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THE SEMI-NORMS ON THE SCHWARTZ SPACE*†

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Abstract

Let $S(R^2)$ be the class of all infinitely differential functions which, as well as their derivatives, are rapidly decreasing on R^2. Here we define a kind of semi-norms which is equivalent to the usual family of semi-norms on the Schwartz space $S(R^2)$.

Keywords Schwartz space; semi-norms; equivalent
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1 Introduction

In the recent years, the Schwartz space as well as their application are concerned in many publication ([1-5]). In this paper, we first give the usual family of semi-norms on the Schwartz space $S(R^2)$. A new family of semi-norms is defined, which is based on the operators we constructed.

Using the new family of semi-norms, we can consider the method to discuss the Schwartz space in terms of the sequential theory.

Let I_+^2 denote the set of all two-tuple of non-negative integers. For $\alpha \in I_+^2$, let

$$|\alpha| = \alpha_1 + \alpha_2. \tag{1.1}$$

For a multi-index α and $x \in R^2$, let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}. \tag{1.2}$$

The Schwartz space $S(R^2)$ is defined to be the class of all infinitely differentiable complex-valued functions φ on R^2 such that

$$\lim_{|x|\to\infty} |x^\alpha D^\beta \varphi| = 0, \tag{1.3}$$

for all multi-indices α and β. The space $S(R^2)$ is closed for the differential operators and multiplication by polynomials.

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2 Some Definitions

In this section, we introduce some definitions. Let R^2 be the 2-dimensional Euclidean space.

Definition 2.1 A semi-norm on a vector space V is a map $\rho : V \to [0,\infty)$ satisfying

(i) $\rho(u+v) \le \rho(u) + \rho(v)$ for $u, v \in V$;

(ii) $\rho(au) = |a|\rho(u)$ for $a \in C$(or R).

A family of semi-norms $\{\rho_\alpha\}_{\alpha\in A}$ is said to be separate points if

(iii) $\rho_\alpha(u) = 0$ for all $\alpha \in A$ implies $u = 0$,

where $\alpha = (\alpha_1, \alpha_2)$ are two-tuple of non-negative integers ([1]).

Definition 2.2 Let $f \in S(R^2)$ and $\|\cdot\|_{\alpha,\beta,\infty}$ be defined by

$$\begin{aligned}\|f\|_{\alpha,\beta,\infty} = \|x^\alpha D^\beta f\|_\infty &= \Big\| x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} f \Big\|_\infty \\ &= \sup_{x_1\in R} \sup_{x_2\in R} \Big| x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial^{(\beta_1+\beta_2)}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} f \Big|. \end{aligned} \tag{2.1}$$

Then $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I_+^2}$ is the usual family of semi-norms on $S(R^2)$.

Definition 2.3 Let V be a vector space and $u \in V$. Let $\{\rho_\alpha\}_{\alpha\in A}$ and $\{d_b\}_{b\in B}$ be two families of semi-norms on a vector space V where A and B are some index sets. The families of semi-norms are equivalent if and only if they satisfy:

(i) For each $a \in A$, there exist $b_1, b_2 \in B$ and $C > 0$, such that

$$\rho_a(u) \le C\big(d_{b_1}(u) + d_{b_2}(u)\big);$$

(ii) for each $b \in B$, there exist $a_1, a_2 \in A$ and $C' > 0$ such that

$$d_b(u) \le C'\big(\rho_{a_1}(u) + \rho_{a_2}(u)\big).$$

Definition 2.4 A family of semi-norms $\{\rho_\alpha\}_{\alpha\in A}$ on a vector space V is called directed if for $\alpha, \beta \in A$ and $u \in V$, there exist $\gamma \in A$ and $C > 0$ such that

$$\rho_\alpha(u) + \rho_\beta(u) \le C\rho_\gamma(u). \tag{2.2}$$

Definition 2.5 Let $f \in S(R^2)$ and $\|\cdot\|_{\alpha,\beta,2}$ be define by

$$\|f\|_{\alpha,\beta,2} = \|x^\alpha D^\beta f\|_2 = \Big(\int_{R^2} |x^\alpha D^\beta f(x)|^2 \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Then $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta\in I_+^2}$ is the usual family of semi-norms on $S(R^2)$.

Definition 2.6 Hölder inequality: Let E be a measurable set of Lebesgue, $x(t)$ and $y(t)$ be measurable functions in E. Then p and q are positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_E |x(t)y(t)|\mathrm{d}t \le \Big(\int_E |x(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} \Big(\int_E |y(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}.$$

Minkowski inequality: Let E is a measurable set of Lebesgue, $x(t)$ and $y(t)$ be measurable functions in E, and $p \ge 1$, then

$$\Big(\int_E |x(t)+y(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} \le \Big(\int_E |x(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}} + \Big(\int_E |y(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}}.$$

3 The Semi-norms on $S(R^2)$

In this section, we discuss the relations of some semi-norms on $S(R^2)$.

In the following lemmas, we prove the family of semi-norms $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I_+^2}$ is equivalent to the family of semi-norms $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta\in I_+^2}$ on $S(R^2)$.

Theorem 3.1 *The families of the semi-norms* $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I_+^2}$ *and* $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta\in I_+^2}$ *on* $S(R^2)$ *are equivalent.*

Proof Let $f \in S(R^2)$, then

$$\begin{aligned}\|f\|^2_{\alpha,\beta,2} &= \int_{R^2} |x^\alpha D^\beta f(x)|^2 \mathrm{d}x \\ &= \int_{R^2} \frac{1}{(1+|x|^2)^2} |(1+|x|^2)x^\alpha D^\beta f(x)|^2 \mathrm{d}x \\ &\le \Big\|\frac{1}{1+|x|^2}\Big\|_2^2 \sup \big|(1+|x|^2)x^\alpha D^\beta f(x)\big|^2.\end{aligned}$$

Note that

$$\begin{aligned}\big|(1+|x|^2)x^\alpha D^\beta f(x)\big| &\le |x^\alpha D^\beta f(x)| + |x|^2 |x^\alpha D^\beta f(x)| \\ &\le \sup_{x\in R^2} |x^\alpha D^\beta f(x)| + |(x_1^2+x_2^2)x^\alpha D^\beta f(x)| \\ &\le \|f\|_{\alpha,\beta,\infty} + |x_1^{\alpha_1+2}x_2^{\alpha_2} D^\beta f(x)| + |x_1^{\alpha_1}x_2^{\alpha_2+2} D^\beta f(x)| \\ &\le \|f\|_{\alpha,\beta,\infty} + \|f\|_{\alpha+2e_1,\beta,\infty} + \|f\|_{\alpha+2e_2,\beta,\infty},\end{aligned}$$

where $e_1 = (1,0)$, $e_2 = (0,1)$. We get

$$\|f\|_{\alpha,\beta,2} \le C(\|f\|_{\alpha,\beta,\infty} + \|f\|_{\alpha+2e_1,\beta,\infty} + \|f\|_{\alpha+2e_2,\beta,\infty}), \tag{3.1}$$

where $C = \|1/(1+|x|^2)\|_2$.

On the other hand, we know

$$D^\beta f(x) = \int_{-\infty}^{x_1} D^{\beta+e_1} f(x(x_1,t_1))\mathrm{d}t_1,$$

where the symbol $x(x_1,t_1)$ means the replacement of x_1 by t_1, that is

$$D^\beta f(x) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} D^{\beta+e} f(t)\mathrm{d}t = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{\partial^{|\beta|+2}}{\partial x_1^{\beta_1+1}\partial x_2^{\beta_2+1}} f(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2.$$

Then

$$|D^\beta f(x)| = \Big|\int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(t)\mathrm{d}t\Big| \le \int_{R^2} |D^{\beta+e} f(t)|\mathrm{d}t$$
$$= \int_{R^2} \Big|\frac{1}{1+|t|^2}(1+|t|^2)D^{\beta+e}f(t)\Big|\mathrm{d}t.$$

By the Hölder inequality, we have

$$\int_{R^2}\Big|\frac{1}{1+|t|^2}(1+|t|^2)D^{\beta+e}f(t)\Big|\mathrm{d}t \le \Big(\int_{R^2}\Big|\frac{1}{1+|t|^2}\Big|^2\mathrm{d}t\Big)^{\frac12}\Big(\int_{R^2}|(1+|t|^2)D^{\beta+e}f(t)|^2\mathrm{d}t\Big)^{\frac12}$$
$$= \Big\|\frac{1}{1+|t|^2}\Big\|_2 \cdot \|(1+|t|^2)D^{\beta+e}f(t)\|_2.$$

Hence

$$|D^\beta f(x)| \le \Big\|\frac{1}{1+|t|^2}\Big\|_2 \cdot \|(1+|t|^2)D^{\beta+2}f(x)\|_2.$$

Using the Minkowski inequality, we obtain

$$\|f\|_{0,\beta,\infty} = \|D^\beta f\|_\infty = \sup_{x\in R^2}|D^\beta f(x)| \le \Big\|\frac{1}{1+|t|^2}\Big\|_2 \cdot \|(1+|t|^2)D^{\beta+e}f(x)\|_2$$
$$= C\|(1+t_1^2+t_2^2)D^{\beta+e}\|_2 \le C(\|f\|_{0,\beta+e,2}+\|f\|_{2e_1,\beta+e,2}+\|f\|_{2e_1,\beta+e,2}). \quad (3.2)$$

Next, consider the relationship between $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I_+^2}$ and $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta\in I_+^2}$ where $\alpha\neq(0,0)$:

If $\alpha=(\alpha_1,0)$ with $\alpha_1\neq 0$,

$$x^\alpha D^\beta f = \int_{-\infty}^{x_1}(\alpha_1 t^{\alpha-e_1}D^\beta f + x^\alpha D^{\beta+e_2}f)(x(x_1,t_1))\mathrm{d}t_1;$$

if $\alpha=(0,\alpha_2)$ with $\alpha_2\neq 0$,

$$x^\alpha D^\beta f = \int_{-\infty}^{x_2}(\alpha_2 x^{\alpha-e_2}D^\beta f + x^\alpha D^{\beta+e_2}f)(x(x_2,t_2))\mathrm{d}t_2;$$

if $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_1\neq 0$, $\alpha_2\neq 0$,

$$x^\alpha D^\beta f = \int_{-\infty}^{x_1}(\alpha_1 t^{\alpha-e_1}D^\beta f + t^\alpha D^{\beta+e_1}f)\mathrm{d}t_1$$
$$= \int_{-\infty}^{x_1}\Big[\int_{-\infty}^{x_2}(\alpha_1\alpha_2 x^{\alpha-e_1-e_2}D^\beta f + \alpha_1 x^{\alpha-e_1}D^{\beta+e_2}f$$
$$+\alpha_2 x^{\alpha-e_2}D^{\beta+e_1}f + x^\alpha D^{\beta+e_1+e_2}f)\mathrm{d}t_2\Big]\mathrm{d}t_1$$
$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}(\alpha_1\alpha_2 x^{\alpha-e_1-e_2}D^\beta f + \alpha_1 x^{\alpha-e_1}D^{\beta+e_2}f$$
$$+\alpha_2 x^{\alpha-e_2}D^{\beta+e_1}f + x^\alpha D^{\beta+e_1+e_2}f)\mathrm{d}t_2\mathrm{d}t_1.$$

Then

$$
\begin{aligned}
|x^\alpha D^\beta f| &= \Big|\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}(\alpha_1\alpha_2 x^{\alpha-e_1-e_2}D^\beta f+\alpha_1 x^{\alpha-e_1}D^{\beta+e_2}f\\
&\quad +\alpha_2 x^{\alpha-e_2}D^{\beta+e_1}f+t^\alpha D^{\beta+e_1+e_2}f)\mathrm{d}t_1\mathrm{d}t_2\Big|\\
&\le \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}|\alpha_1\alpha_2 t^{\alpha-e_1-e_2}D^\beta f|\mathrm{d}t_1\mathrm{d}t_2+\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}|\alpha_1 t^{\alpha-e_1}D^{\beta+e_2}f|\mathrm{d}t_1\mathrm{d}t_2\\
&\quad +\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}|\alpha_2 t^{\alpha-e_2}D^{\beta+e_1}f|\mathrm{d}t_1\mathrm{d}t_2+\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}|t^\alpha D^{\beta+e_1+e_2}f|\mathrm{d}t_1\mathrm{d}t_2\\
&= \alpha_1\alpha_2\Big\|\frac{1}{1+|t|^2}\Big\|_2\cdot\big(\|(1+|t|^2)t^{\alpha-e_1-e_2}D^\beta f\|_2+\|(1+|t|^2)t^{\alpha-e_1}D^{\beta+e_2}f\|_2\\
&\quad +\|(1+|t|^2)\alpha_2 t^{\alpha-e_2}D^{\beta+e_1+e_2}f\|_2\big).
\end{aligned}
$$

By Definition 2.5 and the above inequality, we have

$$
\begin{aligned}
|x^\alpha D^\beta f| \le \alpha_1\alpha_2\Big\|\frac{1}{1+|t|^2}\Big\|\Big[&\|f\|_{\alpha-e_1-e_2,\beta,2}+\sum_{j=1}^{2}\|f\|_{\alpha-e_1-e_2+2e_j,\beta,2}\\
&+\sum_{\{r_1\}\in M_{\alpha,1}}\Big(\|f\|_{\alpha-e_{r_1},\beta+e_{r_2},2}+\sum_{j=1}^{2}\|f\|_{\alpha-e_{r_1}+2e_j,\beta+e_{r_2},2}\Big)\\
&+\|f\|_{\alpha,\beta+e_1+e_2,2}+\sum_{j=1}^{2}\|f\|_{\alpha+2e_j,\beta+e_1+e_2,2}\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
\|f\|_{\alpha,\beta,\infty} &= \sup|x^\alpha D^\beta f|\\
&\le C'\Big[\|f\|_{\alpha-e_1-e_2,\beta,2}+\sum_{j=1}^{2}\|f\|_{\alpha-e_1-e_2+2e_j,\beta,2}\\
&\quad +\sum_{\{r_1\}\in M_{\alpha,1}}\Big(\|f\|_{\alpha-e_{r_1},\beta+e_{r_2},2}+\sum_{j=1}^{2}\|f\|_{\alpha-e_{r_1}+2e_j,\beta+e_{r_2},2}\Big)\\
&\quad +\|f\|_{\alpha,\beta+e_1+e_2,2}+\sum_{j=1}^{2}\|f\|_{\alpha+2e_j,\beta+e_1+e_2,2}\Big],
\end{aligned}
\tag{3.3}
$$

where

$$C'=\Big|\alpha_1\alpha_2\cdot\Big\|\frac{1}{1+|t|^2}\Big\|\Big|_2.$$

(3.1), (3.2) and (3.3) imply $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I_+^2}$ and $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta\in I_+^2}$ are equivalent. We complete the proof.

In the following, we define some operators on $S(R^2)$. And using the operators, we give a family of semi-norms which is equivalent to the usual family of semi-norms on $S(R^2)$. Let

$$A_j = \frac{1}{\sqrt{2}}\Big(x_j + \frac{\partial}{\partial x_j}\Big), \quad A_j^+ = \frac{1}{\sqrt{2}}\Big(x_j - \frac{\partial}{\partial x_j}\Big), \quad \text{for } j = 1, 2.$$

Then for $f, g \in S(R^2)$, using (1.3), we get

$$\begin{aligned}(A_j^+, f, g)_2 &= \int_{R^2} \Big[\frac{1}{\sqrt{2}}\Big(x_j - \frac{\partial}{\partial x_j}\Big) f\Big] \cdot g \mathrm{d}x \\ &= \int_{R^2} \frac{1}{\sqrt{2}} x_j \cdot f \cdot g \mathrm{d}x_j - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}} g \cdot \frac{\partial f}{\partial x_j} \mathrm{d}x_j \\ &= \frac{1}{\sqrt{2}} \int_{R^2} x_j \cdot f \cdot g \mathrm{d}x_j + \frac{1}{\sqrt{2}} \int_{R^2} f \frac{\partial g}{\partial x_j} \mathrm{d}x_j \\ &= \int_{-\infty}^{+\infty} f \cdot \Big[\frac{1}{\sqrt{2}}\Big(x_j + \frac{\partial}{\partial x_j}\Big) g\Big] \mathrm{d}x_j = (f, A_j g)_2.\end{aligned}$$

Thus

$$(A_j^+ f, g)_2 = (f, A_j g)_2, \quad j = 1, 2,$$

where $(\cdot, \cdot)_2$ is the L^2-inner product.

For each j with $1 \le j \le 2$, set

$$N_j = A_j^+ A_j \quad \text{and} \quad N_j^k = N_j(N_j^{k-1}), \quad k = 1, 2, \cdots,$$

where N_j^0 is the identity operator. Now, we can get the conclusion that the Schwartz space $S(R^2)$ is closed for the differential operators and multiplication by polynomials, and is also closed for the differential operators N_j^k ($j = 1, 2$; $k = 0, 1, \cdots$). It is easy to show that for $f, g \in S(R^2)$,

$$(N_j^k f, g)_2 = (f, N_j^k g)_2, \quad k = 0, 1, \cdots.$$

For a multi-index $\beta = (\beta_1, \beta_2) \in I_+^2$, we define an operator $(N+1)^\beta$ by

$$(N+1)^\beta = (N_1 + 1)^{\beta_1} (N_2 + 1)^{\beta_2}.$$

Then for $f, g \in S(R^2)$,

$$\big((N+1)^\beta f, g)\big)_2 = \big(f, (N+1)^\beta g\big)_2.$$

Here we define a family of semi-norms on $S(R^2)$. Let

$$\|f\|_\beta = \|(N+1)^\beta f\|_2 \quad \text{for } f \in S(R^2).$$

Then $\{\|\cdot\|_\beta\}_{\beta \in I_+^2}$ is a directed family of semi-norms on $S(R^2)$.

Next we prove the families of semi-norms $\{\|\cdot\|_\beta\}_{\beta \in I_+^2}$ and $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta \in I_+^2}$ are equivalent on $S(R^2)$.

Theorem 3.2 *The directed family of semi-norms $\{\|\cdot\|_\beta\}_{\beta \in I_+^2}$ is equivalent to $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta \in I_+^2}$ on $S(R^2)$.*

Proof For $f \in S(R^2)$,

$$\|f\|_\beta^2 = \|(N+1)^\beta f\|_2^2 = \big((N+1)^\beta f, (N+1)^\beta f\big)_2,$$

where $\beta = (\beta_1, \beta_2) \in I_+^2$. By the induction, we prove

$$\|f\|_\beta^2 = \|(N_1+1)^{\beta_1}(N_2+1)^{\beta_2} f\|_2^2 = \sum_{\beta'+\beta''\le 2\beta} C_{\beta',\beta''}\|f\|_{\beta',\beta'',2}^2, \tag{3.4}$$

where $2\beta = (2\beta_1, 2\beta_2)$, $C_{\beta',\beta''} > 0$, $\|\cdot\|_{\beta',\beta'',2}$ is given in Definition 2.5,

$$\|f\|_{\beta',\beta'',2} = \Big[\int_{R^2} |x^{\beta'} D^{\beta''} f|^2 \mathrm{d}x\Big]^{\frac{1}{2}}.$$

(i) If $\beta = (0,0)$, then $\beta' = (0,0)$, $\beta'' = (0,0)$ and

$$\|f\|_\beta = \|f\|_2 = \|f\|_{0,0,2}.$$

That is, (3.4) holds for $\beta = (0,0)$.

(ii) If $\beta = e_j$ $(j = 1,2)$, that is $\beta = (1,0)$ or $\beta = (0,1)$, then we have such cases:

$$\beta' = 2e_j,\ \ \beta'' = (0,0); \quad \beta' = e_j,\ \ \beta'' = (0,0); \quad \beta' = e_j,\ \beta'' = e_j;$$
$$\beta' = (0,0),\ \beta'' = (0,0); \quad \beta' = (0,0),\ \beta'' = e_j; \quad \beta' = (0,0),\ \beta'' = 2e_j.$$

Here we consider the case $\beta = e_1 = (1,0)$. For $\beta = (1,0)$,

$$\begin{aligned}
\|f\|_\beta^2 &= \|(N_1+1)f\|_2^2 = \big((N_1+1)f, (N_1+1)f\big)_2\\
&= \frac{1}{2}\Big[(x_1^2 f, x_1^2 f) + 2(x_1 f, x_1 f)_2 + 2\Big(x_1\frac{\partial f}{\partial x_1}, x_1\frac{\partial f}{\partial x_1}\Big)_2\\
&\quad +(f,f)_2 + \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_1}\Big)_2 + \Big(\frac{\partial y}{\partial x_1^2}, \frac{\partial y}{\partial x_1^2}\Big)_2\Big]\\
&= \frac{1}{2}\Big[\|x_1^2 f\|_2^2 + 2\|x_1 f\|_2^2 + 2\Big\|x_1\frac{\partial f}{\partial x_1}\Big\|_2^2 + \|f\|_2^2 + \Big\|\frac{\partial f}{\partial x_1}\Big\|_2^2 + \Big\|\frac{\partial^2 f}{\partial x_1^2}\Big\|_2^2\Big]\\
&= \frac{1}{2}\|f\|_{2e_1,0,2}^2 + \|f\|_{e_1,0,2}^2 + \|f\|_{e_1,e_1,2} + \frac{1}{2}\|f\|_{0,0,2}^2 + \frac{1}{2}\|f\|_{0,e_1,2}^2 + \frac{1}{2}\|f\|_{0,2e_1,2}^2,
\end{aligned}$$

where 0 in $\|\cdot\|_{2e_1,0,2}$ is $(0,0)$.

By the similar calculation, we can show the equality in (3.4) holds for $\beta = e_2 = (0,1)$.

Assume equality (3.4) holds for $\beta = (\beta_1, \beta_2) \in I_+^2$, that is

$$\begin{aligned}
\|f\|_\beta^2 &= \big((N+1)^\beta f, (N+1)^\beta f\big)\\
&= \sum_{\beta'+\beta''\le 2\beta} C_{\beta',\beta''}\Big(x^{\beta'}\frac{\partial^{|\beta''|} f}{\partial x_1^{\beta_1''}\partial x_2^{\beta_2''}}, x^{\beta'}\frac{\partial^{|\beta''|} f}{\partial x_1^{\beta_1''}\partial x_2^{\beta_2''}}\Big)\\
&= \sum_{\beta'+\beta''\le 2\beta} C_{\beta',\beta''}\|f\|_{\beta',\beta'',2}^2,
\end{aligned} \tag{3.5}$$

where $\beta_1, \beta_2 \in \mathrm{N}$.

We discuss the case $\beta + e_j$ $(j = 1, 2)$. We consider the case $j = 1$, that is $\beta + e_1$. By $N_1 = A_1^+ A_1$ and $(A_1^+ f, g) = (f, A_1 g)$,

$$\begin{aligned}
\|f\|^2_{\beta+e_1} &= \|(N_1+1)(N+1)^\beta f\|_2^2 = \big((N_1+1)(N+1)^\beta f, (N_1+1)(N+1)^\beta f\big)_2 \\
&= \big(N_1(N+1)^\beta f, N_1(N+1)^\beta f\big)_2 + \big(N_1(N+1)^\beta f, (N+1)^\beta f\big)_2 \\
&\quad + \big((N+1)^\beta f, N_1(N+1)^\beta f\big)_2 + \big((N+1)^\beta f, (N+1)^\beta f\big)_2 \\
&= \big(N_1(N+1)^\beta f, N_1(N+1)^\beta f\big)_2 + 2\big(A_1(N+1)^\beta f, A_1(N+1)^\beta f\big)_2 \\
&\quad + \big((N+1)^\beta f, (N+1)^\beta f\big)_2. \qquad (3.6)
\end{aligned}$$

By the assertion (3.5), the third term of (3.6) becomes

$$\big((N+1)^\beta f, (N+1)^\beta f\big)_2 = \|(N+1)^\beta f\|_2 = \sum_{\beta'+\beta''\le 2\beta} C_{\beta',\beta''}\|f\|^2_{\beta',\beta'',2}.$$

Because of

$$\begin{aligned}
&\Big(\frac{\partial}{\partial x_1}(N+1)^\beta f, x_1(N+1)^\beta f\Big) = \int_{R^2} \frac{\partial}{\partial x_1}(N+1)^\beta f \cdot \overline{x_1(N+1)^\beta f}\,\mathrm{d}x \\
&= \int_R \Big[(N+1)^\beta f \cdot x_1\overline{(N+1)^\beta f}\Big|_{-\infty}^{+\infty} \\
&\quad - \int_R (N+1)^\beta f \cdot \Big(\overline{(N+1)^\beta f} + x_1\overline{\frac{\partial}{\partial x_1}(N+1)^\beta f}\Big)\mathrm{d}x_1\Big]\mathrm{d}x_2 \\
&= -\iint \Big[(N+1)^\beta f \cdot \overline{(N+1)^\beta f} + x_1(N+1)^\beta f \frac{\partial}{\partial x_1}(N+1)^\beta\Big]\mathrm{d}x_1\mathrm{d}x_2 \\
&= -\big((N+1)^\beta f, (N+1)^\beta f\big)_2 - \Big(x_1(N+1)^\beta f, \frac{\partial}{\partial x_1}(N+1)^\beta f\Big)_2, \qquad (3.7)
\end{aligned}$$

and the assertion (3.5), the second term $\big(A_1(N+1)^\beta f, A_1(N+1)^\beta f\big)$ of (3.6) becomes

$$\begin{aligned}
&\big(A_1(N+1)^\beta f, A_1(N+1)^\beta f\big) \\
&= \frac{1}{2}\Big(\Big(x_1 + \frac{\partial}{\partial x_1}\Big)(N+1)^\beta f, \Big(x_1 + \frac{\partial}{\partial x_1}\Big)(N+1)^\beta f\Big) \\
&= \frac{1}{2}\Big[\big(x_1(N+1)^\beta f, x_1(N+1)^\beta f\big)_2 + \Big(x_1(N+1)^\beta f, \frac{\partial}{\partial x_1}(N+1)^\beta f\Big) \\
&\quad + \Big(\frac{\partial}{\partial x}(N+1)f, x_2(N+1)f\Big) + \Big(\frac{\partial}{\partial x_1}(N+1)^\beta f, \frac{\partial}{\partial x_1}(N+1)^\beta f\Big)\Big] \\
&= \frac{1}{2}\Big[\big(x_1(N+1)^\beta f, x_1(N+1)^\beta f\big) - \big((N+1)^\beta f, (N+1)^\beta f\big) \\
&\quad + \Big(\frac{\partial}{\partial x_1}(N+1)^\beta f, \frac{\partial}{\partial x_1}(N+1)^\beta f\Big)\Big] \\
&= \sum_{\beta'+\beta''\le 2\beta}\big[C_{\beta'+e_1,\beta''}\|f\|^2_{\beta'+e_1,\beta'',2} + C'_{0,\beta',\beta''}\|f\|^2_{\beta',\beta'',2} + C''_{\beta',\beta''+e_1}\|f\|^2_{\beta',\beta''+e_1,2}\big]. \qquad (3.8)
\end{aligned}$$

By the similar calculation, we get

$$\begin{aligned}&\big(N_1(N+1)^\beta f, N_1(N+1)^\beta f\big)_2\\ =&\sum_{\beta'+\beta''\leq 2\beta}\big(C_{\beta'+2e_1,\beta''}\|f\|^2_{\beta'+2e_1,\beta'',2}+C_{\beta'+e_1,\beta''}\|f\|^2_{\beta'+e_1,\beta'',2}+C_{\beta',\beta''}\|f\|^2_{\beta',\beta'',2}\\ &+C_{\beta',\beta''+e_1}\|f\|^2_{\beta',\beta''+e_1,2}+C_{\beta',\beta''+2e_1}\|f\|^2_{\beta',\beta''+2e_1,2}\big).\end{aligned}$$

Hence for $\beta+e_1$,

$$\|f\|^2_{\beta+e_1}=\sum_{\beta'+\beta''\leq 2(\beta+e_1)}C_{\beta',\beta''}\|f\|^2_{\beta',\beta'',2}.$$

Thus equality (3.4) holds. We can conclude that, there exist a $C_r>0$ such that

$$C'_r\|f\|_{\alpha,\beta,2}\leq\|f\|_r\leq\sum_{\alpha+\beta\leq 2r}\|f\|_{\alpha,\beta,2}.$$

That is, the family of semi-norms $\{\|\cdot\|_r\}_{r\in I^2_+}$ is equivalent to $\{\|\cdot\|_{\alpha.\beta,2}\}_{\alpha,\beta\in I^2_+}$ on $S(R^2)$. The proof is completed.

By Theorems 3.1 and 3.2, we obtain

Theorem 3.3 *The directed family of semi-norms* $\{\|\cdot\|_\beta\}_{\beta\in I^2_+}$ *is equivalent to* $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I^2_+}$ *on* $S(R^2)$.

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