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THE EULER EQUATION AND ONSAGER CONJECTURE*†

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Abstract

In this paper, we introduce the progress of the Euler equation and Onsager conjecture. We also introduce the Euler's life, the researches about the incompressible Euler equation, and the Onsager conjecture.

Keywords incompressible Euler equation; Onsager conjecture; boundary layer; Lax pair; inviscid limit

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As the number theory, the Euler equation in the incompressible flow is the source of many branches of mathematics. Linear partial differential equation, dynamic system, nonlinear partial differential equation, geometric partial differential equation, harmonic analysis and the completely integrable system and other mathematical branches can be traced back to the Euler equation. Close to the weather forecast, as far as the supernova explosion theory, the influence of Euler equation has been run throughout all subjects outside mathematics. From the Euler equation, we can get the challenge problem and the inspiration.

1 Introduction about the Euler's Life

Leohard Euler was a Swiss mathematician, physicist, astronomer, logician and engineer. L. Euler was born on 15 April 1707, in Basel. He was died on 18 September 1783, in Saint Petersburg, Russian Empire. He was born in a clergyman's family.

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When he was 15, he got a bachelor's degree from the University of Basel, and got his master degree in the next year. Although his father wanted him to study theology, but he was interested in mathematics. He was guided by Bornoulli. At the age of 18, he gave up the idea of becoming a pastor, specialized in mathematics, and began to publish articles. In 1727, he was invited by Petersburg Academy to Russia. In 1731, he succeeded Bernoulli as a professor of Physics. He went into research with great energy. For 14 years in Russia, he did a great deal of work in analysis, theory of numbers, and mechanics. He also solved many practical problems such as geology and shipbuilding at the request of the Russian government. A lot of writing caused eye trouble that blinded him in the right eye in 1735. In 1741, he was invited to the Berlin Academy of Sciences by the great emperor of Prussia for 25 years. During his stay in Berlin, his researches were more extensive, involving planetary motion, rigid body transport, thermodynamics, demography, and mathematical models of fluids. These work and his mathematical research promoted each other. Euler's achievements in mathematical equations, surface geometry and other mathematical fields was pioneering at this time. In 1766 he returned to Petersburg. A serious illness make his left eye completely blind in 1771. However, because of his amazing memory and mental arithmetic skills, his creativity continued to work. He completed a great deal of scientific work through the discussion with his assistant until the last minute of his life. Euler is one of the most outstanding figures in Mathematics in eighteenth Century. Not only did he make great contributions to mathematics, but also he applied mathematics to almost the entire field of physics. He is an incomparable prolific author. He wrote a great deal of textbooks about mechanics, analysis, geometry, calculus. "The introduction to infinitesimal analysis", "The principles of calculus", and "The principles of integral calculus" all became the classical works of mathematics. In addition to textbooks, he wrote creative papers at the rate of eight hundred pages a year at his work. He wrote 856 papers and 31 monographs. His complete works are in 74 volumes.

Euler's greatest contribution was to expand the field of calculus and lay the foundation for the emergence and development of some important branches of analysis (such as infinite series, differential equations) and differential geometry.

Euler transformed an infinite series from a general tool into an important research project. His best result is to compute the number of functions in even numbers:

$$\xi(2k) = \sum_{n=1}^{\infty} n^{-2k} = a_{2k}\pi^{2k}.$$

He proved that a_{2k} was a rational number, and was expressed by Bernoulli numbers. He studied the harmonic series and calculated the Euler constant exactly as the value of

$$r = \lim_{n\to+\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)$$

(Its value is approximately as $0.57721566490\cdots$).

In the middle of the eighteenth Century, Euler and other mathematicians created the discipline of differential equations in the process of solving physics problems. In the ordinary differential equation, he gave a complete solving method to the n-order linear homogeneous equation with constant coefficients. For the non-homogeneous equation, he gave a method to reduce the order of the equation. In the partial differential equation, Euler took into account the problem of two-dimensional object shaking, and he reduced to the Bessel equation, whose solution is the first kind of Bessel's function. Euler also obtained one- or two- and three-dimensional wave equations and gave a solving method. It is worth mentioning that the first paper of pure mathematical research on partial differential equations is Euler's book "The research of the integral method equations". Euler also studied the method of representation of functions by trigonometric series and the series method to solve differential equations.

Differential geometry is the study of the nature of the curve varies from point to point. Euler introduced the parametric equation of space curve, and gave the analytic expression of curvature radius of space curve. In 1766, he published the book "the study of curves on surfaces", and established the theory of surfaces. He represented the curve as $z = f(x, y)$ and introduced a series of standard symbols which can be used to denote the partial derivation of z with respect to x, y. These symbols are still common today. He got the curvature formula of the intersection line on the surface in arbitrary cross section. This work is the most important contribution of Euler to differential geometry, and is a milestone in the history of differential geometry. In the variational theory, through the study of function extremum, after solving the extremal problem of general function, Euler studied "the steepest descent" in 1734, and successfully found the ordinary differential equation of extremal function which must be satisfied, namely the Euler equation. In 1758, he named the new discipline calculus calculus. In the theory of numbers, the two complement law was first discovered by Euler. He also introduced the Euler functions in the number theory which was named after him, and established the basis of analytic number theory.

2 Four Characters of the Incompressible Euler Equation

For the incompressible, we can summary the following four characters.

(1) The Euler equation is a complete integrable system.

The incompressible Euler equation not only is brief in form, but also captures the essential characteristics of the ideal fluid.

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \tag{2.1}$$

$$\nabla \cdot u = 0, \tag{2.2}$$

where $u = (u_j)$ represents the velocity vector with n components. $u \cdot \nabla = \sum\limits_{j=1}^{n} u_j \partial_j$ denotes the first order differential operator with the coefficient u_j. The every component $u_j = u_j(x,t)$ in the velocity vector is a function of the spacial variable x and time variable t, where $x \in R^n$, $t \in R$. $\partial_t = \frac{\partial}{\partial_t}$, $\partial_j = \frac{\partial}{\partial_j}$ denote the partial derivative. (2.2) is the incompressible condition, while the divergence $\nabla \cdot u = \sum\limits_{j=1}^{n} \partial_j u_j$ is equal to 0. The function $p(x,t)$ appeared in (2.1) is a scaled function which denotes as the pressure. The Euler equation is nonlinear, and the second order nonlinear.

The Euler equation is nonlocal, and invariant under the Galileo's transformation. If the coordinate system rotates in constant v, then in the velocity y coordinate, the speed can be expressed as $u(x,t) + v$. If the coordinate system rotates at a certain angle, then the velocity will rotate at the same angle. Equation (2.1) discussed the fluid motion in a Euler coordinate system (that is, a fixed spatial coordinate system). The Euler equation is hyperbolic under some sense. The information of a particle is transmitted by a fluid. Consider time dependent mappings $X : R^n \to R^n$,

$$a \mapsto X(a,t), \quad X(a,0) = a.$$

This mapping describes the trajectory of a fluid particle that has been tagged, and a is used as a sign to recognize the particle of the fluid. The motion law of a particle under the action of a velocity field can be represented by ordinary differential equations as:

$$\partial_t X = u(X,t). \tag{2.3}$$

Equation (2.1) is converted into Newton's second law under the Lagrange coordinate as

$$\partial_t^2 X + \nabla_x p(x,t) = 0. \tag{2.4}$$

Accordingly, the incompressible condition (2.2) is

$$\det(\nabla_a X) = 1, \tag{2.5}$$

under in the Lagrange coordinate. The incompressibility means that the mapping $a \mapsto X(a,t)$ remains invariant in volume. The graduate $\nabla_x p$ in (2.4) can be calculated by $\Delta p = \nabla \cdot (u \cdot \nabla u)$.

Compared with the general incompressible fluid, the Euler equation described very effectively the motion of a fluid with constant density without any action. If the interaction between the fluids is considered and the solution is dissipated, we can derive the Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \tag{2.6}$$

which also contains the incompressible condition (2.2). Here $\nu > 0$ is the motion dissipation constant, which is a constant under the normal temperature. The second order operator needs a boundary condition. Therefore, the Laplace term in equation (2.6) has the essential physical meaning. The Navier-Stokes equation is well described the Newton flow by considering the force and boundary conditions, such as considering additional stresses, which are well described for non Newton flow.

We can write the Euler equation as the vorticity formation

$$\partial_t \Omega + (u\cdot)\nabla\Omega - (\Omega\cdot)\nabla u = 0, \tag{2.7}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the vorticity vector, $u = (u_1, u_2, u_3)$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Omega = \nabla \times u$, $\nabla \cdot u = 0$, and u can be solved from Ω by the Biot-Savart formula.

For two dimensional flow, $u = (u_1(t,x,y), u_2(t,x,y), 0)$, $\Omega = (0, 0, \Omega(t,x,y))$,

$$\partial_t \Omega + [\psi, \Omega] = 0, \tag{2.8}$$

where ψ is the flow function, $u = -\partial_y \psi$, $v = \partial_x \psi$, $\Omega = \partial_x v - \partial_y u = \Delta\psi$,

$$[f, g] = (\partial_x f)(\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g).$$

The Lax pair of equation (2.8) is

$$\begin{cases} L\psi = \lambda\psi, \\ \partial_t \psi = A\psi, \end{cases} \tag{2.9}$$

where

$$L\varphi = [\Omega, \varphi], \quad A\varphi = [\psi, \varphi].$$

Here λ is the complex number, and φ is the complex function. Using the Lax pair, we can find Darboux transform, infinite conserved low, soliton solution and inverse scattering method. The Lax pair in two dimensional case was proposed by M. Vishihk and Friedlander [15] in 1993, Friedlander and Vishik [5] in 1990, while the three dimensional Lax pair was proved by Yanguang Li [7-12].

(2) It has continuous solution, and does not contain shock wave solution.

Onsager Conjecture[13] *If the Hölder continuous index of the weak solution to the Euler equation satisfies $\alpha > \frac{1}{3}$, then the energy is conversed. If $\alpha \le \frac{1}{3}$, it will occur turbulence or abnormal dissipation. The Hölder continuous index $\alpha = \frac{1}{3}$ is corresponding to the index $-\frac{5}{3}$ of the energy spectrum $E(k) = c\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ in the well-known Kolmogorov-Obukhov power law. Here $\varepsilon > 0$ is the non empty kinetic energy* *dissipation rate*

$$\varepsilon = \nu\langle|\nabla u|^2\rangle,$$

where $\langle\cdot\rangle$ *denotes the mathematical expectation of the invariant measure for the solution of the NS equation at large Reynolds number. Specially,*

$$\langle((u(x+re)-u(x))\cdot e)^p\rangle \sim (\varepsilon r)^{\frac{p}{3}}.$$

If $p=2$, *this is the Kolmogorov* $\frac{2}{3}$ *law.*

So far, the first conjecture has been proved, and the second one has new process recently.

(3) It has the abnormal dissipation phenomenon.

Inviscid limit, which is the limit of the vanishing tending to 0 about the solution to the Navier-Stokes equation, has a special kind singularity. When making the limit, the type of the original equation is changed. In the bounded domain, the boundary layer can explain why the inviscid limit is the singular limit. The Navier-Stokes equation is a second order partial differential equation, which needs the boundary conditions, such as the homogeneous Dirichlet boundary condition

$$u|_{\partial\Omega} = 0.$$

For the Euler equation, the boundary condition is that the quality path is tangent to the region boundary

$$u\cdot\mathbf{N}|_{\partial\Omega} = 0,$$

where $\mathbf{N}$ is the normal vector of the boundary. In the inviscid limit, there is a thin layer near the boundary. In this thin layer, there will be a great shift of viscous flow into cohesionless flow.

One of the basic problem in the inviscid limit is what changes the ideal energy conversed quantity will make. There is an important conserved quantity of the smooth solution to the two dimensional Euler equation, that is, the eddy energy

$$\int|\omega(x,t)|^2\mathrm{d}x.$$

Considering the Navier-Stokes equation without outer force, which has the initial eddy energy in L^2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}|\omega(x,t)|^2\mathrm{d}x + \nu\int_{R^2}|\nabla\omega(x,t)|^2\mathrm{d}x = 0, \tag{2.10}$$

where the expression $\nu\int_{R^2}|\nabla\omega(x,t)|^2\mathrm{d}x$ is called as the instantaneous velocity of eddy energy dissipation.

If the gradient of the vorticity is bounded in L^2, then this expression tends to 0 as $\nu \to 0$. Since the gradient of temperature may be very large, the limit of the upper equation is not necessarily zero, this is called abnormal dissipation. According to the theory of turbulence, long time behavior of the two-dimensional NS equation with the external force, due to reverse cascade action of the kinetic energy, the energy will gather and grow unlimitedly in big scale, many small vortex swirls into the maelstrom. If the viscosity is zero, and the outer force does not vanish, it can not exist a stable inviscid state. Both the experimental and numerical calculations show that the instantaneous velocity of the eddy energy dissipation of the NS equation is bounded, and the lower bound also exists with the high Reynolds number (very small viscosity), that is

$$\lim_{\nu \to 0} \nu \int_{R^2} |\nabla \omega(x,t)|^2 \mathrm{d}x > \varepsilon_0 > 0.$$

(4) The solution of the Euler equation is more unstable than the actual situation.

For the Euler equation, the more suitable stability concept is a generalized case, namely the stability of the orbit. For complex fluids with Euler equations including the stochastic PDE model and the mixed system (PDE coupled stochastic differential equations), their stability theory is very complex.

3 Onsager Conjecture

In 1949, L. Onsager [13] pointed out one of his conjectures in the statistical mechanics paper, which is called the Onsager conjecture now. He was a Nobel prize winner in chemistry and he was good at mathematics. He firstly worked in Brown university, and later worked in Yellow university. In 1994, P. Constantin, E, Titi [4] proved that for the Hölder index $\alpha > \frac{1}{3}$, the Onsager conjecture is true, namely the energy is conserved. For the case of $\alpha \le \frac{1}{3}$, this conjecture is completely proved. The failure of conservation of energy may lead to the phenomenon of the eddy currents, which can be seen in the following theorem.

Theorem 3.1[1] *Let $e : [0,1] \to R$ be a positive function and ε be a positive constant. Then there exists a continuous vector field $v \in C^{\frac{1}{5}-\varepsilon}(T^3 \times [0,1], R^3)$ and a continuous scaled field $p \in C^{\frac{2}{5}-\varepsilon}(T^3 \times [0,T], R)$ which are solutions to the incompressible Euler equation*

$$\begin{aligned} &\partial_t v + \mathrm{div}(v \otimes v) + \nabla p = 0, \\ &\mathrm{div} v = 0, \end{aligned} \tag{3.1}$$

in the distribution, such that

$$e(t) = \int |v|^2(x,t)\mathrm{d}t, \quad \textit{for any } t \in [0,1].$$

Theorem 3.2[3] *For any $\alpha < \frac{1}{3}$, the incompressible Euler equation* (3.1) *has a non-zero weak solution*

$$v \in C_{tx}^{\alpha}(R \times (R/Z)^3), \quad p \in C_{tx}^{2\alpha}(R \times (R/Z)^3)$$

such that $v \equiv 0$ in finite time, specially, the energy conservation of the solution is not true.

Theorem 3.3[3] *Let (v,p) be the weak solution of the incompressible Euler equation* (3.1)*, $v \in C_t C_x^{\frac{1}{3}}$, then*

$$e(t) = \int_{T^d} \frac{|v(x,t)|^2}{2} \mathrm{d}x \in C^1(t), \quad d \geq 2.$$

4 The Non-uniqueness of Solutions of Three Dimensional Euler Equations

A. Scheffer (1993), A. Shnirelman (1997) proved the non uniqueness of solutions of three dimensional Euler equations. Later on, more complete results have been obtained.

Theorem 4.1[14] *There exists a weak solution to the incompressible Euler equation* (3.1)*, $v(x,t) \in L^2(T^2 \times R)$ and $C > 0$ such that*

$$v(x,t) = 0, \quad |t| > C.$$

Theorem 4.2[6] *Let $v \in L^\infty(R_x^n \times R_t; R^n)$, $p \in L^\infty(R_x^n \times R_t)$ be the weak solutions to equation* (3.1) *in the distribution, then v is not identical to* 0*, supp v and supp p are compact in the space-time $R_x^n \times R_t$, moreover it yields*

$$\int |v(x,t)|^2 \mathrm{d}x = 1, \quad a.e.\ t \in [-1,1], \tag{4.1}$$

$$v(x,t) = 0, \quad |t| > 1. \tag{4.2}$$

5 Several Significant Open Problems

In this section, we give some important open problems about the Euler equation.

Question 1: The Rayleigh-Taylor instability and Kelvin-Helmaltz instability are open problem in the mixing layer of the Euler equation.

Question 2: The blow-up of the Euler equation is still one of the most challenging and most significant problem in the nonlinear PDE field. In the domain with boundary, our research on the inviscid limit (more precisely, the limit of the high Reynolds number) is still immature. Specially, the inviscid limit in the bounded region is extremely challenging. We do not know whether there is a weak solution of energy dissipation in the whole space (in the proper function space). In 2014, J. Bourgain and Dong Li (Invent. Math.) proved the illposedness of the solution to the 2D Euler equation in the whole space and the Sobolev space $W^{\frac{d}{p}+1}$ $(1 \leq p < +\infty,\ d = 2)$.

Question 3: The statistical characteristic properties of the long time inviscid limit (including the Kolmogorov-Obukhov energy spectrum) are still an important open question.

Question 4: We do not know how the coefficient of viscosity of the NS equation corresponds to the regularity of the Euler equation.

Question 5: Although the theory about the Prandtl boundary layer have some progress, there exist some very important open problems.

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