210} \frac{(r+1)(r+2)}{l} - a_{111}(r+1) = 0. \quad (3.6)
$$

By the barycenter value, we get a 4th equation,

$$
v\Big(\frac{\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3}{3}\Big) = \frac{1}{3^{3r+3}} (a_{021} + a_{102} + a_{210} + a_{111}) = 0. \quad (3.7)
$$

Adding above four equations, as $r = 2k$ and $l = k+3$, we obtain

$$
a_{021} = a_{102} = a_{210} = \frac{(r+1)l}{(r+1)(r+2) - l(r+2) + l(l-1)} a_{111}.
$$

By (3.7),

$$
a_{111} = 0, \quad \text{and} \quad a_{021} = a_{102} = a_{210} = 0.
$$

So $v = 0$ in this third case. The proof is completed.

# 4 Quasi-optimal Approximation

In this section, we derive a quasi–optimal convergence of finite element solutions. The analysis in some sense is standard. By the usual Strang lemma,

$$\begin{aligned}\|\nabla_h^m(u_m-u_h)\|_0 \le C &\inf_{v_{m,h}\in V_m(\mathcal{T}_h)} \|\nabla_h^m(u_m-v_{m,h})\|_0\\ &+C\sup_{0\ne v_{m,h}\in V_m(\mathcal{T}_h)} \frac{(\nabla_h^m u_m,\nabla_h^m v_{m,h})-(f,v_{m.h})}{\|\nabla_h^m v_{m,h}\|_0}. \end{aligned}\tag{4.1}$$

The first term on the right–hand side of (4.1) is the approximation error term which can be estimated by a standard argument while the second term on the right–hand side of (4.1) is usual referred to as the consistent error term. For the analysis, we need a finite element subspace, say $V_m^c(\mathcal{T}_h)$, of $H_0^m(\Omega)$. In fact, a function $v\in P_{4m-3}(K)$ can be uniquely defined by the following degrees of freedom [3]:

- The value of $\nabla^\ell v$, $\ell=0,\cdots,2m-2$, at the three vertices of element $K$;
- the $i$-th order (edge) normal derivative at each of $i$ distinct points in the interior of each edge for $i\le m-1$;
- the value at $\frac{(m-2)(m-1)}{2}$ distinct points in the interior of each triangle, chosen so that if a polynomial of degree $m-3$ vanishes at all the points, it vanishes identically.

Then the $H^m$ conforming finite element space $V_m^c(\mathcal{T}_h)$ can be defined as

$$\begin{aligned}V_m^c(\mathcal{T}_h):=\{v\in H_0^m(\Omega),\ &v|_K\in P_{4m-3}(K) \text{ for any element } K,\\ &v \text{ is continuous with respect to degrees of freedom}\\ &\text{on the internal interface of the mesh}\}.\end{aligned}\tag{4.2}$$

It follows from [4, 8, 15, 17] that there exists an operator $\Pi_m^c: V_m(\mathcal{T}_h)\to V_m^c(\mathcal{T}_h)$ such that

$$\sum_{K\in\mathcal{T}_h}\sum_{j=0}^{m-1}\big(h_K^{2(j-m)}\|\nabla_h^j(v_{m,h}-\Pi_m^c v_{m,h})\|_{0,K}^2\big)+\|\nabla_h^m\Pi_m^c v_{m,h}\|_0^2\le C\|\nabla_h^m v_{m,h}\|_0^2,\tag{4.3}$$

for any $v_{m,h}\in V_m(\mathcal{T}_h)$. Therefore, $\Pi_m^c$ is a uniformly bounded operator. Given $e=K_1\cap K_2$, define $\omega_e=K_1\cap K_2$. Given $\omega\subset\Omega$ and $g\in L^2(\omega)$, define the integral mean over $\omega$ of $g$ by

$$\Pi_\omega^0 g=\frac{1}{|\omega|}\int_\omega g\,\mathrm{d}s,$$

which allows for defining the piecewise constant projection operator $\Pi^0$:

$$\Pi^0 g|_K=\Pi_K^0(g|_K) \text{ for any } K\in\mathcal{T}_h \text{ and } g\in L^2(\Omega).$$

**Theorem 4.1** *Let* $u_m \in H_0^m(\Omega)$ *and* $u_{m,h} \in V_m(\mathcal{T}_h)$ *be the solutions of problems* (2.9) *and* (2.10) *respectively. It holds that*

$$C\|\nabla_h^m(u_m-u_{m,h})\|_0 \leq \inf_{v_{m,h}\in V_m(\mathcal{T}_h)} \|\nabla_h^m(u_m - v_{m,h})\|_0 + \|\nabla^m u_m - \Pi^0\nabla^m u_m\|_0 \\ +\Big(\sum_{e\in\mathcal{E}_h(\Omega)} \|\nabla^m u_m - \Pi_{\omega_e}^0 \nabla^m u_m\|_{0,\omega_e}^2\Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h} h_K^{2m}\|f\|_{0,K}^2\Big)^{1/2}. \tag{4.4}$$

**Remark 4.1** Notice that in this theorem only the basic $H^m$ regularity is needed for the exact solution $u_m$. See [7, 9, 13] for some related references on this aspect.

**Proof of Theorem 4.1** By (4.1), we only need to analyze the consistent error term. Given any $s_{m,h}, v_{m,h} \in V_m(\mathcal{T}_h)$, let $\Pi_m^c v_{m,h} \in V_m^c(\mathcal{T}_h)$ be defined in (4.3). Then,

$$\begin{aligned}(\nabla^m u_m, \nabla_h^m v_{m,h}) - (f, v_{m,h}) &= \big(\nabla_h^m(u_m - s_{m,h}), \nabla_h^m(v_{m,h} - \Pi_m^c v_{m,h})\big) \\ &\quad + \big(\nabla_h^m s_{m,h}, \nabla_h^m(v_{m,h} - \Pi_m^c v_{m,h})\big) - \big(f, (v_{m,h} - \Pi_m^c v_{m,h})\big) \\ &=: I_1 + I_2 + I_3.\end{aligned} \tag{4.5}$$

By (4.3), the first term $I_1$ can be bounded as

$$I_1 \leq C\|\nabla_h^m(u_m - s_{m,h})\|_0 \|\nabla_h^m v_{m,h}\|_0, \tag{4.6}$$

while the third term $I_3$ has the following estimate

$$I_3 \leq C\Big(\sum_{K\in\mathcal{T}_h} h_K^{2m}\|f\|_{0,K}^2\Big)^{1/2} \|\nabla_h^m v_{m,h}\|_0. \tag{4.7}$$

Next, we analyze the second term $I_2$. A series of integration by part leads to

$$\begin{aligned}&\big(\nabla_h^m s_{m,h}, \nabla_h^m(v_{m,h} - \Pi_m^c v_{m,h})\big) \\ =& \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h^m s_{m,h}\} \cdot \mathbf{n} : [\nabla_h^{m-1}(v_{m,h} - \Pi_m^c v_{m,h})] \mathrm{d}s \\ &+ \sum_{e\in\mathcal{E}_h(\Omega)} \int_e [\nabla_h^m s_{m,h}] \cdot \mathbf{n} : \{\nabla_h^{m-1}(v_{m,h} - \Pi_m^c v_{m,h})\} \mathrm{d}s \\ &- \sum_{e\in\mathcal{E}_h} \int_e \{\mathrm{div}\nabla_h^m s_{m,h}\} \cdot \mathbf{n} : [\nabla_h^{m-2}(v_{m,h} - \Pi_m^c v_{m,h})] \mathrm{d}s \\ &- \sum_{e\in\mathcal{E}_h(\Omega)} \int_e [\mathrm{div}\nabla_h^m s_{m,h}] \cdot \mathbf{n} : \{\nabla_h^{m-2}(v_{m,h} - \Pi_m^c v_{m,h})\} \mathrm{d}s \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \{\mathrm{divdiv}\nabla_h^m s_{m,h}\} \cdot \mathbf{n} : [\nabla_h^{m-3}(v_{m,h} - \Pi_m^c v_{m,h})] \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\sum_{e\in\mathcal{E}_h(\Omega)}\int_e[\mathrm{div}\mathrm{div}\nabla_h^m s_{m,h}]\cdot\mathbf{n}:\{\nabla_h^{m-3}(v_{m,h}-\Pi_m^c v_{m,h})\}\mathrm{d}s\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&+(-1)^{m-1}\sum_{e\in\mathcal{E}_h}\int_e\{\underbrace{\mathrm{div}\cdots\mathrm{div}}_{m-1}\nabla_h^m s_{m,h}\}\cdot\mathbf{n}:[(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
&+(-1)^{m-1}\sum_{e\in\mathcal{E}_h(\Omega)}\int_e[\underbrace{\mathrm{div}\cdots\mathrm{div}}_{m-1}\nabla_h^m s_{m,h}]\cdot\mathbf{n}:\{(v_{m,h}-\Pi_m^c v_{m,h})\}\mathrm{d}s\\
&+\sum_{K\in\mathcal{T}_h}((-1)^m\underbrace{\mathrm{div}\cdots\mathrm{div}}_{m}\nabla^m s_{m,h},v_{m,h}-\Pi_m^c v_{m,h})_{0,K}. \qquad (4.8)
\end{aligned}
$$

In the rest proof, we estimate the terms on the right-hand side of (4.8). First, the trace theorem and the inverse estimate yield, for $\ell\geq 2$,

$$
\begin{aligned}
&\int_e\{\underbrace{\mathrm{div}\cdots\mathrm{div}}_{\ell-1}\nabla_h^m s_{m,h}\}\cdot\mathbf{n}:[\nabla_h^{m-\ell}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
=&\int_e\{\underbrace{\mathrm{div}\cdots\mathrm{div}}_{\ell-1}(\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m)\}\cdot\mathbf{n}:[\nabla_h^{m-\ell}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
\leq&\, Ch_e^{-\ell}\|\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m\|_{0,\omega_e}\|\nabla_h^{m-\ell}(v_{m,h}-\Pi_m^c v_{m,h})\|_{0,\omega_e}.
\end{aligned}
$$

This and (4.3) show that

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e\{\underbrace{\mathrm{div}\cdots\mathrm{div}}_{\ell-1}\nabla_h^m s_{m,h}\}\cdot\mathbf{n}:[\nabla_h^{m-\ell}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
=&\sum_{e\in\mathcal{E}_h}\int_e\{\underbrace{\mathrm{div}\cdots\mathrm{div}}_{\ell-1}(\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m)\}\cdot\mathbf{n}:[\nabla_h^{m-\ell}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
\leq&\, C\|\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m\|_0\|\nabla_h^m v_{m,h}\|_0. \qquad (4.9)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h(\Omega)}\int_e[\underbrace{\mathrm{div}\cdots\mathrm{div}}_{\ell-1}\nabla_h^m s_{m,h}]\cdot\mathbf{n}:\{\nabla_h^{m-\ell}(v_{m,h}-\Pi_m^c v_{m,h})\}\mathrm{d}s\\
\leq&\, C\|\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m\|_0\|\nabla_h^m v_{m,h}\|_0. \qquad (4.10)
\end{aligned}
$$

It remains to analyze the first two terms and the last term on the right hand–side of (4.8). For the first term, it follows from the trace theorem and the inverse estimate that

$$
\begin{aligned}
&\int_e\{\nabla_h^m s_{m,h}\}\cdot\mathbf{n}:[\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
=&\int_e\{\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m\}\cdot\mathbf{n}:[\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\
\leq&\, Ch_e^{-1}\|\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m\|_{0,\omega_e}\|\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})\|_{0,\omega_e}.
\end{aligned}
$$

Here we use the fact that
$$\int_e [\nabla_h^{m-1} v_{m,h}] \mathrm{d}s = 0.$$
Together with (4.3), it states that
$$\begin{aligned}&\sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h^m s_{m,h}\}\cdot \mathbf{n} : [\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})]\mathrm{d}s\\ \le{}& C\|\nabla_h^m s_{m,h}-\Pi^0\nabla^m u_m\|_0\|\nabla_h^m v_{m,h}\|_0.\end{aligned}\tag{4.11}$$
Since $[\Pi^0_{\omega_e}\nabla^m u_m]=0$ for any internal edge $e$, the trace theorem and the inverse estimate lead to
$$\begin{aligned}&\int_e [\nabla_h^m s_{m,h}]\cdot \mathbf{n} : \{\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})\}\mathrm{d}s\\ ={}&\int_e [\nabla_h^m s_{m,h}-\Pi^0_{\omega_e}\nabla^m u_m]\cdot \mathbf{n} : \{\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})\}\mathrm{d}s\\ \le{}& Ch_e^{-1}\|\nabla_h^m s_{m,h}-\Pi^0_{\omega_e}\nabla^m u_m\|_{0,\omega_e}\|\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})\|_{0,\omega_e},\end{aligned}$$
which, plus (4.3), yields
$$\begin{aligned}&\sum_{e\in\mathcal{E}_h(\Omega)}\int_e [\nabla_h^m s_{m,h}]\cdot \mathbf{n} : \{\nabla_h^{m-1}(v_{m,h}-\Pi_m^c v_{m,h})\}\mathrm{d}s\\ \le{}& C\Big(\sum_{e\in\mathcal{E}_h(\Omega)}\|\nabla_h^m s_{m,h}-\Pi^0_{\omega_e}\nabla^m u_m\|^2_{0,\omega_e}\Big)^{1/2}\|\nabla_h^m v_{m,h}\|_0.\end{aligned}\tag{4.12}$$
We turn to the last term which can be bounded by the element-wise inequality and (4.3) as follows
$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}((-1)^m \underbrace{\mathrm{div}\cdots\mathrm{div}}_{m}\nabla^m s_{m,h}, v_{m,h}-\Pi_m^c v_{m,h})_{0,K}\\ ={}&\sum_{K\in\mathcal{T}_h}((-1)^m \underbrace{\mathrm{div}\cdots\mathrm{div}}_{m}(\nabla^m s_{m,h}-\Pi^0\nabla^m u_m), v_{m,h}-\Pi_m^c v_{m,h})_{0,K}\\ \le{}& C\|\nabla^m s_{m,h}-\Pi^0\nabla^m u_m\|_0\|\nabla_h^m v_{m,h}\|_0.\end{aligned}\tag{4.13}$$
Since $s_{m,h}$ is arbitrary, the desired estimate follows from (4.6), (4.7), (4.8)-(4.13), and the triangle inequality. The proof is completed.

# 5 Numerical Tests

**5.1 Numerical test 1** We apply $H^3$ non-conforming finite element method (2.2) to solve the tri-harmonic equation

$$\begin{cases} (-\Delta)^3 u = f & \text{in } (0,1)\times(0,1), \\ u = \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial^2 u}{\partial \mathbf{n}^2} = 0 & \text{on the boundary,} \end{cases}$$

with exact solution

$$u = 2^8 (x - x^2)^3 (y - y^2)^3. \tag{5.1}$$

We use the nested refined, uniform grids shown in Figure 5. The errors and the orders of convergence are displayed in Table 1. The optimal order convergence is achieved, under the consistent error limitation.

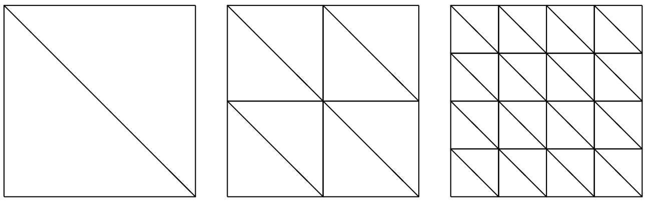

Figure 5: The first three levels of grids, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ .

Table 1: The error $e_h = u - u_h$ and the order of convergence, by $V_3$ element (2.2), for solution (5.1).

| $\mathcal{T}_k$ | $\|e_h\|_0$ | $h^n$ | $\|e_h\|_{1,h}$ | $h^n$ | $\|e_h\|_{2,h}$ | $h^n$ | $\|e_h\|_{3,h}$ | $h^n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.54397 | 0.0 | 2.47921 | 0.0 | 8.4032 | 0.0 | 40.666 | 0.0 |
| 2 | 0.03064 | 4.1 | 0.15218 | 4.0 | 1.0941 | 2.9 | 9.654 | 2.1 |
| 3 | 0.01403 | 1.1 | 0.07170 | 1.1 | 0.5084 | 1.1 | 6.967 | 0.5 |
| 4 | 0.00430 | 1.7 | 0.02190 | 1.7 | 0.1508 | 1.8 | 3.948 | 0.8 |
| 5 | 0.00115 | 1.9 | 0.00585 | 1.9 | 0.0401 | 1.9 | 2.051 | 0.9 |
| 6 | 0.00029 | 2.0 | 0.00149 | 2.0 | 0.0102 | 2.0 | 1.036 | 1.0 |

**5.2 Numerical test 2** We apply $H^4$ non-conforming finite element method (2.3) to solve the 4-harmonic equation

$$\begin{cases} (-\Delta)^4 u = f & \text{in } (0,1)\times(0,1), \\ u = \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial^2 u}{\partial \mathbf{n}^2} = \dfrac{\partial^3 u}{\partial \mathbf{n}^3} = 0 & \text{on the boundary,} \end{cases}$$

with exact solution

$$u = 2^{10} (x - x^2)^4 (y - y^2)^4. \tag{5.2}$$

The errors and the orders of convergence are displayed in Table 2. The optimal order convergence is achieved, under the consistent error limitation.

**5.3 Numerical test 3** We apply $H^5$ non-conforming finite element method (2.4) to solve the 5-harmonic (order 10 PDE) equation

Table 2: The error $e_h = u - u_h$ and the order of convergence, by $V_4$ element (2.3), for 4-harmonic solution (5.2).

| $\mathcal{T}_k$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_{1,h}$ | $h^n$ | $|e_h|_{2,h}$ | $h^n$ | $|e_h|_{3,h}$ | $h^n$ | $|e_h|_{4,h}$ | $h^n$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00484 | 0.0 | 0.02699 | 0.0 | 0.2743 | 0.0 | 2.242 | 0.0 | 35.90807 | 0.0 |
| 2 | 0.00394 | 0.3 | 0.02406 | 0.2 | 0.2474 | 0.1 | 2.634 | 0.0 | 40.65954 | 0.0 |
| 3 | 0.00271 | 0.5 | 0.01553 | 0.6 | 0.1221 | 1.0 | 1.287 | 1.0 | 27.77417 | 0.5 |
| 4 | 0.00089 | 1.6 | 0.00508 | 1.6 | 0.0399 | 1.6 | 0.416 | 1.6 | 14.01396 | 1.0 |
| 5 | 0.00024 | 1.9 | 0.00136 | 1.9 | 0.0107 | 1.9 | 0.110 | 1.9 | 6.91730 | 1.0 |
| 6 | 0.00006 | 2.0 | 0.00035 | 2.0 | 0.0027 | 2.0 | 0.028 | 2.0 | 3.43714 | 1.0 |
| 7 | 0.00002 | 2.0 | 0.00009 | 2.0 | 0.0007 | 2.0 | 0.007 | 2.0 | 1.71544 | 1.0 |

$$\begin{cases} (-\Delta)^5 u = f & \text{in } (0,1)\times(0,1), \\ u = \dfrac{\partial^i u}{\partial \mathbf{n}^i} = 0 & \text{on the boundary}, \quad i = 1,2,3,4, \end{cases}$$

with exact solution

$$u = 2^{14}(x - x^2)^5 (y - y^2)^5. \tag{5.3}$$

The errors and the orders of convergence are displayed in Table 3. The optimal order convergence is achieved, under the consistent error limitation.

Table 3: The error $e_h = u - u_h$ and the order of convergence, by $V_5$ ($P_7$) element (2.4), for 5-harmonic solution (5.3).

| $\mathcal{T}_k$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_{1,h}$ | $h^n$ | $|e_h|_{2,h}$ | $h^n$ | $|e_h|_{3,h}$ | $h^n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.00440 | 0.0 | 0.06001 | 0.0 | 0.2842 | 0.0 | 2.655 | 0.0 |
| 2 | 0.00214 | 1.0 | 0.88918 | 0.0 | 0.2433 | 0.2 | 3.590 | 0.0 |
| 3 | 0.00212 | 0.0 | 0.00019 | — | 0.1215 | 1.0 | 1.363 | 1.4 |
| 4 | 0.00077 | 1.5 | 0.00485 | 0.0 | 0.0424 | 1.5 | 0.449 | 1.6 |
| 5 | 0.00020 | 1.9 | 0.00126 | 1.9 | 0.0111 | 1.9 | 0.117 | 1.9 |

| $\mathcal{T}_k$ | $|e_h|_{4,h}$ | $h^n$ | $|e_h|_{5,h}$ | $h^n$ |
|---|---|---|---|---|
| 1 | 56.7108 | 0.0 | 590.506 | 0.0 |
| 2 | 80.3451 | 0.0 | 1081.068 | 0.0 |
| 3 | 30.7082 | 1.4 | 723.191 | 0.6 |
| 4 | 8.0391 | 1.9 | 313.370 | 1.2 |
| 5 | 2.1981 | 1.9 | 149.955 | 1.1 |

**5.4 Numerical test 4** We apply $H^6$ non-conforming finite element method (2.5) to solve the 6-harmonic (order 12 PDE) equation

$$\begin{cases} (-\Delta)^6 u = f & \text{in } (0,1)\times(0,1), \\ u = \dfrac{\partial^i u}{\partial \mathbf{n}^i} = 0 & \text{on the boundary}, \quad i = 1,2,\cdots,5, \end{cases}$$

with exact solution

$$u = 2^{18}(x - x^2)^6 (y - y^2)^6. \tag{5.4}$$

The errors and the orders of convergence are displayed in Table 4. The optimal order convergence is achieved, under the consistent error limitation.

Table 4: The error $e_h = u - u_h$ and the order of convergence, by $V_6$ ($P_9$) element (2.5), for 6-harmonic solution (5.4).

| $\mathcal{T}_k$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_{1,h}$ | $h^n$ | $|e_h|_{2,h}$ | $h^n$ | $|e_h|_{3,h}$ | $h^n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.1267 | 0.0 | 0.4895 | 0.0 | 14.7042 | 0.0 | 295.1211 | 0.0 |
| 2 | 0.0252 | 2.3 | 7.4484 | 0.0 | 1.9216 | 2.9 | 25.7602 | 3.5 |
| 3 | 0.0333 | 0.0 | 0.0656 | 6.8 | 2.2800 | 0.0 | 26.4761 | 0.0 |
| 4 | 0.0083 | 2.0 | 0.0638 | 0.0 | 0.6696 | 1.8 | 8.3561 | 1.7 |
| 5 | 0.0039 | 1.1 | 0.0286 | 1.2 | 0.2919 | 1.2 | 3.5617 | 1.2 |
| $\mathcal{T}_k$ | $|e_h|_{4,h}$ | $h^n$ | $|e_h|_{5,h}$ | $h^n$ | $|e_h|_{6,h}$ | $h^n$ | | |
| 1 | 3662.1828 | 0.0 | 28952.5084 | 0.0 | 296063.7189 | 0.0 | | |
| 2 | 348.0689 | 3.4 | 5385.4426 | 2.4 | 104537.9869 | 1.5 | | |
| 3 | 346.8000 | 0.0 | 5059.5643 | 0.1 | 99668.5074 | 0.1 | | |
| 4 | 117.1285 | 1.6 | 1783.9195 | 1.5 | 30779.2503 | 1.7 | | |
| 5 | 49.2540 | 1.2 | 758.6861 | 1.2 | 13664.7306 | 1.2 | | |

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