NORM RETRIEVAL BY PROJECTIONS ON INFINITE-DIMENSIONAL HILBERT SPACES[*][†]

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Abstract

We study the norm retrieval by projections on an infinite-dimensional Hilbert space H. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and $W_i = \{e_i\}^{\perp}$ for all $i \in I$. We show that $\{W_i\}_{i\in I}$ does norm retrieval if and only if I is an infinite subset of **N**. We also give some properties of norm retrieval by projections.

Keywords norm retrieval; phase retrieval; frames; Hilbert spaces

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1 Introduction

Signal reconstruction is an important problem in engineering and has a wide variety of applications. Recovering a signal when there is partial loss of information is a significant challenge. Partial loss of phase information occurs in application areas such as speech recognition [4, 12, 13], and optics applications such as X-ray crystallography [3, 10, 11], and there is a need to do phase retrieval efficiently. The concept of phase retrieval for Hilbert space frames was introduced in 2006 by Balan, Casazza, and Edidin [2], and since then it has become an active area of research in signal processing and harmonic analysis.

Phase retrieval has been defined for vectors as well as for projections and in general deals with recovering the phase of a signal given its intensity measurements from a redundant linear system. Phase retrieval by projections, where the signal is projected onto some higher dimensional subspaces and has to be recovered from the norms of the projections of the vectors onto the subspaces, appears in real life problems such as crystal twinning [9]. We refer the readers to [5] for the detailed study of phase retrieval by projections.

In this paper, we consider the notion of norm retrieval which was recently introduced by Bahmanpour et. al. [1], and is the problem of retrieving the norm of a vector given the absolute value of its intensity measurements. Norm retrieval arises naturally from phase retrieval when one utilizes both a collection of subspaces and their orthogonal complements. Let $\{e_i\}_{i \in I}$ be an orthonormal basis in Hilbert space H. Let $W_i = \{e_i\}^{\perp}$ for all $i \in I$. [7] discussed that $\{W_i\}_{i \in I}$ does norm retrieval or cannot do norm retrieval in a finite dimensional Hilbert space H. [7] also gave some properties of norm retrieval. We will discuss the same problems but in an infinite-dimensional Hilbert space H. We will also give some results similar to these in [7].

2   Norm Retrieval

Firstly, we give some definitions.

Definition 2.1 A family of vectors $\{f_i\}_{i \in I}$ in the infinite-dimensional Hilbert space H is a frame if there are constants $0 < A \leq B < +\infty$ so that for all $x \in H$,

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B\|x\|^2.$$

If $A = B$, the frame is called an A-tight frame (or a tight frame).

Note that the theory of frames in Hilbert spaces presents a central tool in mathematics and engineering, and has been developed rather rapidly in the past decade [6,8].

Definition 2.2 Let $\{W_i\}_{i \in I}$ be a collection of subspaces in the infinite-dimensional Hilbert space H and $\{P_i\}_{i \in I}$ be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i \in I}$ (or $\{P_i\}_{i \in I}$) yields phase retrieval if for all $x, y \in H$ and $i \in I$, $\|P_i x\| = \|P_i y\|$, then $x = cy$ for some scalar c with $|c| = 1$.

In particular, a set of vectors $\{f_i\}_{i \in I}$ in H does phase retrieval, if for $x, y \in H$ and $i \in I$, $|\langle x, f_i \rangle| = |\langle y, f_i \rangle|$, then $x = cy$ for some scalar c with $|c| = 1$.

Definition 2.3 Let $\{W_i\}_{i \in I}$ be a collection of subspaces in the infinite-dimensional Hilbert space H and $\{P_i\}_{i \in I}$ be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i \in I}$ (or $\{P_i\}_{i \in I}$) yields norm retrieval if for all $x, y \in H$ and $i \in I$, $\|P_i x\| = \|P_i y\|$, then $\|x\| = \|y\|$.

In particular, a set of vectors $\{f_i\}_{i \in I}$ in H does norm retrieval, if for $x, y \in H$ and $i \in I$, $|\langle x, f_i \rangle| = |\langle y, f_i \rangle|$, then $\|x\| = \|y\|$.

Remark 2.1 It is immediate that a collection of subspaces (or orthogonal projections) or a family of vectors doing phase retrieval does norm retrieval.

It is easy to see that tight frames do norm retrieval.

Theorem 2.1 *Tight frames do norm retrieval.*

Proof Let $\{f_i\}_{i=1}^{\infty}$ in the infinite-dimensional Hilbert space H be an A-tight

frame. Suppose that $x, y \in H$ satisfy $|\langle x, f_i \rangle| = |\langle y, f_i \rangle|$ for all i. Then

$$A\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, f_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle y, f_i \rangle|^2 = A\|y\|^2,$$

thus $\|x\| = \|y\|$. Therefore $\{f_i\}_{i=1}^{\infty}$ does norm retrieval. The proof is completed.

Since an orthonormal basis is a tight frame, we have the following corollary.

Corollary 2.1 *Orthonormal bases do norm retrieval.*

The following proposition is obvious.

Proposition 2.1 *Let $\{P_i\}_{i \in I}$ be a collection of orthogonal projections on the infinite-dimensional Hilbert space H and $\{f_i\}_{i \in I}$ be a family of vectors in H.*

(1) If $\{P_i\}_{i \in I}$ does norm retrieval, then $\{P_i\}_{i \in I} \cup \{Q_j\}_{j \in J}$ does norm retrieval for any orthogonal projections $\{Q_j\}_{j \in J}$ on H.

(2) If $\{f_i\}_{i \in I}$ does norm retrieval, then $\{f_i\}_{i \in I} \cup \{g_j\}_{j \in J}$ does norm retrieval for any vectors $\{g_j\}_{j \in J}$ in H.

In particular, if a family of vectors $\{h_i\}_{i \in I}$ contains a tight frame, then it does norm retrieval.

The following theorem provides a sufficient condition under which the subspaces spanned by the canonical basis vectors do norm retrieval.

Theorem 2.2 *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the infinite-dimensional Hilbert space H. Let $\{W_i\}_{i \in I}$ be subspaces of H with $W_i = \overline{\mathrm{span}}\{e_j : j \in I_i\}$, $I_i \subseteq \mathbf{N}$. If there exists an m such that for all j, $|\{i : j \in I_i\}| = m$, then $\{W_i\}_{i \in I}$ does norm retrieval.*

Proof Let P_i be orthogonal projections onto W_i for $i \in I$. Suppose that for $x, y \in H$ and $i \in I$, $\|P_i x\| = \|P_i y\|$. Since

$$\sum_{i=1}^{\infty} \|P_i x\|^2 = \sum_{i=1}^{\infty} \sum_{j \in I_i} |\langle P_i x, e_j \rangle|^2 = \sum_{i=1}^{\infty} \sum_{j \in I_i} |\langle x, P_i e_j \rangle|^2$$

$$= \sum_{i=1}^{\infty} \sum_{j \in I_i} |\langle x, e_j \rangle|^2 = m \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2 = m\|x\|^2$$

and $\sum_{i=1}^{\infty} \|P_i y\|^2 = m\|y\|^2$, it follows that $\|x\| = \|y\|$. Thus $\{W_i\}_{i \in I}$ does norm retrieval. The proof is completed.

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the infinite-dimensional Hilbert space H. Let $W_i = \{e_i\}^{\perp}$ for all $i \in I \subseteq \mathbf{N}$. The following two theorems show that $\{W_i\}_{i \in I}$ does norm retrieval if and only if I is an infinite subset of $\mathbf{N}$.

Theorem 2.3 *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the infinite-dimensional Hilbert space H. Suppose that I is an infinite subset of $\mathbf{N}$. Let $W_i = \{e_i\}^{\perp}$ for all $i \in I$. Then $\{W_i\}_{i \in I}$ does norm retrieval.*

Proof Let P_i be orthogonal projections onto W_i for $i \in I$. Suppose that for $x, y \in H$ and $i \in I$, $\|P_ix\| = \|P_iy\|$. Then

$$\sum_{j \neq i} |\langle x, e_j \rangle|^2 = \|P_ix\|^2 = \|P_iy\|^2 = \sum_{j \neq i} |\langle y, e_j \rangle|^2.$$

Let $i_0 = \min\{i : i \in I\}$. For each $i \in I$ with $i > i_0$, we have

$$
\begin{aligned}
|\langle x, e_i \rangle|^2 - |\langle x, e_{i_0} \rangle|^2 &= \sum_{j \neq i_0} |\langle x, e_j \rangle|^2 - \sum_{j \neq i} |\langle x, e_j \rangle|^2 \\
&= \sum_{j \neq i_0} |\langle y, e_j \rangle|^2 - \sum_{j \neq i} |\langle y, e_j \rangle|^2 \\
&= |\langle y, e_i \rangle|^2 - |\langle y, e_{i_0} \rangle|^2,
\end{aligned}
$$

then $|\langle x, e_i \rangle|^2 - |\langle y, e_i \rangle|^2 = |\langle x, e_{i_0} \rangle|^2 - |\langle y, e_{i_0} \rangle|^2 = K$. Thus $|\langle x, e_i \rangle|^2 = |\langle y, e_i \rangle|^2 + K$ for all $i \in I$. Now from

$$\sum_{i \in I} K \leq \sum_{i \in I} (|\langle y, e_i \rangle|^2 + K) = \sum_{i \in I} |\langle x, e_i \rangle|^2 \leq \|x\|^2 < +\infty,$$

we can conclude that $K = 0$. Then $|\langle x, e_{i_0} \rangle|^2 = |\langle y, e_{i_0} \rangle|^2$. Therefore

$$\|x\| = \left(\sum_{j \neq i_0} |\langle x, e_j \rangle|^2 + |\langle x, e_{i_0} \rangle|^2 \right)^{\frac{1}{2}} = \left(\sum_{j \neq i_0} |\langle y, e_j \rangle|^2 + |\langle y, e_{i_0} \rangle|^2 \right)^{\frac{1}{2}} = \|y\|.$$

Thus $\{W_i\}_{i \in I}$ does norm retrieval. The proof is completed.

Theorem 2.4 *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the infinite-dimensional Hilbert space H. Suppose that I is a finite subset of $\mathbf{N}$. Let $W_i = \{e_i\}^{\perp}$ for all $i \in I$. Then $\{W_i\}_{i \in I}$ cannot do norm retrieval.*

Proof Without loss of generality, we suppose $I = \{1, 2, \cdots, N\}$ $(N > 1)$. Let P_i be orthogonal projections onto W_i for $i \in I$. Let $x = \sum_{i=1}^{N+1} e_i$ and $y = \sqrt{\frac{N}{N-1}} \sum_{i=1}^{N} e_i$. Then

$$\|P_ix\|^2 = \sum_{j \neq i} |\langle x, e_j \rangle|^2 = \sum_{j \neq i, 1 \leq j \leq N+1} |\langle e_j, e_j \rangle|^2 = N$$

and

$$\|P_iy\|^2 = \sum_{j \neq i} |\langle y, e_j \rangle|^2 = \frac{N}{N-1} \sum_{j \neq i, 1 \leq j \leq N} |\langle e_j, e_j \rangle|^2 = N,$$

so $\|P_ix\| = \|P_iy\|$ for all $i \in I$. But $\|x\| = \sqrt{N+1}$ and $\|y\| = \sqrt{\frac{N^2}{N-1}}$. Thus $\{W_i\}_{i \in I}$ cannot do norm retrieval. The proof is completed.

Now, we strengthen the above result but do not require that the vectors are orthogonal. To prove this, we need the following lemma.

Lemma 2.1 *Let $\{f_i\}_{i=1}^N$ be independent vectors in the infinite-dimensional real Hilbert space H. Then there are a vector $f \in \mathrm{span}\{f_i : i = 1, 2, \cdots, N\}$, with $\|f\| = 1$, and a constant $c > 0$ satisfying:*

$$|\langle f, f_i \rangle| = c, \quad for\ all\ i = 1, 2, \cdots, N.$$

Proof Let $H_1 = \mathrm{span}\{f_i : i = 1, 2, \cdots, N\}$. Then H_1 is a closed subspace of H. Thus H_1 is a Hilbert space. There exists a unitary operator T from Hilbert space $\mathbf{R}^N$ onto H_1. Let $\phi_i = T^{-1}f_i$, $i = 1, 2, \cdots, N$. Then $\{\phi_i\}_{i=1}^N \subseteq \mathbf{R}^N$ are independent. By [[7], Lemma 3.7], there exist a vector $\phi \in \mathbf{R}^N$, with $\|\phi\| = 1$, and a constant $c > 0$ satisfying $|\langle \phi, \phi_i \rangle| = c$ for all $i = 1, 2, \cdots, N$. Let $f = T\phi$. Then $f \in H_1$ and

$$|\langle f, f_i \rangle| = |\langle T\phi, T\phi_i \rangle| = |\langle T^*T\phi, \phi_i \rangle| = |\langle \phi, \phi_i \rangle| = c, \quad \text{for all } i = 1, 2, \cdots, N.$$

The proof is completed.

Theorem 2.5 *Let $\{f_i\}_{i=1}^N$ be independent vectors in the infinite-dimensional real Hilbert space H with $\|f_i\| = 1$. Let $W_i = \{f_i\}^{\perp}$ for all $i = 1, 2, \cdots, N$. Then $\{W_i\}_{i=1}^N$ cannot do norm retrieval.*

Proof By Lemma 2.1, there are a vector $f \in \mathrm{span}\{f_i : i = 1, 2, \cdots, N\}$ with $\|f\| = 1$, and a constant $c > 0$ satisfying $|\langle f, f_i \rangle| = c$ for all $i = 1, 2, \cdots, N$. Let P_i be orthogonal projections onto W_i, for all $i = 1, 2, \cdots, N$. Choose an $x \in \bigcap_{i=1}^N W_i$ with $\|x\| = 1$. Let $y = cx + f$. Since $\langle x, f_i \rangle = 0$ for each $i = 1, 2, \cdots, N$, we have $\langle x, f \rangle = 0$. Thus

$$\|y\| = \sqrt{\|cx\|^2 + \|f\|^2} = \sqrt{c^2 + 1} \neq 1.$$

But for all $i = 1, 2, \cdots, N$,

$$\begin{aligned}
\|P_i y\|^2 &= \|y\|^2 - \|P_i^{\perp} y\|^2 = c^2 + 1 - \left\| \frac{\langle y, f_i \rangle}{\|f_i\|^2} f_i \right\|^2 \\
&= c^2 + 1 - |c\langle x, f_i \rangle + \langle f, f_i \rangle|^2 = c^2 + 1 - c^2 \\
&= 1 = \|x\|^2 = \|P_i x\|^2,
\end{aligned}$$

then $\|P_i x\| = \|P_i y\|$. Thus $\{W_i\}_{i=1}^N$ cannot do norm retrieval. The proof is completed.

At last, we show that the property of independent for the vectors in the above result is necessary.

Example 2.1 In the infinite-dimensional real Hilbert space H, three proper subspaces of codimension one can do norm retrieval.

Proof Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in H. Let

$$f_1 = e_1, \quad f_2 = e_2, \quad f_3 = \frac{e_1 - e_2}{\sqrt{2}}.$$

Let $W_i = \{f_i\}^{\perp}$ and P_i be orthogonal projections onto W_i for $i = 1, 2, 3$. We claim $\{W_i\}_{i=1}^{3}$ does norm retrieval. Firstly, for any $z = \sum\limits_{i=1}^{\infty} z_i e_i \in H$, we have

$$\|P_1 z\|^2 = z_2^2 + \sum_{i=3}^{\infty} z_i^2, \quad \|P_2 z\|^2 = z_1^2 + \sum_{i=3}^{\infty} z_i^2,$$

$$\|P_3 z\|^2 = \left(\frac{z_1 + z_2}{\sqrt{2}}\right)^2 + \sum_{i=3}^{\infty} |z_i|^2 = \frac{z_1^2 + 2 z_1 z_2 + z_2^2}{2} + \sum_{i=3}^{\infty} z_i^2.$$

Then

$$\|P_1 z\|^2 - \|P_2 z\|^2 = z_2^2 - z_1^2$$

and

$$\frac{\|P_1 z\|^2 + \|P_2 z\|^2}{2} - \|P_3 z\|^2 = z_1 z_2.$$

Now suppose that $x = \sum\limits_{i=1}^{\infty} x_i e_i \in H$, $y = \sum\limits_{i=1}^{\infty} y_i e_i \in H$ satisfy $\|P_i x\| = \|P_i y\|$ for all $i = 1, 2, 3$. Then

$$x_2^2 - x_1^2 = y_2^2 - y_1^2 = K \quad \text{and} \quad x_1 x_2 = y_1 y_2.$$

So

$$x_1^4 + K x_1^2 = x_1^2 (x_1^2 + K) = x_1^2 x_2^2 = y_1^2 y_2^2 = y_1^2 (y_1^2 + K) = y_1^4 + K y_1^2$$

and

$$x_2^4 - K x_2^2 = (x_2^2 - K) x_2^2 = x_1^2 x_2^2 = y_1^2 y_2^2 = (y_2^2 - K) y_2^2 = y_2^4 - K y_2^2.$$

Thus

$$(x_1^2 + y_1^2)(x_1^2 - y_1^2) = x_1^4 - y_1^4 = -K(x_1^2 - y_1^2)$$

and

$$(x_2^2 + y_2^2)(x_2^2 - y_2^2) = x_2^4 - y_2^4 = K(x_2^2 - y_2^2).$$

If $x_1^2 \neq y_1^2$ and $x_2^2 \neq y_2^2$, then $K = -(x_1^2 + y_1^2) \leq 0$ and $K = x_2^2 + y_2^2 \geq 0$, so $K = 0$. Thus $x_1^2 = y_1^2 = x_2^2 = y_2^2 = 0$, a contradiction. Therefore $x_i^2 = y_i^2$ for $i = 1$ or 2. Since

$$\sum_{j \neq i} x_j^2 = \|P_i x\|^2 = \|P_i y\|^2 = \sum_{j \neq i} y_j^2,$$

$$\|x\| = \left(x_i^2 + \sum_{j \neq i} x_j^2\right)^{\frac{1}{2}} = \left(y_i^2 + \sum_{j \neq i} y_j^2\right)^{\frac{1}{2}} = \|y\|.$$

Thus $\{W_i\}_{i=1}^{3}$ does norm retrieval.

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