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PROPERTIES OF TENSOR COMPLEMENTARITY PROBLEM AND SOME CLASSES OF STRUCTURED TENSORS[*†]

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Abstract

This paper deals with the class of Q-tensors, that is, a Q-tensor is a real tensor $\mathcal{A}$ such that the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$:

$$\text{finding an } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geq \mathbf{0}, \mathbf{q}+\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top(\mathbf{q}+\mathcal{A}\mathbf{x}^{m-1}) = 0,$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$. Several subclasses of Q-tensors are given: P-tensors, R-tensors, strictly semi-positive tensors and semi-positive R_0-tensors. We prove that a nonnegative tensor is a Q-tensor if and only if all of its principal diagonal entries are positive, and so the equivalence of Q-tensor, R-tensors, strictly semi-positive tensors was showed if they are nonnegative tensors. We also show that a tensor is an R_0-tensor if and only if the tensor complementarity problem $(\mathbf{0}, \mathcal{A})$ has no non-zero vector solution, and a tensor is a R-tensor if and only if it is an R_0-tensor and the tensor complementarity problem $(\mathbf{e}, \mathcal{A})$ has no non-zero vector solution, where $\mathbf{e} = (1, 1 \cdots, 1)^\top$.

Keywords Q-tensor; R-tensor; R_0-tensor; strictly semi-positive; tensor complementarity problem

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1 Introduction

Throughout this paper, we use small letters $x, u, v, \alpha, \cdots$, for scalars, small bold

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letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. All the tensors discussed in this paper are real. Let $I_n := \{1, 2, \cdots, n\}$, $\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n)^\top; x_i \in \mathbb{R}, i \in I_n\}$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n; x \geq \mathbf{0}\}$, $\mathbb{R}^n_- := \{\mathbf{x} \in \mathbb{R}^n; x \leq \mathbf{0}\}$, $\mathbb{R}^n_{++} := \{\mathbf{x} \in \mathbb{R}^n; x > \mathbf{0}\}$, $\mathbf{e} = (1, 1, \cdots, 1)^\top$, and $\mathbf{x}^{[m]} = (x_1^m, x_2^m, \cdots, x_n^m)^\top$ for $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$, where $\mathbb{R}$ is the set of real numbers, $\mathbf{x}^\top$ is the transposition of a vector $\mathbf{x}$, and $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) means $x_i \geq 0$ ($x_i > 0$) for all $i \in I_n$.

Let $A = (a_{ij})$ be an $n \times n$ real matrix. A is said to be a **Q-matrix** iff the linear complementarity problem, denoted by $(\mathbf{q}, A)$,

$$\text{finding a } \mathbf{z} \in \mathbb{R}^n \text{ such that } \mathbf{z} \geq \mathbf{0}, \mathbf{q} + A\mathbf{z} \geq \mathbf{0}, \text{ and } \mathbf{z}^\top(\mathbf{q} + A\mathbf{z}) = 0, \tag{1.1}$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$. We say that A is a **P-matrix** iff for any nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$, there exists an $i \in I_n$ such that $x_i(Ax)_i > 0$. It is well-known that A is a P-matrix if and only if the linear complementarity problem $(\mathbf{q}, A)$ has a unique solution for all $\mathbf{q} \in \mathbb{R}^n$. Xiu and Zhang [1] also gave the necessary and sufficient conditions of P-matrices. A good review of P-matrices and Q-matrices can be found in the books by Berman and Plemmons [2], and Cottle, Pang and Stone [3].

Q-matrices and P(P_0)-matrices have a long history and wide applications in mathematical sciences. Pang [4] showed that each semi-monotone R_0-matrix is a Q-matrix. Pang [5] gave a class of Q-matrices which includes N-matrices and strictly semi-monotone matrices. Murty [6] showed that a nonnegative matrix is a Q-matrix if and only if all its diagonal entries are positive. Morris [7] presented two counterexamples of the Q-Matrix conjectures: a matrix is Q-matrix solely by considering the signs of its subdeterminants. Cottle [8] studied some properties of complete Q-matrices, a subclass of Q-matrices. Kojima and Saigal [9] studied the number of solutions to a class of linear complementarity problems. Gowda [10] proved that a symmetric semi-monotone matrix is a Q-matrix if and only if it is an R_0-matrix. Eaves [11] obtained the equivalent definition of strictly semi-monotone matrices, a main subclass of Q-matrices.

On the other hand, motivated by the discussion on positive definiteness of multivariate homogeneous polynomial forms [12-14], in 2005, Qi [15] introduced the concept of positive (semi-)definite symmetric tensors. In the same time, Qi also introduced eigenvalues, H-eigenvalues, E-eigenvalues and Z-eigenvalues for symmetric tensors. It was shown that an even order symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues or Z-eigenvalues are positive (nonnegative) ([15, Theorem 5]). Various structured tensors have been studied well, such as, Zhang, Qi and Zhou [16] and Ding, Qi and Wei [17] for M-tensors, Song and Qi [18] for P-(P_0)tensors and B-(B_0)tensors, Qi and Song [19] for positive (semi-)definition of

B-(B_0)tensors, Song and Qi [20] for infinite and finite dimensional Hilbert tensors, Song and Qi [33] for structure properties and an equivalent definition of (strictly) copositive tensors, Chen and Qi [22] for Cauchy tensor, Song and Qi [23] for E-eigenvalues of weakly symmetric nonnegative tensors and so on. Beside automatical control, positive semi-definite tensors have been applied in magnetic resonance imaging [24-27] and spectral hypergraph theory [28-30]. Recently, Song and Qi [31] extended the linear complementarity problem to the tensor complementarity problem, a special class of nonlinear complementarity problems, denoted by TCP($\mathbf{q}, \mathcal{A}$): finding an $\mathbf{x} \in \mathbb{R}^n$ such that

$$\text{TCP}(\mathbf{q}, \mathcal{A}) \qquad \mathbf{x} \geq \mathbf{0}, \ \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \ \text{and } \mathbf{x}^\top(\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0$$

or showing that no such vector exists.

Very recently, an n-person noncooperative game was converted by Huang and Qi [32] to a tensor complementarity problem. Furthermore, they gave the equivalence between a Nash equilibrium point of the multilinear game and a solution of the tensor complementarity problem. The equivalence between (strictly) semi-positive tensors and (strictly) copositive tensors in the case of symmetry was showed by Song and Qi [33]. The existence and uniqueness of solution to TCP($\mathbf{q}, \mathcal{A}$) with some special tensors were discussed by Che, Qi, Wei [34]. The boundedness of the solution set of the TCP($\mathbf{q}, \mathcal{A}$) was studied by Song and Yu [35]. The sparsest solutions to TCP($\mathbf{q}, \mathcal{A}$) with a Z-tensor and its method to calculate were obtained by Luo, Qi and Xiu [36]. The equivalent conditions for solution to TCP($\mathbf{q}, \mathcal{A}$) were showed by Gowda, Luo, Qi and Xiu [37] for a Z-tensor $\mathcal{A}$. The global uniqueness of solution of TCP($\mathbf{q}, \mathcal{A}$) was considered by Bai, Huang and Wang [38] for a strong P-tensor $\mathcal{A}$. The solvability of TCP($\mathbf{q}, \mathcal{A}$) was given by Wang, Huang and Bai [39] for a class of exceptionally regular tensors $\mathcal{A}$. The properties of TCP($\mathbf{q}, \mathcal{A}$) was studied by Ding, Luo and Qi [40] for a new class of P-tensor $\mathcal{A}$. The nice properties of the several classes of Q-tensors were presented by Suo and Wang [41]. The properties and algorithm of the tensor eigenvalue complementarity problem were studied by Song and Qi [42], Ling, He, Qi [43,44], Chen, Yang, Ye [45], respectively.

The following questions are natural. Can we extend the concept of Q-matrices to Q-tensors? If this can be done, are those nice properties of Q-matrices still true for Q-tensors?

In this paper, we introduce the concept of Q-tensors (Q-hypermatrices) and study some subclasses and nice properties of such tensors.

In Section 2, we extend the concept of Q-matrices to Q-tensors. Serval main subclasses of Q-matrices also are extended to the corresponding subclasses of Q-tensors: R-tensors, R_0-tensors, semi-positive tensors, strictly semi-positive tensors.

We give serval examples to verify that the class of R-(R_0-)tensors properly contains strictly semi-positive tensors as a subclass, while the class of P-tensors is a subclass of strictly semi-positive tensors. Some basic definitions and facts also are given in this section.

In Section 3, we study some properties of Q-tensors. Firstly, the equivalent definition of R-tensors is given: a tensor is an R_0-tensor if and only if the tensor complementarity problem $(\mathbf{0}, \mathcal{A})$ has not non-zero vector solution and a tensor is an R-tensor if and only if it is an R_0-tensor and the tensor complementarity problem $(\mathbf{e}, \mathcal{A})$ has not non-zero vector solution, where $\mathbf{e} = (1, 1 \cdots, 1)^\top$. Subsequently, we prove that each R-tensor is certainly a Q-tensor and each semi-positive R_0-tensor is an R-tensor. Thus, we show that every P-tensor is a Q-tensor. We will show that a nonnegative tensor is a Q-tensor if and only if all of its principal diagonal elements are positive, and so the relationship of several structured sensors are given. It will be proved that $\mathbf{0}$ is the unique feasible solution to the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ for $\mathbf{q} \geq \mathbf{0}$ if $\mathcal{A}$ is a non-negative Q-tensor.

2 Preliminaries

In this section, we define the notation and collect some basic definitions and facts, which will be used later on.

A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m}$, where $i_j \in I_n$ for $j \in I_m$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. If the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a **symmetric tensor**. The zero tensor in $T_{m,n}$ is denoted by $\mathcal{O}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with its ith component as

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i := \sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

for $i \in I_n$. We now give the definition of Q-tensors, which are natural extensions of Q-matrices.

Definition 2.1 Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. We say that $\mathcal{A}$ is a **Q-tensor** iff the tensor complementarity problem, denoted by $(\mathbf{q}, \mathcal{A})$,

$$\text{finding an } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geq \mathbf{0},\ \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \text{ and } \mathbf{x}^\top(\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0, \tag{2.1}$$

has a solution for each vector $\mathbf{q} \in \mathbb{R}^n$.

Definition 2.2 Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. We say that $\mathcal{A}$ is

(i) an **R-tensor** iff the following system is inconsistent

$$\begin{cases} 0 \neq \mathbf{x} \geq 0, \quad t \geq 0, \\ \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i + t = 0 \;\text{ if } x_i > 0, \\ \left(\mathcal{A}\mathbf{x}^{m-1}\right)_j + t \geq 0 \;\text{ if } x_j = 0; \end{cases} \tag{2.2}$$

(ii) an $\mathbf{R}_0$**-tensor** iff system (2.2) is inconsistent for $t = 0$.

Clearly, Definition 2.2 is a natural extension of the definition of Karamardian's class of regular matrices [46].

Definition 2.3 Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. $\mathcal{A}$ is said to be

(i) **semi-positive** iff for each $\mathbf{x}\geq 0$ and $\mathbf{x}\neq\mathbf{0}$, there exists an index $k\in I_n$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k \geq 0;$$

(ii) **strictly semi-positive** iff for each $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k > 0;$$

(iii) a **P-tensor** (Song and Qi [18]) iff for each $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, there exists an $i \in I_n$ such that

$$x_i \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i > 0;$$

(iv) a $\mathbf{P}_0$**-tensor**(Song and Qi [18]) iff for every $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, there exists an $i \in I_n$ such that $x_i \neq 0$ and

$$x_i \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i \geq 0.$$

Clearly, each P_0-tensor is certainly semi-positive. The concept of P-(P_0-)tensor is introduced by Song and Qi [18]. Furthermore, Song and Qi [18] studied some nice properties of such a class of tensors. The definition of (strictly) semi-positive tensor is a natural extension of the concept of (strictly) semi-positive (or semi-monotone) matrices [11,47].

It follows from Definitions 2.2 and 2.3 that each P-tensor must be strictly semi-positive and every strictly semi-positive tensor is certainly both R-tensor and R_0-tensor. Now we give several examples to demonstrate that the above inclusions are proper.

Example 2.1 Let $\widehat{\mathcal{A}} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $a_{i_1\cdots i_m} = 1$ for all $i_1, i_2, \cdots, i_m \in I_n$. Then

$$\left(\widehat{\mathcal{A}}\mathbf{x}^{m-1}\right)_i = (x_1 + x_2 + \cdots + x_n)^{m-1}$$

for all $i \in I_n$ and hence $\widehat{\mathcal{A}}$ is strictly semi-positive. However, $\widehat{\mathcal{A}}$ is not a P-tensor (for example, $x_i\left(\widehat{\mathcal{A}}\mathbf{x}^{m-1}\right)_i = 0$ for $\mathbf{x} = (1, -1, 0, \cdots, 0)^\top$ and all $i \in I_n$).

Example 2.2 Let $\widetilde{\mathcal{A}} = (a_{i_1i_2i_3}) \in T_{3,2}$ and $a_{111} = 1$, $a_{122} = -1$, $a_{211} = -2$, $a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then

$$\widetilde{\mathcal{A}}\mathbf{x}^2 = \begin{pmatrix} x_1^2 - x_2^2 \\ -2x_1^2 + x_2^2 \end{pmatrix}.$$

Clearly, $\widetilde{\mathcal{A}}$ is not strictly semi-positive (for example, $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_1 = 0$ and $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_2 = -1$ for $\mathbf{x} = (1,1)^\top$).

$\widetilde{\mathcal{A}}$ is an R_0-tensor. In fact,

(i) if $x_1 > 0$, $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_1 = x_1^2 - x_2^2 = 0$, then $x_2^2 = x_1^2$, so $x_2 > 0$, but $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = 0$, then $x_1^2 = \frac{1}{2}x_2^2 > 0$, but $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_1 = x_1^2 - x_2^2 = -\frac{1}{2}x_2^2 < 0$.

$\widetilde{\mathcal{A}}$ is not an R-tensor. In fact, if $x_1 > 0$, $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_1 + t = x_1^2 - x_2^2 + t = 0$, then $x_2^2 = x_1^2 + t > 0$, so $x_2 > 0$, $\left(\widetilde{\mathcal{A}}\mathbf{x}^2\right)_2 + t = -2x_1^2 + x_2^2 + t = -x_1^2 + 2t$. Take $x_1 = a > 0$, $t = \frac{1}{2}a^2$ and $x_2 = \frac{\sqrt{6}}{2}a$. That is, $\mathbf{x} = a(1, \frac{\sqrt{6}}{2})^\top$ and $t = \frac{1}{2}a^2$ solve system (2.2).

Example 2.3 Let $\overline{\mathcal{A}} = (a_{i_1i_2i_3}) \in T_{3,2}$ and $a_{111} = -1$, $a_{122} = 1$, $a_{211} = -2$, $a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then

$$\overline{\mathcal{A}}\mathbf{x}^2 = \begin{pmatrix} -x_1^2 + x_2^2 \\ -2x_1^2 + x_2^2 \end{pmatrix}.$$

Clearly, $\overline{\mathcal{A}}$ is not strictly semi-positive (for example, $\mathbf{x} = (1,1)^\top$).

$\overline{\mathcal{A}}$ is an R-tensor. In fact,

(i) if $x_1 > 0$, $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_1 + t = -x_1^2 + x_2^2 + t = 0$, then $x_2^2 = x_1^2 - t$, but $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_2 + t = -2x_1^2 + x_2^2 + t = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_2 + t = -2x_1^2 + x_2^2 + t = 0$, then $x_1^2 = \frac{1}{2}(x_2^2 + t) > 0$, but $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_1 + t = -x_1^2 + x_2^2 + t = \frac{1}{2}(x_2^2 + t) > 0$.

$\overline{\mathcal{A}}$ is an R_0-tensor. In fact,

(i) if $x_1 > 0$, $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_1 = -x_1^2 + x_2^2 = 0$, then $x_2^2 = x_1^2$, so $x_2 > 0$, but $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = -x_1^2 < 0$;

(ii) if $x_2 > 0$, $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_2 = -2x_1^2 + x_2^2 = 0$, then $x_1^2 = \frac{1}{2}x_2^2 > 0$, but $\left(\overline{\mathcal{A}}\mathbf{x}^2\right)_1 = -x_1^2 + x_2^2 = \frac{1}{2}x_2^2 > 0$.

Lemma 2.1[2] *Let* $S = \left\{\mathbf{x} \in \mathbb{R}_+^{n+1}; \sum\limits_{i=1}^{n+1} x_i = 1\right\}$. *Assume that* $F : S \to \mathbb{R}^{n+1}$ *is* *continuous on* S. *Then there exists an* $\overline{\mathbf{x}} \in S$ *such that*

$$\mathbf{x}^\top F(\overline{\mathbf{x}}) \geq \overline{\mathbf{x}}^\top F(\overline{\mathbf{x}}) \quad \textit{for all } \mathbf{x} \in S, \tag{2.3}$$

$$(F(\overline{\mathbf{x}}))_k = \min_{i \in I_{n+1}} (F(\overline{\mathbf{x}}))_i = \omega \quad \textit{if } x_k > 0, \tag{2.4}$$

$$(F(\overline{\mathbf{x}}))_k \geq \omega \quad \textit{if } x_k = 0. \tag{2.5}$$

Recall that a tensor $\mathcal{C} \in T_{m,r}$ is called **a principal sub-tensor** of a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ $(1 \leq r \leq n)$ iff there is a set J that composed of r elements in I_n such that

$$\mathcal{C} = (a_{i_1 \cdots i_m}), \text{ for all } i_1, i_2, \cdots, i_m \in J.$$

The concept was first introduced and used in [15] for symmetric tensor. We denote by $\mathcal{A}_r^J$ the principal sub-tensor of a tensor $\mathcal{A} \in T_{m,n}$ such that the entries of $\mathcal{A}_r^J$ are indexed by $J \subset I_n$ with $|J| = r$ $(1 \leq r \leq n)$, and denote by $\mathbf{x}_J$ the r-dimensional sub-vector of a vector $\mathbf{x} \in \mathbb{R}^n$, with the components of $\mathbf{x}_J$ indexed by J. Note that for $r = 1$, the principal sub-tensors are just the diagonal entries.

Definition 2.4 Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$. $\mathcal{A}$ is said to be

(i) **copositive** if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}_+^n$;

(ii) **strictly copositive** if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}_+^n \setminus \{0\}$.

The concept of (strictly) copositive tensors was first introduced by Qi in [48]. Song and Qi [33] showed their equivalent definition and some special structures. The following lemma is one of the structure conclusions of (strictly) copositive tensors in [33].

Lemma 2.2[33] *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$. *Then:*

(i) *If* $\mathcal{A}$ *is copositive, then* $a_{ii \cdots i} \geq 0$ *for all* $i \in I_n$;

(ii) *if* $\mathcal{A}$ *is strictly copositive, then* $a_{ii \cdots i} > 0$ *for all* $i \in I_n$.

Definition 2.5 Given a function $F : \mathbb{R}_+^n \to \mathbb{R}^n$, the nonlinear complementarity problem, denoted by NCP(F), is to

$$\text{find a vector } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geq \mathbf{0}, F(\mathbf{x}) \geq \mathbf{0}, \text{ and } \mathbf{x}^\top F(\mathbf{x}) = 0. \tag{2.6}$$

It is well known that the nonlinear complementarity problems have been widely applied to the field of transportation planning, regional science, socio-economic analysis, energy modeling, and game theory. So over the past decades, the solutions of nonlinear complementarity problems have been rapidly studied in its theory of existence, uniqueness and algorithms.

Definition 2.6[49,50] A mapping $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(i) pseudo-monotone on K if for all vectors $\mathbf{x}, \mathbf{y} \in K$,

$$(\mathbf{x}-\mathbf{y})^\top F(\mathbf{y}) \geq 0 \ \Rightarrow \ (\mathbf{x}-\mathbf{y})^\top F(x) \geq 0;$$

(ii) monotone on K if

$$(F(\mathbf{x}) - F(\mathbf{y}))^\top (\mathbf{x}-\mathbf{y}) \geq 0, \quad \text{for any } x, y \in K;$$

(iii) strictly monotone on K if

$$(F(\mathbf{x}) - F(\mathbf{y}))^\top (\mathbf{x}-\mathbf{y}) > 0, \quad \text{for any } x, y \in K \text{ and } x \neq y;$$

(iv) strongly monotone on K if there exists a constant $c > 0$ such that

$$(F(\mathbf{x}) - F(\mathbf{y}))^\top (\mathbf{x}-\mathbf{y}) \geq c\|\mathbf{x}-\mathbf{y}\|^2;$$

(v) a P_0 function on K if for all pairs of distinct vectors $\mathbf{x}$ and $\mathbf{y}$ in K, there exists a $k \in I_n$ such that

$$x_k \neq y_k \quad \text{and} \quad (x_k - y_k)(F(\mathbf{x}) - F(\mathbf{y}))_k \geq 0;$$

(vi) a P function on K if for all pairs of distinct vectors $\mathbf{x}$ and $\mathbf{y}$ in K,

$$\max_{k \in I_n} (x_k - y_k)\,(F(\mathbf{x}) - F(\mathbf{y}))_k > 0;$$

(vii) a uniformly P function on K if there exists a constant $c > 0$ such that for all pairs of vectors $\mathbf{x}$ and $\mathbf{y}$ in K,

$$\max_{k \in I_n} (x_k - y_k)\,(F(\mathbf{x}) - F(\mathbf{y}))_k \geq c\|x-y\|^2.$$

It follows from the above definition of the monotonicity and P properties that the following relations hold (see [49,50] for more details):

$$\begin{array}{ccccc}
\text{strongly monotone} & \Rightarrow & \text{strictly monotone} & \Rightarrow & \text{monotone} & \Rightarrow & \text{pseudo-monotone} \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\text{uniformly P function} & \Rightarrow & \text{P function} & \Rightarrow & \mathrm{P}_0 \text{ function} & &
\end{array}$$

Now we give an example to certify the function deduced by an R-tensor is neither pseudo-monotone nor a P_0 function.

Example 2.4 Let $\overline{\mathcal{A}}$ be an R-tensor defined by Example 2.3 and $F(\mathbf{x}) = \overline{\mathcal{A}}\mathbf{x}^2 + \mathbf{q}$, where $\mathbf{q} = (\frac{1}{2}, \frac{1}{2})^\top$. Then F is neither pseudo-monotone nor a P_0 function. In fact,

$$F(\mathbf{x}) = \overline{\mathcal{A}}\mathbf{x}^2 + \mathbf{q} = \begin{pmatrix} -x_1^2 + x_2^2 + \frac{1}{2} \\ -2x_1^2 + x_2^2 + \frac{1}{2} \end{pmatrix}.$$

Let $\mathbf{x} = (1, 0)^\top$ and $\mathbf{y} = (0, \frac{1}{4})^\top$. Then

$$\mathbf{x}-\mathbf{y}=\begin{pmatrix}1\\ -\frac{1}{4}\end{pmatrix},\quad F(\mathbf{x})=\begin{pmatrix}-\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\quad\text{and}\quad F(\mathbf{y})=\begin{pmatrix}\frac{9}{16}\\ \frac{9}{16}\end{pmatrix}.$$

Clearly, we have

$$(\mathbf{x}-\mathbf{y})^\top F(\mathbf{y})=1\times\frac{9}{16}-\frac{1}{4}\times\frac{9}{16}>0.$$

However,

$$(\mathbf{x}-\mathbf{y})^\top F(\mathbf{x})=-\frac{1}{2}-\frac{1}{4}\times\Big(-\frac{3}{2}\Big)<0,$$

hence F is not pseudo-monotone.

Take $\mathbf{x}=(1,1)^\top$ and $\mathbf{y}=(0,\frac{1}{4})^\top$. Then

$$\mathbf{x}-\mathbf{y}=\begin{pmatrix}1\\ -\frac{1}{4}\end{pmatrix},\quad F(\mathbf{x})=\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}\quad\text{and}\quad F(\mathbf{y})=\begin{pmatrix}\frac{9}{16}\\ \frac{9}{16}\end{pmatrix}.$$

Clearly, we have

$$(x_1-y_1)\,(F(\mathbf{x})-F(\mathbf{y}))_1=1\times\Big(\frac{1}{2}-\frac{9}{16}\Big)<0$$

and

$$(x_2-y_2)\,(F(\mathbf{x})-F(\mathbf{y}))_2=\frac{3}{4}\times\Big(-\frac{1}{2}-\frac{9}{16}\Big)<0,$$

hence F is not a P_0 function.

Remark 2.1 Let $\mathcal{A}\in T_{m,n}$ and $F(\mathbf{x})=\mathcal{A}\mathbf{x}^{m-1}$. Taking $\mathbf{y}=\mathbf{0}$ and $\mathbf{x}\in\mathbb{R}^n_+$ in Definition 2(vi), we obtain that $\mathcal{A}$ is a P-tensor if F is a P function. So $\mathcal{A}$ must be an R-tensor if $F(\mathbf{x})=\mathcal{A}\mathbf{x}^{m-1}$ is a P function. Example 2.1 means that the inverse implication is not true.

Next we will show our main result: Each R-tensor $\mathcal{A}$ is a Q-tensor. That is, the nonlinear complementarity problem:

$$\text{finding an } \mathbf{x}\in\mathbb{R}^n \text{ such that } \mathbf{x}\geq\mathbf{0}, F(\mathbf{x})=\mathcal{A}\mathbf{x}^{m-1}+\mathbf{q}\geq\mathbf{0},\ \text{and } \mathbf{x}^\top F(\mathbf{x})=0, \tag{2.7}$$

has a solution for each vector $\mathbf{q}\in\mathbb{R}^n$.

3 Tensor Complementarity Problem and Some Classes of Structured Tensors

We first give the equivalent definition of R_0-tensor (R-tensor) by means of the tensor complementarity problem.

Theorem 3.1 *Let $\mathcal{A}=(a_{i_1\cdots i_m})\in T_{m,n}$. Then:*

(i) *$\mathcal{A}$ is an R_0-tensor if and only if the tensor complementarity problem $(\mathbf{0},\mathcal{A})$ has a unique solution $\mathbf{0}$;*

(ii) *$\mathcal{A}$ is an R-tensor if and only if it is an R_0-tensor and the tensor complementarity problem $(\mathbf{e},\mathcal{A})$ has a unique solution $\mathbf{0}$, where $\mathbf{e}=(1,1\cdots,1)^\top$.*

Proof (i) The tensor complementarity problem $(\mathbf{0}, \mathcal{A})$ has not non-zero vector solution if and only if the system

$$\begin{cases} 0 \neq \mathbf{x} = (x_1, \cdots, x_n)^\top \geq 0, \\ \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = 0 \ \text{ if } x_i > 0, \\ \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i \geq 0 \ \text{ if } x_i = 0 \end{cases}$$

has no solution. So the conclusion is proved.

(ii) It follows from Definition 2.2 that the necessity is obvious ($t = 1$).

Conversely suppose $\mathcal{A}$ is not an R-tensor. Then there exists an $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ satisfying the system (2.2). That is, the tensor complementarity problem $(t\mathbf{e}, \mathcal{A})$ has non-zero vector solution $\mathbf{x}$ for some $t \geq 0$. We have $t > 0$ since $\mathcal{A}$ is an R_0-tensor. So the tensor complementarity problem $(\mathbf{e}, \mathcal{A})$ has non-zero vector solution $\dfrac{\mathbf{x}}{\sqrt[m-1]{t}}$, a contradiction. The proof is completed.

Now we give the following result which can be obtained by Theorem 3.1 together with the main results of Karamardian [51]. For completeness, we give another proof using the similar proof technique in Berman and Plemmons [2, Theorem 3.6].

Corollary 3.1 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ be an R-tensor. Then $\mathcal{A}$ is a Q-tensor. That is, the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ has a solution for all $\mathbf{q} \in \mathbb{R}^n$.*

Proof Let the mapping $F : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ be defined by

$$F(\mathbf{y}) = \begin{pmatrix} \mathcal{A}\mathbf{x}^{m-1} + s\mathbf{q} + s\mathbf{e} \\ s \end{pmatrix}, \tag{3.1}$$

where $\mathbf{y} = (\mathbf{x}, s)^\top$, $\mathbf{x} \in \mathbb{R}^n_+$, $s \in \mathbb{R}_+$ and $\mathbf{e} = (1, 1, \cdots, 1)^\top \in \mathbb{R}^n$, $\mathbf{q} \in \mathbb{R}^n$. Obviously, $F : S \to \mathbb{R}^{n+1}$ is continuous on the set $S = \left\{\mathbf{x} \in \mathbb{R}^{n+1}_+; \sum\limits_{i=1}^{n+1} x_i = 1\right\}$. It follows from Lemma 2.1 that there exists a $\widetilde{\mathbf{y}} = (\widetilde{\mathbf{x}}, \widetilde{s})^\top \in S$ such that

$$\mathbf{y}^\top F(\widetilde{\mathbf{y}}) \geq \widetilde{\mathbf{y}}^\top F(\widetilde{\mathbf{y}}) \quad \text{for all } \ \mathbf{y} \in S, \tag{3.2}$$

$$(F(\widetilde{\mathbf{y}}))_k = \min_{i \in I_{n+1}} (F(\widetilde{\mathbf{y}}))_i = \omega \quad \text{if } \ \widetilde{y}_k > 0, \tag{3.3}$$

$$(F(\widetilde{\mathbf{y}}))_k \geq \omega \quad \text{if } \ \widetilde{y}_k = 0. \tag{3.4}$$

We claim $\widetilde{s} > 0$. Suppose $\widetilde{s} = 0$. Then the fact that $\widetilde{y}_{n+1} = \widetilde{s} = 0$ together with (3.4) implies that

$$\omega \leq (F(\widetilde{\mathbf{y}}))_{n+1} = \widetilde{s} = 0,$$

and so for $k \in I_n$,

$$(F(\widetilde{\mathbf{y}}))_k = \left(\mathcal{A}\widetilde{\mathbf{x}}^{m-1}\right)_k = \omega \quad \text{if } \ \widetilde{x}_k > 0,$$
$$(F(\widetilde{\mathbf{y}}))_k = \left(\mathcal{A}\widetilde{\mathbf{x}}^{m-1}\right)_k \geq \omega \quad \text{if } \ \widetilde{x}_k = 0.$$

That is, for $t=-\omega\geq 0$,

$$\begin{aligned}&\left(\mathcal{A}\widetilde{\mathbf{x}}^{m-1}\right)_k+t=0 \quad \text{if} \;\; \widetilde{x}_k>0,\\ &\left(\mathcal{A}\widetilde{\mathbf{x}}^{m-1}\right)_k+t\geq 0 \quad \text{if} \;\; \widetilde{x}_k=0.\end{aligned}$$

This obtains a contradiction with the definition of R-tensor $\mathcal{A}$, which completes the proof of the claim.

Now we show that the tensor complementarity problem $(\mathbf{q},\mathcal{A})$ has a solution for all $\mathbf{q}\in\mathbb{R}^n$. In fact, if $\mathbf{q}\geq\mathbf{0}$, clearly $\mathbf{z}=\mathbf{0}$ and $\mathbf{w}=\mathcal{A}\mathbf{z}^{m-1}+\mathbf{q}=\mathbf{q}$ solve $(\mathbf{q},\mathcal{A})$. Next we consider $\mathbf{q}\in\mathbb{R}^n/\mathbb{R}^n_+$. It follows from (3.1), (3.3) and (3.4) that

$$(F(\widetilde{\mathbf{y}}))_{n+1}=\min_{i\in I_{n+1}}(F(\widetilde{\mathbf{y}}))_i=\omega=\widetilde{s}=\widetilde{y}_{n+1}>0,$$

and for $i\in I_n$,

$$\begin{aligned}&(F(\widetilde{\mathbf{y}}))_i=\left(\mathcal{A}\widetilde{\mathbf{x}}^{m-1}\right)_i+\widetilde{s}q_i+\widetilde{s}=\omega=\widetilde{s} \quad \text{if} \;\; \widetilde{y}_i=\widetilde{x}_i>0,\\ &(F(\widetilde{\mathbf{y}}))_i=\left(\mathcal{A}\widetilde{\mathbf{x}}^{m-1}\right)_i+\widetilde{s}q_i+\widetilde{s}\geq\omega=\widetilde{s} \quad \text{if} \;\; \widetilde{y}_i=\widetilde{x}_i=0.\end{aligned}$$

Thus for $\mathbf{z}=\frac{\widetilde{\mathbf{x}}}{\widetilde{s}^{\frac{1}{m-1}}}$ and $i\in I_n$, we have

$$\begin{aligned}&\left(\mathcal{A}\mathbf{z}^{m-1}\right)_i+q_i=0 \quad \text{if} \;\; z_i>0,\\ &\left(\mathcal{A}\mathbf{z}^{m-1}\right)_i+q_i\geq 0 \quad \text{if} \;\; z_i=0,\end{aligned}$$

hence,

$$\mathbf{z}\geq\mathbf{0}, \quad \mathbf{w}=\mathbf{q}+\mathcal{A}\mathbf{z}^{m-1}\geq\mathbf{0}, \quad \text{and} \quad \mathbf{z}^\top\mathbf{w}=0.$$

So we obtain a feasible solution $(\mathbf{z},\mathbf{w})$ of the tensor complementarity problem $(\mathbf{q},\mathcal{A})$, and then $\mathcal{A}$ is a Q-tensor. The theorem is proved.

Corollary 3.2 *Each strictly semi-positive tensor is a Q-tensor, and so is P-tensor. That is, the tensor complementarity problem $(\mathbf{q},\mathcal{A})$ has a solution for all $\mathbf{q}\in\mathbb{R}^n$ if $\mathcal{A}$ is either a P-tensor or a strictly semi-positive tensor.*

Theorem 3.2 *Let an R_0-tensor $\mathcal{A}(\in T_{m,n})$ be semi-positive. Then $\mathcal{A}$ is an R-tensor, and hence $\mathcal{A}$ is a Q-tensor.*

Proof Suppose that $\mathcal{A}$ is not an R-tensor. Let system (2.2) have a solution $\overline{\mathbf{x}}\geq 0$ and $\overline{\mathbf{x}}\neq 0$. If $t=0$, this contradicts the assumption that $\mathcal{A}$ is an R_0-tensor. So we must have $t>0$. Then for $i\in I_n$, we have

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i+t=0 \quad \text{if} \;\; x_i>0,$$

hence,

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i=-t<0 \quad \text{if} \;\; x_i>0,$$

which contradicts the assumption that $\mathcal{A}$ is semi-positive. So $\mathcal{A}$ is an R-tensor, and hence $\mathcal{A}$ is a Q-tensor by Corollary 3.1. The proof is completed.

So, the following relationship of several classes of structured sensors hold:

$$
\begin{array}{ccccccc}
 & & & & \text{Semi-positive } \mathrm{R}_0\text{-Tensors} & & \\
 & & & & \Downarrow & & \\
\text{P-Tensors} & \Rightarrow & \text{Strictly Semi-positive Tensors} & \Rightarrow & \text{R-Tensors} & \Rightarrow & \text{Q-Tensors} \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\mathrm{P}_0\text{-Tensors} & \Rightarrow & \text{Semi-positive Tensors} & & \mathrm{R}_0\text{-Tensors} & &
\end{array}
$$

Theorem 3.3 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ with $\mathcal{A} \geq \mathcal{O}$ ($a_{i_1\cdots i_m} \geq 0$ for all $i_1 \cdots i_m \in I_n$). Then $\mathcal{A}$ is a Q-tensor if and only if $a_{ii\cdots i} > 0$ for all $i \in I_n$.*

Proof Sufficiency If $a_{ii\cdots i} > 0$ for all $i \in I_n$ and $\mathcal{A} \geq \mathcal{O}$, then it follows from Definition 2.3 of the strictly semi-positive tensor that $\mathcal{A}$ is strictly semi-positive, hence $\mathcal{A}$ is a Q-tensor by Corollary 3.2.

Necessity Suppose that there exists a $k \in I_n$ such that $a_{kk\cdots k} = 0$. Let $\mathbf{q} = (q_1, \cdots, q_n)^\top$ with $q_k < 0$ and $q_i > 0$ for all $i \in I_n$ and $i \neq k$. Since $\mathcal{A}$ is a Q-tensor, the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ has at least one solution. Let $\mathbf{z}$ be a feasible solution to $(\mathbf{q}, \mathcal{A})$. Then

$$\mathbf{z} \geq \mathbf{0}, \quad \mathbf{w} = \mathcal{A}\mathbf{z}^{m-1} + \mathbf{q} \geq \mathbf{0} \quad \text{and} \quad \mathbf{z}^\top \mathbf{w} = 0. \tag{3.5}$$

Clearly, $\mathbf{z} \neq \mathbf{0}$. Since $\mathbf{z} \geq \mathbf{0}$ and $\mathcal{A} \geq 0$ together with $q_i > 0$ for each $i \in I_n$ with $i \neq k$, we must have

$$w_i = \left(\mathcal{A}\mathbf{z}^{m-1}\right)_i + q_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} z_{i_2} \cdots z_{i_m} + q_i > 0 \quad \text{for } i \neq k \text{ and } i \in I_n.$$

It follows from (3.5) that

$$z_i = 0 \quad \text{for } i \neq k \text{ and } i \in I_n.$$

Thus, we have

$$w_k = \left(\mathcal{A}\mathbf{z}^{m-1}\right)_k + q_k = \sum_{i_2,\cdots,i_m=1}^{n} a_{ki_2\cdots i_m} z_{i_2} \cdots z_{i_m} + q_k = a_{kk\cdots k} z_k^{m-1} + q_k = q_k < 0,$$

since $a_{kk\cdots k} = 0$. This contradicts the fact that $\mathbf{w} \geq \mathbf{0}$, so $a_{ii\cdots i} > 0$ for all $i \in I_n$. The proof is completed.

Corollary 3.3 *Let a non-negative tensor $\mathcal{A}$ be a Q-tensor. Then all principal sub-tensors of $\mathcal{A}$ are also Q-tensors.*

Corollary 3.4 *Let a non-negative tensor $\mathcal{A}$ be a Q-tensor. Then $\mathbf{0}$ is the unique feasible solution to the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ for $\mathbf{q} \geq \mathbf{0}$.*

Proof It follows from Theorem 3.3 that $a_{ii\cdots i} > 0$ for all $i \in I_n$, and hence

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_1}\cdots x_{i_m} = a_{ii\cdots i}x_i^{m-1} + \sum_{(i_2,\cdots,i_m)\neq(i,\cdots,i)} a_{ii_2\cdots i_m} x_{i_1}\cdots x_{i_m}.$$

If $\mathbf{x} = (x_1, \cdots, x_n)^\top$ is any feasible solution to the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$, then we have

$$\mathbf{x} \geq \mathbf{0}, \quad \mathbf{w} = \mathcal{A}\mathbf{x}^{m-1} + \mathbf{q} \geq \mathbf{0} \quad \text{and} \quad \mathbf{x}^\top \mathbf{w} = \mathcal{A}\mathbf{x}^m + \mathbf{x}^\top \mathbf{q} = 0. \tag{3.6}$$

Suppose $x_i > 0$ for some $i \in I_n$. Then

$$w_i = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i + q_i = a_{ii\cdots i}x_i^{m-1} + \sum_{(i_2,\cdots,i_m)\neq(i,\cdots,i)} a_{ii_2\cdots i_m} x_{i_1}\cdots x_{i_m} + q_i > 0,$$

hence, $\mathbf{x}^\top \mathbf{w} = x_i w_i + \sum\limits_{k\neq i} x_k w_k > 0$. This contradicts the fact that $\mathbf{x}^\top \mathbf{w} = 0$. Consequently, $x_i = 0$ for all $i \in I_n$. The proof is completed.

Following the above conclusions together with Theorems 3.2 and 3.4 of Song and Qi [33], the following results are obvious.

Corollary 3.5 *Let $\mathcal{A}$ be a non-negative tensor. Then the following are equivalent:*

(i) *$\mathcal{A}$ is a Q-tensor;*

(ii) *$\mathcal{A}$ is a R-tensor;*

(iii) *$\mathcal{A}$ is a strictly semi-positive tensor;*

(iv) *$a_{ii\cdots i} > 0$ for all $i \in I_n$.*

Corollary 3.6 *Let $\mathcal{A}$ be a symmetric and non-negative tensor. Then the following are equivalent:*

(i) *$\mathcal{A}$ is a Q-tensor;*

(ii) *$\mathcal{A}$ is a R-tensor;*

(iii) *$\mathcal{A}$ is a strictly semi-positive tensor;*

(iv) *$\mathcal{A}$ is a strictly copositive tensor;*

(v) *$a_{ii\cdots i} > 0$ for all $i \in I_n$.*

Question 3.1 Let $\mathcal{A}$ be a Q-tensor.

- Whether or not a nonzero solution $\mathbf{x}$ to Tensor Complementarity Problem $(\mathbf{0}, \mathcal{A})$ contains at least two nonzero components if $\mathcal{A}$ is a semi-positive Q-tensor;

- Whether or not there are some relation between the eigenvalue of (symmetric) Q-tensor and the feasible solution of Tensor Complementarity Problem $(\mathbf{q}, \mathcal{A})$.

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