Ann. of Appl. Math.
33:4(2017), 340-352

EXISTENCE OF SOLUTIONS FOR NONLOCAL BOUNDARY VALUE PROBLEM OF FRACTIONAL DIFFERENTIAL EQUATIONS ON THE INFINITE INTERVAL*

Abdellatif Ghendir Aoun†

(*Ecole Normale Supérieure, 16050 Kouba, Algiers, Algeria*)

Abstract

In this paper, we study a fractional differential equation

$$ {}^cD^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0, \quad t \in (0, +\infty) $$

satisfying the boundary conditions:

$$ u'(0) = 0, \quad \lim_{t\to+\infty} {}^cD^{\alpha-1}_{0^+}u(t) = g(u), $$

where $1 < \alpha \leqslant 2$, ${}^cD^{\alpha}_{0^+}$ is the standard Caputo fractional derivative of order α. The main tools used in the paper is a contraction principle in the Banach space and the fixed point theorem due to D. O'Regan. Under a compactness criterion, the existence of solutions are established.

Keywords boundary value problem; fractional differential equation; infinite interval; nonlocal condition; fixed point theorem

2000 Mathematics Subject Classification 34A08; 34B40

1 Introduction

In this paper, we consider the following boundary value problem (BVP for short)

$$ \begin{cases} {}^cD^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0, \quad t \in (0, +\infty), \\ u'(0) = 0, \quad \lim\limits_{t\to+\infty} {}^cD^{\alpha-1}_{0^+}u(t) = g(u), \end{cases} \tag{1.1} $$

where $1 < \alpha \leqslant 2$ and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $g : X \to \mathbb{R}$ are the given functions such that X is a suitable Banach space.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, electrical circuits, biology, and so on, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory

*Manuscript received April 13, 2017; Revised August 14, 2017

†Corresponding author. E-mail: ghendirmaths@gmail.com

and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Further, the concept of nonlocal boundary conditions has been introduced to extend the study of classical boundary value problems. This notion is more precise for describing nature phenomena than the classical notion because additional information is taken into account. For the importance of nonlocal conditions in different fields, in the paper, we let $g(u)=\sum\limits_{i=1}^{p} c_i u(\xi_i)$, where $c_1, c_2, \cdots, c_p$ are given constants with $p\in\mathbb{N}^*$, and $0<\xi_1<\xi_2<\ldots<\xi_p<+\infty$ as an application of the results we will get.

The problem studied in this paper is very well motivated in relationship with several previous contributions. The main difficulty in treating this class of the fractional differential equations is the possible lack of compactness due to the infinite interval, besides the boundary condition which prevents from proving compactness conditions. In order to overcome these difficulties, the authors use a special Banach space which can establish some similar inequalities as finite interval. These better properties can be guarantee that the operator is completely continuous.

The mathematical investigation of such problems has been the subject of several research works during the last years (see, e.g., [1-5,7,9-16]).

In [1], Arara, Benchohra, Hamidi and Nieto discussed the existence of bounded solutions, using Schauder's fixed point theorem, of the following problem on an unbounded domain:

$$\begin{cases} {}^cD^{\alpha}_{0+}y(t)=f(t,y(t))=0, \quad t\in J:=[0,+\infty),\\ u(0)=y_0, \quad y \text{ is bounded on } J,\end{cases}$$

where $1<\alpha\leqslant 2$ and $y_0\in\mathbb{R}$.

In [5], Liang and Zhang considered the m-point BVP of fractional differential equation on unbounded interval:

$$\begin{cases} D^{\alpha}_{0+}u(t)+a(t)f(t,u(t))=0, \quad t\in(0,+\infty),\\ u(0)=0, \quad u'(0)=0, \quad D^{\alpha-1}_{0+}u(+\infty)=\sum\limits_{i=1}^{m-2}\beta_i u(\xi_i),\end{cases}$$

where $2<\alpha\leqslant 3$. Using a fixed point theorem for operators on a cone of a Banach space, sufficient conditions for the existence of multiple positive solutions were established.

Su and Zhang [9] discussed the existence of unbounded solutions of the following BVP using Schauder's fixed point theorem:

$$\begin{cases} D^{\alpha}_{0+}u(t)+f(t,u(t),D^{\alpha-1}_{0+}u(t))=0, \quad t\in(0,+\infty),\\ u(0)=0, \quad u'(0)=0, \quad D^{\alpha-1}_{0+}u(\infty)=u_\infty, \quad u_\infty\in\mathbb{R},\end{cases}$$

where $1 < \alpha \leqslant 2$.

Also a fixed point theorem is employed in Zhao and Ge [16] to show the existence of positive solutions of the fractional order differential equation:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, +\infty), \\ u(0) = 0, \quad \lim\limits_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = \beta u(\xi), \end{cases}$$

where $1 < \alpha < 2$, $\xi > 0$.

The work presented in this paper is the designation of boundary value problem of fractional order on the infinite interval, the main tool used is the fixed point theorem attributed to D. O'Regan and the result obtained is for the existence solution of problem (1.1). As we known, $[0, +\infty)$ is noncompact, in order to overcome these difficulties, a special Banach space is introduced so that we can establish some similar inequalities, which guarantee that the functionals defined on $[0, +\infty)$ have better properties and then we can proceed with the fixed point D. O'Regan theorem.

Motivated by the papers [1,5,9,16], we consider the fractional BVP (1.1) on the half-line. The plan of the paper is as follows. First we present in Section 2 some definitions and lemmas which are crucial to our discussion. Related lemmas necessary to the fixed point formulation are given in Section 3. Section 4 is devoted to the main existence theorem of solutions. We end the paper by an example of application to illustrate the theoretical result.

2 Preliminaries

We start with some definitions and lemmas on the fractional calculus (see [4]).

One of the basic tools of the fractional calculus is the Gamma function which extends the factorial to positive real numbers (and even complex numbers with positive real parts).

Definition 2.1 For $\alpha > 0$, the Euler gamma function is defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \mathrm{e}^{-t} \mathrm{d}t.$$

Properties 2.1 For every $\alpha > 0$ and n being a positive integer, we have

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)},$$

hence

$$\Gamma(\alpha+n) = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\Gamma(\alpha).$$

Specially,

$$\Gamma(1) = \int_0^{+\infty} e^{-t} \mathrm{d}t = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

$$\Gamma(n+1)=n!,\quad \Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}(2n)!}{2^{2n}n!}.$$

Definition 2.2 The fractional integral of order $\alpha \geqslant 0$ for a function h is defined as

$$I_{0+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\mathrm{d}s,$$

provided the right side is point-wise defined on $(0,+\infty)$.

Definition 2.3 For a function h given on the interval $[0,+\infty)$, the Caputo fractional derivative of order $\alpha \geqslant 0$ of h is defined by

$${}^{c}D_{0+}^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)\mathrm{d}s,$$

where $n=[\alpha]+1$.

Lemma 2.1[4] *Let $\alpha \geqslant 0$, then*

$$I_{0+}^{\alpha}\,{}^{c}D_{0+}^{\alpha}h(t)=h(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

for some $c_i\in\mathbb{R},\ i=0,1,2,\cdots,n-1,\ n=[\alpha]+1$.

Lemma 2.2[4] *Let $\beta \geqslant \alpha \geqslant 0$ and $h\in L^1[0,+\infty)$, then ${}^{c}D_{0+}^{\alpha}I_{0+}^{\beta}h(t)=I_{0+}^{\beta-\alpha}h(t)$.*

We introduce the fixed point theorem which was established by O'Regan. This theorem will be adopted to prove the main results.

Theorem 2.1[8] *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0\in U$, $F(\overline{U})$ is bounded and $F:\overline{U}\to C$ is given by $F=F_1+F_2$, where $F_1:\overline{U}\to E$ is continuous and completely continuous and $F_2:\overline{U}\to E$ is a nonlinear contraction (that is, there exists a continuous nondecreasing function $\phi:[0,+\infty)\to[0,+\infty)$ satisfying $\phi(z)<z$ for $z\geqslant 0$, such that $\|F_2(x)-F_2(y)\|\leqslant \phi(\|x-y\|)$, for all $x,\,y\in\overline{U}$). Then either,*

(A1) *F has a fixed point in $\overline{U}$, or*

(A2) *there is a point $u\in\partial U$ and $\lambda\in(0,\ 1)$ with $u=\lambda F(u)$.*

Remark 2.1 $\overline{U}$ and ∂U respectively, represent the closure and boundary of U.

3 Related Lemmas

Consider a space X defined by

$$X=\left\{u\in C([0,\,+\infty),\mathbb{R}),\qquad \sup_{t\in[0,+\infty)}\left|\frac{u(t)}{1+t^{\alpha-1}}\right| \text{ exists}\right\},$$

with the norm

$$\|u\|_X=\sup_{t\in[0,+\infty)}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|.$$

Lemma 3.1 *$(X, \|\cdot\|_X)$ is a Banach space.*

Proof Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the space $(X, \|\cdot\|_X)$. Then for any $\epsilon \geqslant 0$, there exists an $N \geqslant 0$ such that $\|u_n - u_m\|_X < \epsilon$ for any $n, m \geqslant N$, that is, for any $\epsilon \geqslant 0$, there exists an $N \geqslant 0$ such that $\left|\frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u_m(t)}{1+t^{\alpha-1}}\right| < \epsilon$, for any $t \in [0, +\infty)$ and $n, m \geqslant N$. Thus $\left\{\frac{u_n(t)}{1+t^{\alpha-1}}\right\}_{n\in\mathbb{N}}$ for $t \in [0, +\infty)$, is a Cauchy sequence in $\mathbb{R}$, too.

So, there exists a $\frac{u(t)}{1+t^{\alpha-1}} \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} \left|\frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u(t)}{1+t^{\alpha-1}}\right| = 0,\ t \in [0, +\infty)$, with $u \in X$ which implies $\lim\limits_{n\to+\infty} \|u_n - u\|_X = 0$. So $(X, \|\cdot\|_X)$ is a Banach space. The proof is completed.

Now we list some conditions in this section for convenience:

(H1) The function $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exists a nonnegative function $\varphi \in L^1[0, +\infty)$ such that

$$|f(t, (1+t^{\alpha-1})x) - f(t, (1+t^{\alpha-1})y)| \leqslant \varphi(t)|x-y|, \quad \text{for } t \in [0, +\infty), \text{ and all } x, y \in \mathbb{R}.$$

(H3) There exists a positive constant $l < 1$ such that $|g(u) - g(v)| \leqslant l\|u - v\|_X$, for all $u, v \in X$.

Lemma 3.2 *Let $h(t) \in L^1[0, +\infty)$. If* (H2) *holds, then the following BVP*

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0, +\infty), \\ u'(0) = 0, \quad \lim\limits_{t\to+\infty} {}^cD_{0^+}^{\alpha-1} u(t) = g(u) \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\mathrm{d}s + g(u) + \int_0^{+\infty} h(s)\mathrm{d}s.$$

Proof By Lemma 2.1 and from ${}^cD_{0^+}^{\alpha} u(t) + h(t) = 0$, we have

$$u(t) = -I_{0^+}^{\alpha} h(t) + c_0 + c_1 t \quad \text{for some } c_0, c_1 \in \mathbb{R}.$$

So the solution of (3.1) can be written as

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\mathrm{d}s + c_0 + c_1 t$$

and

$$u'(t) = -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} h(s)\mathrm{d}s + c_1.$$

From $u'(0) = 0$ we known that $c_1 = 0$. On the other hand by Lemma 2.2, we have

$${}^cD_{0^+}^{\alpha-1} u(t) = -{}^cD_{0^+}^{\alpha-1} I_{0^+}^{\alpha} h(t) + c_0 = -I_{0^+}^{1} h(t) + c_0 = -\int_0^t h(s)\mathrm{d}s + c_0,$$

together with $\lim\limits_{t\to+\infty} {}^cD_{0^+}^{\alpha-1}u(t) = g(u)$,

$$c_0 = g(u) + \int_0^{+\infty} h(s)\mathrm{d}s.$$

So

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\mathrm{d}s + g(u) + \int_0^{+\infty} h(s)\mathrm{d}s,$$

which verifies the existence of the solution.

Next, we will prove the uniqueness of the solution. To do this, assume that $u(t)$ and $v(t)$ are two solutions of the BVP (3.1). Then from (H3) we have

$$\|u-v\|_X \leqslant |g(u)-g(v)| \leqslant l\|u-v\|_X,$$

which imply $\|u-v\|_X = 0$, then $u \equiv v$.

Now, define the following operators $T_1,\ T_2,\ T$ by

$$\begin{aligned}
&(T_1u)(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))\mathrm{d}s + \int_0^{+\infty} f(s,u(s))\mathrm{d}s,\\
&(T_2u)(t) = g(u),\\
&(Tu)(t) = (T_1u)(t) + (T_2u)(t), \quad \text{for } u \in X.
\end{aligned}$$

The BVP (1.1) has a solution u if and only if u solves the operator equation $u = Tu$.

We will prove the existence of a fixed point of T. For this we verify that the operator T satisfies all conditions of Theorem 2.1.

Since the Arzela-Ascoli theorem fails in the space X, we need a modified compactness criterion to prove that T_1 is compact.

Lemma 3.3[6] *Let* $B = \{u \in X,\ \|u\|_X < k\}$ *be an open ball in* X $(k > 0)$ *and* $B_1 = \left\{\frac{u(t)}{1+t^{\alpha-1}},\ u \in B\right\}$. *If* B_1 *is equicontinuous on any compact intervals of* $[0,+\infty)$ *and equiconvergent at infinity, then* B *is relatively compact on* X.

Definition 3.1 B_1 is called equiconvergent at infinity if and only if for all $\epsilon \geqslant 0$, there exists a $\delta = \delta(\epsilon) \geqslant 0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \epsilon, \quad \text{for all } t_1,\ t_2 \geqslant \delta \text{ and } u \in B.$$

Remark 3.1 Equivalently, B_1 is equiconvergent at infinity if for all $\epsilon \geqslant 0$, there exists a $\delta = \delta(\epsilon) \geqslant 0$ such that

$$\left|\frac{u(t)}{1+t^{\alpha-1}} - u(+\infty)\right| < \epsilon, \quad \text{for all } t \geqslant \delta \text{ and } u \in B,$$

with $u(+\infty) = \lim\limits_{t\to+\infty} \frac{u(t)}{1+t^{\alpha-1}}$.

Let $\Omega = \{u \in X, \|u\|_X < r\}$ $(r > 0)$ be an open ball of radius r in X, and there exists a $u_0 \in \Omega$ such that $f(t, u_0(t)) = 0$, $t \in [0, +\infty)$.

Lemma 3.4 *Under assumptions* (H1) *and* (H2), $T_1 : \overline{\Omega} \to X$ *is completely continuous.*

Proof Step 1 We show that T_1 is continuous.

Let $u_n \to u$ as $n \to +\infty$ in $\overline{\Omega}$. From (H1) and (H2), we have

$$\begin{aligned}
&\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \int_0^{+\infty} \frac{1}{1+t^{\alpha-1}}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
\leqslant\ & \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} |f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
\leqslant\ & \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} \left|f\left(s, \frac{(1+s^{\alpha-1})u_n(s)}{1+s^{\alpha-1}}\right) - f\left(s, \frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right)\right|\mathrm{d}s \\
\leqslant\ & \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} \varphi(s)\left|\frac{u_n(s)}{1+s^{\alpha-1}} - \frac{u(s)}{1+s^{\alpha-1}}\right|\mathrm{d}s \\
\leqslant\ & \left(\frac{1}{\Gamma(\alpha)}+1\right)\|u_n - u\|_X \int_0^{+\infty} \varphi(s)\mathrm{d}s < +\infty.
\end{aligned}$$

Hence, from the continuity of f, we have

$$\|T_1u_n - T_1u\|_X = \sup_{t\in[0,+\infty)}\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \to 0,$$

uniformly as $n \to +\infty$, as claimed.

Step 2 We show that $T_1 : \overline{\Omega} \to X$ is relatively compact.

From (H1) and (H2), we have the estimates:

$$\begin{aligned}
\|T_1u\|_X &= \sup_{t\in[0,+\infty)}\left|\frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} |f(s,u(s))|\mathrm{d}s \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} \left(|f(s,u(s)) - f(s,u_0(s))| + |f(s,u_0(s))|\right)\mathrm{d}s \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} \left|f\left(s, \frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}\right) - f\left(s, \frac{(1+s^{\alpha-1})u_0(s)}{1+s^{\alpha-1}}\right)\right|\mathrm{d}s \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty} \left|\frac{u(s)}{1+s^{\alpha-1}} - \frac{u_0(s)}{1+s^{\alpha-1}}\right|\varphi(s)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\left(\left|\frac{u(s)}{1+s^{\alpha-1}}\right|+\left|\frac{u_0(s)}{1+s^{\alpha-1}}\right|\right)\varphi(s)\mathrm{d}s\\
&\leqslant 2r\left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\varphi(s)\mathrm{d}s<+\infty, \quad \text{for } u\in\Omega.
\end{aligned}
$$

Hence, $T_1\Omega$ is uniformly bounded.

Next, we show that $T_1\Omega$ is equicontinuous on any compact interval of $[0,+\infty)$.

For any $b\geqslant 0$, $t_1,t_2\in[0,b]$ $(t_1<t_2)$, and for $u\in\Omega$, we have

$$
\begin{aligned}
&\left|\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}-\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}\right|\\
=&\left|-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s))\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s))\mathrm{d}s\right.\\
&\left.+\int_0^{+\infty}\left(\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right)f(s,u(s))\mathrm{d}s\right|\\
=&\left|\frac{1}{\Gamma(\alpha)}\left(-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s))\mathrm{d}s+\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s))\mathrm{d}s\right.\right.\\
&\left.-\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s))\mathrm{d}s+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s))\mathrm{d}s\right)\\
&\left.+\int_0^{+\infty}\left(\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right)f(s,u(s))\mathrm{d}s\right|\\
=&\left|\frac{1}{\Gamma(\alpha)}\left[\int_{t_2}^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s))\mathrm{d}s+\int_0^{t_1}\left(\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right)f(s,u(s))\mathrm{d}s\right]\right.\\
&\left.+\int_0^{+\infty}\frac{t_1^{\alpha-1}-t_2^{\alpha-1}}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}f(s,u(s))\mathrm{d}s\right|\\
\leqslant&\frac{1}{\Gamma(\alpha)}\left(\int_{t_2}^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|f(s,u(s))|\mathrm{d}s+\int_0^{t_1}\frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}|f(s,u(s))|\mathrm{d}s\right.\\
&\left.+\int_0^{t_1}\frac{|t_2^{\alpha-1}(t_1-s)^{\alpha-1}-t_1^{\alpha-1}(t_2-s)^{\alpha-1}|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}|f(s,u(s))|\mathrm{d}s\right)\\
&+\int_0^{+\infty}\frac{|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}|f(s,u(s))|\mathrm{d}s\\
\leqslant&\frac{2r}{\Gamma(\alpha)}\left(\int_{t_2}^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\varphi(s)\mathrm{d}s+\int_0^{t_1}\frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}\varphi(s)\mathrm{d}s\right.\\
&\left.+\int_0^{t_1}\frac{|t_2^{\alpha-1}(t_1-s)^{\alpha-1}-t_1^{\alpha-1}(t_2-s)^{\alpha-1}|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}\varphi(s)\mathrm{d}s\right)\\
&+2r\int_0^{+\infty}\frac{|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}\varphi(s)\mathrm{d}s\to 0,
\end{aligned}
$$

uniformly as $|t_1 - t_2| \to 0$. Hence

$$\left|\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}} - \frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}\right| \to 0 \quad \text{as } |t_1-t_2| \to 0, \quad \text{for all } u \in \Omega.$$

This shows that $T_1\Omega$ is locally equicontinuous on $[0, +\infty)$.

Step 3 $T_1\Omega$ is equiconvergent at infinity.

For any $u \in \Omega$, we have

$$\int_0^{+\infty} f(s,u(s))\mathrm{d}s \leqslant 2r \int_0^{+\infty} \varphi(s)\mathrm{d}s < +\infty.$$

Hence

$$\begin{aligned} u(+\infty) &= \lim_{t\to+\infty} \frac{(T_1u)(t)}{1+t^{\alpha-1}} \\ &= \lim_{t\to+\infty} \int_0^t -\frac{1}{\Gamma(\alpha)} \times \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} f(s,u(s))\mathrm{d}s \\ &\quad + \lim_{t\to+\infty} \int_0^{+\infty} \frac{1}{1+t^{\alpha-1}} f(s,u(s))\mathrm{d}s \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} f(s,u(s))\mathrm{d}s < +\infty, \end{aligned}$$

and

$$\begin{aligned} \left|\frac{(T_1u)(t)}{1+t^{\alpha-1}} - u(+\infty)\right| = \Bigg| &-\frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} f(s,u(s))\mathrm{d}s + \int_0^{+\infty} \frac{1}{1+t^{\alpha-1}} f(s,u(s))\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} f(s,u(s))\mathrm{d}s \Bigg|. \end{aligned}$$

So

$$\left|\frac{(T_1u)(t)}{1+t^{\alpha-1}} - u(+\infty)\right| \to 0 \quad \text{as } t \to +\infty.$$

Then $T_1\Omega$ is equiconvergent at infinity.

By Lemma 3.3, we deduce that $T_1 : \overline{\Omega} \to X$ is relatively compact, which ends the proof of the lemma.

Lemma 3.5 *If* (H3) *holds, then $T_2 : \overline{\Omega} \to X$ is contractive.*

Proof From (H3), T_2 is contractive:

$$\begin{aligned} \|T_2u - T_2v\|_X &= \sup_{t\in[0,+\infty)} \left|\frac{(T_2u)(t)}{1+t^{\alpha-1}} - \frac{(T_2v)(t)}{1+t^{\alpha-1}}\right| \\ &= \sup_{t\in[0,+\infty)} \frac{1}{1+t^{\alpha-1}} |g(u) - g(v)| \\ &\leqslant l\|u-v\|_X, \quad \text{for all } u, v \in \Omega. \end{aligned}$$

4 Main Result

Theorem 4.1 *Assume that* (H1)-(H3) *hold and that*

(H4) $l+2\left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\varphi(s)\mathrm{d}s < 1$, *where the constant* l *is in assumption* (H3).

(H5) $g(0)=0$.

Then problem (1.1) *has at least one solution.*

Proof Consider the following BVP

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad t\in(0,+\infty), \\ u'(0)=0, \quad \lim\limits_{t\to+\infty}{}^cD_{0^+}^{\alpha-1}u(t)=g(u), \end{cases} \tag{4.1}$$

for $\lambda\in(0,1)$.

The existence of a solution of problem (4.1) is equivalent to that of a fixed point of equation $u=\lambda Tu$.

Suppose that there exists a number $r_0>0$ such that $\Omega_{r_0}=\{u\in X,\ \|u\|_X<r_0\}$ and there exists $u_0\in\Omega_{r_0}$ such that $f(t,u_0(t))=0$, $t\in[0,+\infty)$.

By Lemma 3.4, the operator $T_1:\overline{\Omega}_{r_0}\to X$ is completely continuous, and Lemma 3.5 implies that the operator $T_2:\overline{\Omega}_{r_0}\to X$ is contractive. So it remains to prove that $u\neq\lambda Tu$ for $u\in\partial\Omega_{r_0}$ and $\lambda\in(0,1)$.

Arguing by contradiction, if there exists a $u\in\partial\Omega_{r_0}$ with $u=\lambda Tu$, then for $\lambda\in(0,1)$ we have

$$\begin{aligned}
\|u\|_X &= \sup_{t\in[0,+\infty)}\left|\frac{(\lambda Tu)(t)}{1+t^{\alpha-1}}\right| \leqslant \sup_{t\in[0,+\infty)}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s))|\mathrm{d}s+\int_0^{+\infty}\frac{1}{1+t^{\alpha-1}}|f(s,u(s))|\mathrm{d}s+\frac{1}{1+t^{\alpha-1}}|g(u)| \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}|f(s,u(s))|\mathrm{d}s+|g(u)| \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\left(|f(s,u(s))-f(s,u_0(s))|+|f(s,u_0(s))|\right)\mathrm{d}s \\
&\quad +|g(u)-g(0)|+|g(0)| \\
&\leqslant 2r_0\left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\varphi(s)\mathrm{d}s+lr_0.
\end{aligned}$$

So

$$r_0\leqslant 2r_0\left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\varphi(s)\mathrm{d}s+lr_0.$$

Hence

$$2\left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\varphi(s)\mathrm{d}s+l\geqslant 1,$$

which contradicts condition (H4). According to Theorem 2.1 we conclude that the BVP (1.1) has at least one solution.

As first application, we consider the function g of the form

$$g(u) = \sum_{i=1}^{p} c_i u(\xi_i),$$

where $c_1, c_2, \cdots, c_p$ are given constants with $p \in \mathbb{N}^*$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_p < +\infty$.

Consider the multi-point fractional BVP:

$$\begin{cases} {}^cD_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, +\infty), \\ u'(0) = 0, \quad \lim\limits_{t \to +\infty} {}^cD_{0+}^{\alpha-1} u(t) = \sum\limits_{i=1}^{p} c_i u(\xi_i), \end{cases} \tag{4.2}$$

with $1 < \alpha \leqslant 2$. Then the following result is a direct consequence of Theorem 4.1.

Corollary 4.1 *Assume that* (H1)-(H4) *hold and suppose that* $0 < \sum\limits_{i=1}^{p} c_i \left(1 + \xi_i^{\alpha-1}\right) < 1$. *Then problem* (4.2) *has at least one solution.*

Remark 4.1 In Corollary 4.1, we have

$$|g(u) - g(v)| \leqslant \sum_{i=1}^{p} c_i \left(1 + \xi_i^{\alpha-1}\right) \|u - v\|_X.$$

Indeed, since that $0 < \sum\limits_{i=1}^{p} c_i \left(1 + \xi_i^{\alpha-1}\right) < 1$ and with l instead of $\sum\limits_{i=1}^{p} c_i \left(1 + \xi_i^{\alpha-1}\right)$, that is (H3), we get that the operator T_2 is a contraction. In addition condition (H5) holds.

The rest of the proof is as that of Theorem 4.1.

5 Example

Example 5.1 Consider the following BVP on the half line

$$\begin{cases} {}^cD_{0+}^{\frac{3}{2}} u(t) + \dfrac{u(t)}{(1 + \sqrt{t})(10 + t)^2} = 0, \quad t \in (0, +\infty), \\ u'(0) = 0, \quad \lim\limits_{t \to +\infty} {}^cD_{0+}^{\frac{1}{2}} u(t) = \dfrac{1}{5} u(1) + \dfrac{1}{20} u(2). \end{cases} \tag{5.1}$$

In this case, $\alpha = \frac{3}{2}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $g(u) = \frac{1}{5} u(1) + \frac{1}{20} u(2)$.

We apply Corollary 4.1 to show that problem (5.1) has at least one solution.

Let $f(t, u(t)) = \frac{u(t)}{(1+\sqrt{t})(10+t)^2}$ and $\varphi(t) = \frac{1}{(10+t)^2}$. Then

(H1) $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2)

$$\left|f\left(t,(1+t^{\frac{1}{2}})x\right)-f\left(t,(1+t^{\frac{1}{2}})y\right)\right|=\left|\frac{1}{(10+t)^2}(x-y)\right|\leqslant\varphi(t)|x-y|,\ \text{on } [0,+\infty)\times\mathbb{R}$$

with $\varphi\in L^1[0,+\infty)$.

(H3) $0<\frac{1}{5}(2)+\frac{1}{20}(1+\sqrt{2})<1$, $|g(u)-g(v)|\leqslant\frac{9+\sqrt{2}}{20}\|u-v\|_X$, for all $u,v\in X$.

(H4)

$$\left(\frac{9+\sqrt{2}}{20}\right)+2\left(\frac{1}{\Gamma(\alpha)}+1\right)\int_0^{+\infty}\varphi(s)\mathrm{d}s=\frac{8+(13+\sqrt{2})\sqrt{\pi}}{20\sqrt{\pi}}<1.$$

By Corollary 4.1, we deduce that the BVP (5.1) has at least one solution.

References

[1] A. Arara, M. Benchohra, N. Hamidi, J.J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal.*, **72**(2010),580-586.

[2] Z. Bai and H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311**(2005),495-505.

[3] C.S. Goodrich, Existence of a positive solution to systems of differential equations of fractional order, *Computers and Mathematics with Applications*, **62**(2011),1251-1268.

[4] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol.204. Elsevier Science B. V., Amsterdam, 2006.

[5] S. Liang and J. Zhang, Existence of multiple positive solutions for m-point fractional boundary value problems of on an infinite interval, *Mathematical and Computer Modelling*, **54**(2011),1334-1346.

[6] Y.S. Liu, Boundary value problem for second order differential equations on unbounded domain, *Acta. Anal. Funct. Appl.*, **4**(2002),211-216.

[7] X. Liu and M. Jia, Multiple solutions of nonlocal boundary value problems for fractional differential equations on the half-line, *Electronic Journal of Qualitative Theory of Differential Equations*, **56**(2011),1-14.

[8] D. O'Regan, Fixed-point theory for the sum of two operators, *Appl. Math. Lett.*, **9**:1(1996),1-8.

[9] X. Su and S. Zhang, Unbounded solutions to a boundary value problem of fractional order on the half-line, *Computers and Mathematics with Applications*, **61**(2011),1079-1087.

[10] C. Shen, H. Zhou and L. Yang, On the existence of solution to a boundary value problem of fractional differential equation on the infinite interval, *Boundary Value Problems*, (2015), 2015.241, DOI:10.1186/s13661-015-0509-z.

[11] Y. Wang and L. Liu, Positive solutions for fractional m-point boundary value problem in Banach spaces, *Acta. Math. Sci.*, **32**(2012),246-256.

[12] L. Wang and X. Zhang, Positive solutions of m-point boundary value problems for a class of nonlinear fractional differential equations, *J. Appl. Math. Comput.*, **42**(2013), 387-399.

[13] B. Yan, D .O'Regan and R.P. Agarwal, Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity, *Journal of Computational and Applied Mathematics*, **197**(2006),365-386.

[14] C. Yu, J. Wang and Y. Guo, Solvability for integral boundary value problems of fractional differential equation on infinite intervals, *J. Nonlinear Sci. Appl.*, **9**(2016),160-170.

[15] C. Yu, Z. Li, H. Wei and J. Wang, Existence and uniqueness of solutions for second-order m-point boundary value problems at resonance on infinite interval, *J. Hebei Univ. Sci. Tech.*, **34**(2013),7-14.

[16] X. Zhao, W. Ge, Unbounded solutions for a fractional boundary value problem on the infinite interval, *Acta. Appl. Math.*, **109**(2010),495-505.

(*edited by Mengxin He*)