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SEMICLASSICAL LIMIT TO THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION*†

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Abstract

In this paper, we investigate the semiclassical limit of the generalized nonlinear Schrödinger equation for initial data with Sobolev regularity. Also, we will analyze the structure of the fluid dynamical system with quantum effect corresponding to the semiclassical limit of the generalized nonlinear Schrödinger equation.

Keywords quantum hydrodynamics; dispersive limit; compressible Euler equation

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1 Introduction

Hydrodynamics equations with quantum effect describe the hydrodynamical properties and states of some important physical phenomena such as semiconductor, superconductor and superflow. This kind of equations have theoretical significance and practical value. From the semiclassical limit of the nonlinear Schrodinger (NLS) equation with Plank constant h, we can derive various hydrodynamics equations with quantum effect when $h \to 0$.

It is well known that the quantum hydrodynamics equations (QHD) can be derived based on the moment method, which is analogous to the derivation of the compressible Euler equation from the Boltzmann equation by taking the zeroth, first and second order velocity moments of the quantum Boltzmann equation and

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resulting in a hydrodynamical model which then has to be closed in an approximate way, that is, a reasonable macroscopic approximation for the quantum heat flow tensor has to be derived by using additional (quantum) physical properties of the particle ensembles. Moreover, in the case of high electric fields, small mean-free-path asymptotics have been used to derive QHD-models.

When the time and distance scales are large enough relative to the Plank constant h, the system will approximately obey the laws of classical, Newtonian mechanics. That is, quantum mechanics becomes Newtonian mechanics as $h \to 0$. The asymptotics of quantum variables as $h \to 0$ are known as semiclassical expressing this limiting behavior.

In the semiclassical limit or WKB limit and when ∇_x and ∂_t scale like ϵ as $\epsilon \to 0$ (ϵ is the scaled Planck constant), the quantum-mechanical pressure becomes negligible. The isentropic compressible Euler equation can be formally recovered from the nonlinear Schrödinger equation in this limit. This fact was proven rigorously by Jin, Levermore and McLaughlin [5, 6] for the one-dimensional integrable case using the inverse scattering technique and by Grenier [3] for higher dimensions in situations where no vortices are involved.

Very similar model equations have been used for quite a while in other areas of theoretical and computational physics, for instance, in superfluidity [11, 12] and in superconductivity [2].

2 Semiclassical Limit to the Nonlinear Schrödinger Equation in Short Time Range

In this section, we consider the following nonlinear Schrödinger (NLS) equation with rapidly oscillating data

$$ih\partial_t\psi_h + \frac{h^2}{2}\Delta_x\psi_h + f(|\psi_h|^2)\psi_h = 0, \tag{2.1}$$

$$\psi_h(0,x) = a^0(x,h)\exp\Big(\frac{iS^0(x)}{h}\Big), \tag{2.2}$$

where $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$, $S^0(x) \in H^s(\mathbb{R}^d)$ for s large enough. And a^0 is a function, polynomial in h with coefficients of Sobolev regularity in x. h is the Plank constant and ψ_h is the wave function.

We will study the semiclassical limit of equation (2.1)-(2.2) and determine the limiting dynamics of any function of the field ψ_h as $h \to 0$.

Remark 2.1 When $f(x) = x$, equation (2.1) appears in the phenomenological description of superfluidity of an almost ideal Bose gas [10]. In this case, the squared modulus of the wave function $\psi\bar{\psi}$ is interpreted as the particle number density in the condensate state, while the gradient of the phase is proportional to the superfluid velocity $u = \nabla arg\psi$. Moreover, the nonlinear Schrödinger equation is very helpful for the mathematical analysis of the isentropic irrotational QHD-system [4–6, 15].

Employing the WKB method, we will look for the solution to (2.1)-(2.2) having the following form

$$\psi_h(x,t) = a(x,t,h)\exp\Big(\frac{iS(x,t)}{h}\Big), \tag{2.3}$$

where

$$a(x,t,h) = \sum_{j=0}^{+\infty} a_j(x,t)h^j, \tag{2.4}$$

and $a_j(x,t)$ satisfies certain equations so that we could solve (2.1) locally in time.

Let

$$\begin{aligned} v &= \nabla_x S + \frac{h}{2i\rho}(\bar{a}\nabla_x a - a\nabla_x \bar{a}),\\ \rho &= |a|^2, \end{aligned}$$

then (2.1) is transformed to

$$\partial_t \rho + \nabla_x \cdot (\rho v) = 0, \tag{2.5}$$

$$\partial_t\, v + \nabla_x\Big(\frac{|v|^2}{2} + f(\rho)\Big) = \frac{h^2}{2}\nabla_x\Big(\frac{1}{\sqrt{\rho}}\Delta_x\sqrt{\rho}\Big). \tag{2.6}$$

The above equation is a perturbation of the following isentropic compressible Euler equation

$$\partial_t \rho + \nabla_x \cdot (\rho v) = 0, \tag{2.7}$$

$$\partial_t\, v + \nabla_x\Big(\frac{|v|^2}{2} + f(\rho)\Big) = 0. \tag{2.8}$$

If $f' > 0$, then equation (2.7)-(2.8) admits a local smooth solution in $[0, T^*]$ for T^* sufficiently small. In fact, we have the following theorems.

Theorem 2.1 *Suppose $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$, $f' > 0$. $s > \frac{d}{2} + 2$. Let $S^0(x) \in H^s(\mathbb{R}^d)$ and $a^0(x,h)$ be uniformly bounded in $H^s(\mathbb{R}^d)$ with respect to x. Then there exists a constant $T > 0$ such that equation (2.1)-(2.2) admits a solution $\psi_h = a_h(x,t)\exp(iS_h(x,t)/h)$, where a_h and S_h are bounded in $L^\infty([0,T]; H^s)$.*

Theorem 2.2 *Under the assumption of Theorem 2.1, assume further $a^0(x,h) \xrightarrow{H^s(\mathbb{R}^d)} a^0$ as $h \to 0$ and equation (2.7)-(2.8) with initial data $(\rho(0,x), v(0,x)) =$*

$(|a^0(x)|^2, \nabla_x S^0(x))$ admits a solution $(\rho, v) \in L^\infty([0,T], H^{s+2})$. Then, equation (2.1) admits a formal solution $\psi_h(x,t) = a_h(x,t)\exp(iS_h(x,t)/h)$ on $[0,T]$ satisfying the initial condition (2.2) for h small enough, where a_h and S_h are uniformly bounded in $L^\infty([0,T], H^s)$ with respect to h.

Theorem 2.3 *Under the assumption of Theorem 2.2, if $a^0(x,h)$ admits the following expansion*

$$a^0(x,h) = \sum_{j=0}^{N} a_j^0(x)h^j + h^N r_N(x,h), \tag{2.9}$$

where $N \in \mathbb{N}$, $s - 2N - 2 - d/2 > 0$ and r_N satisfy

$$\lim_{h\to 0} ||r_N||_{H^s(\mathbb{R}^d)} = 0. \tag{2.10}$$

Then for interval $[0,T]$ given in Theorem 2.2, one has as $h \to 0$

$$a_h(x,t)\exp\Big(\frac{iS_h(x,t)}{h}\Big) = \sum_{j=0}^{N} a_j(x,t)h^j \exp\Big(\frac{iS_h(x,t)}{h}\Big) + h^N r_N(x,t), \tag{2.11}$$

where S and a_j are determined by the WKB method and

$$\lim_{h\to 0} ||r_N||_{L^\infty([0,T], H^{s-2N-2-d/2}(\mathbb{R}^d))} = 0. \tag{2.12}$$

Proof of Theorem 2.1 Suppose

$$\psi_h(x,t) = a_h(x,t)\exp\Big(\frac{iS_h(x,t)}{h}\Big). \tag{2.13}$$

Substituting (2.13) into (2.1) yields

$$-ih\partial_t a_h + \partial_t S_h a_h - \frac{h^2}{2}\Delta_x a_h - ih\nabla_x S_h \cdot \nabla_x a_h - \frac{ih}{2} a_h \Delta_x S_h + \frac{1}{2} a_h |\nabla_x S_h|^2 + a_h f(|a_h|^2) = 0,$$

which can be rewritten as

$$\begin{aligned} &\partial_t S_h + \frac{1}{2}|\nabla_x S_h|^2 + f(|a_h|^2) = 0, \\ &\partial_t a_h - \frac{ih}{2}\Delta_x a_h + \nabla_x S_h \cdot \nabla_x a_h + \frac{1}{2} a_h \Delta_x S_h = 0. \end{aligned}$$

Setting $\omega_h = \nabla_x S_h$, we have

$$\begin{aligned} &\partial_t \omega_h + \omega_h \cdot \nabla_x \omega_h + f'(|a_h|^2)\nabla_x |a_h|^2 = 0, \\ &\partial_t a_h + \omega_h \cdot \nabla_x a_h + \frac{1}{2} a_h \nabla_x \omega_h = \frac{ih}{2}\Delta_x a_h. \end{aligned}$$

Let $a_h = a_h^1 + i a_h^2$. One obtains

$$\partial_t a_h^1 + \sum_{j=1}^d \omega_h^j \partial_j a_h^1 + \frac{1}{2} a_h^1 \sum_{j=1}^d \partial_j \omega_h^j = -\frac{h}{2} \Delta_x a_h^2,$$

$$\partial_t a_h^2 + \sum_{j=1}^d \omega_h^j \partial_j a_h^2 + \frac{1}{2} a_h^2 \sum_{j=1}^d \partial_j \omega_h^j = \frac{h}{2} \Delta_x a_h^1,$$

$$\partial_t \omega_h^i + f'(|a_h^1|^2 + |a_h^2|^2)(2a_h^1 \partial_i a_h^1 + 2a_h^2 \partial_i a_h^2) + \sum_{j=1}^d \omega_h^j \partial_j \omega_h^i = 0,$$

where ω_h^i is the i-th component of ω_h. We can rewritten the above equation as

$$\partial_t u_h + \sum_{i=1}^d A^i(u_h) \partial_i u_h = hL(u_h), \tag{2.14}$$

where

$$u_h = \begin{pmatrix} a_h^1 \\ a_h^2 \\ \omega_h^1 \\ \omega_h^2 \\ \vdots \\ \omega_h^d \end{pmatrix}, \quad L(u_h) = \begin{pmatrix} -\frac{1}{2}\Delta_x a_h^2 \\ \frac{1}{2}\Delta_x a_h^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$A(u,\xi) = \sum_{j=1}^d \xi_j A^j(u_h) = \begin{pmatrix} \sum\limits_{i=1}^d \xi_i \omega_h^i & 0 & \frac{1}{2}\xi_1 a_h^1 & \frac{1}{2}\xi_2 a_h^1 & \dots \\ 0 & \sum\limits_{i=1}^d \xi_i \omega_h^i & \frac{1}{2}\xi_1 a_h^2 & \frac{1}{2}\xi_2 a_h^1 & \dots \\ 2\xi_1 a_h^1 f' & 2\xi_1 a_h^2 f' & \sum\limits_{i=1}^d \xi_i \omega_h^i & 0 & \dots \\ 2\xi_2 a_h^1 f' & 2\xi_2 a_h^2 f' & 0 & \sum\limits_{i=1}^d \xi_i \omega_h^i & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The matrix $A(u_h, \xi)$ can be symmetrized for $f' > 0$ by

$$S = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{4}f' & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{4}f' \end{pmatrix}.$$

Let $u_h = (a_h^1, a_h^2, \omega_h)$ be a solution of (2.14). The classical energy estimate $(S\partial_x^\alpha u_h, \partial_x^\alpha u_h)$ leads to

$$\partial_t(S\partial_x^\alpha u_h, \partial_x^\alpha u_h) = (\partial_t S \partial_x^\alpha u_h, \partial_x^\alpha u_h) + 2(S\partial_t \partial_x^\alpha u_h, \partial_x^\alpha u_h)$$

with S symmetry.

For the first term, we have

$$(\partial_t S \partial_x^\alpha u_h, \partial_x^\alpha u_h) \leq |\partial_t S|_{L^\infty} ||\partial_x^\alpha u_h||_{L^2}^2.$$

Since

$$|\partial_t S|_{L^\infty} \leq C(|u_h|_{L^\infty}) |\partial_t u_h|_{L^\infty},$$

employing the Sobolev embedding and the equation (2.14) yield

$$|\partial_t u_h|_{L^\infty} \leq C(||u_h||_s) ||u_h||_s,$$

where $s > \frac{d}{2} + 2$, $||u_h||_s^2 = \sum\limits_{|\alpha| \leq s} ||\partial_x^\alpha u_h||_{L^2}^2$.

For the second term, one easily obtains

$$(S\partial_t \partial_x^\alpha u_h, \partial_x^\alpha u_h) = h(SL(\partial_x^\alpha u_h), \partial_x^\alpha u_h) - \left(S\partial_x^\alpha \left(\sum_{i=1}^d A^i(u_h)\partial_i u_h \right), \partial_x^\alpha u_h \right). \tag{2.15}$$

Using integration by parts leads to

$$h(SL(\partial_x^\alpha u_h), \partial_x^\alpha u_h) = -\frac{h}{2} \int (\partial_x^\alpha a_h^1 \Delta_x \partial_x^\alpha a_h^2 - \partial_x^\alpha a_h^2 \Delta_x \partial_x^\alpha a_h^1) = 0.$$

The second term on the right hand side of (2.15) can be written as

$$\begin{aligned}
&\left(S\partial_x^\alpha \left(\sum_{i=1}^d A^i(u_h)\partial_i u_h \right), \partial_x^\alpha u_h \right) \\
=& \left(S \sum_{i=1}^d A^i(u_h) \partial_i \partial_x^\alpha u_h, \partial_x^\alpha u_h \right) \\
&+ \left(S \left(\partial_x^\alpha \left(\sum_{i=1}^d A^i(u_h)\partial_i u_h \right) - \sum_{i=1}^d A^i(u_h)\partial_i \partial_x^\alpha u_h \right), \partial_x^\alpha u_h \right).
\end{aligned}$$

Invoking the symmetry of $SA^i(u_h)$ yields

$$\begin{aligned}
&\left(S \sum_{i=1}^d A^i(u_h) \partial_i \partial_x^\alpha u_h, \partial_x^\alpha u_h \right) \\
=& -\sum_{i=1}^d (\partial_i(SA^i(u_h))\partial_x^\alpha u_h, \partial_x^\alpha u_h) - \sum_{i=1}^d (SA^i(u_h)\partial_i \partial_x^\alpha u_h, \partial_x^\alpha u_h),
\end{aligned}$$

therefore, one finds

$$\left|\left(S\sum_{i=1}^{d}A^i(u_h)\partial_i\partial_x^\alpha u_h,\partial_x^\alpha u_h\right)\right| \le C(|u_h|_{L^\infty})||\partial_x^\alpha u_h||_{L^2}^2|\nabla_x u_h|_{L^\infty}$$
$$\le C(|u_h|_{L^\infty})||\partial_x^\alpha u_h||_{L^2}^2||u_h||_s.$$

By the commutator estimates, there holds

$$\left|\left(S\left(\partial_x^\alpha\left(\sum_{i=1}^{d}A^i(u_h)\partial_i u_h\right)-\sum_{i=1}^{d}A^i(u_h)\partial_i\partial_x^\alpha u_h\right),\partial_x^\alpha u_h\right)\right| \le C(||u_h||_s)||u_h||_s^2.$$

Consequently, we find

$$\partial_t\sum_{|\alpha|\le s}(S\partial_x^\alpha u_h,\partial_x^\alpha u_h)\le C(||u_h||_s)||u_h||_s^2,$$

where $s>\frac{d}{2}+2$. We thus complete the proof by use of the Gronwall inequality.

Proof of Theorem 2.2 Assume there exists a solution $(\rho,v)\in L^\infty([0,T],H^{s+2}(\mathbb{R}^d))$ of (2.7)-(2.8) on $[0,T]$ with $s>\frac{d}{2}+2$ and the initial condition

$$\rho=|\lim_{h\to 0}a_h^0|^2,\quad v=\nabla_x S^0.$$

We will prove there exists a solution of (2.14) in a interval $[0,T)$ with T independent of h for h small enough and the solution is uniformly bounded in $L^\infty([0,T];H^s)$. The formal limit of equation (2.14) is

$$\partial_t u+\sum_{i=1}^{d}A^i(u)\partial_i u=0, \tag{2.16}$$

where $u=(a_1,a_2,\omega)$ admits a solution in the maximal interval $[0,T']$ with $T'\ge T$.

Set $v_h=u_h-u$, then we find

$$\partial_t v_h+\sum_{i=1}^{d}A^i(u+v_h)\partial_i v_h+\sum_{i=1}^{d}(A^i(u+v_h)-A^i(u))\partial_i u=hL(v_h)+hL(u).$$

The matrix $\sum\limits_{i=1}^{d}A^i(u+v_h)\xi_i$ is symmetrizable and we can do similar energy estimates. The term $\sum\limits_{i=1}^{d}A^i(u+v_h)\partial_i v_h$ can be handled by

$$\left|\left(S\left(\partial_x^\alpha\left(\sum_{i=1}^{d}A^i(u+v_h)-A^i(u)\right)\partial_i u\right),\partial_x^\alpha v_h\right)\right|\le C(||v_h||_s,||u||_s)||v_h||_s^2.$$

Note that

$$(hSL(\partial_x^\alpha v_h)+hSL(\partial_x^\alpha u),\partial_x^\alpha v_h)=(hL(\partial_x^\alpha u),\partial_x^\alpha v_h)\le h||u||_{\alpha+2}||v_h||_\alpha,$$

so

$$\partial_t \sum_{|\alpha|\le s} (S\partial_x^\alpha v_h, \partial_x^\alpha v_h) \le C(||v_h||_s, ||u||_{s+2})||v_h||_s^2 + h||u||_{s+2}||v_h||_s,$$

where $s > \frac{d}{2} + 2$. $||v_h(t=0)||_s \to 0$, $h \to 0$. Therefore

$$||v_h||_s \le C(h), \quad t \in [0,T],$$

where $C(h) \to 0$ and $h \to 0$. We thus complete the proof of Theorem 2.2.

Proof of Theorem 2.3 We prove the theorem in four steps.

1. Zero order approximation

From Theorem 2.2, we know that a_h and ω_h are uniformly bounded in $L^\infty([0,T]; H^s(\mathbb{R}^d))$. Therefore, $\partial_t a_h$ and $\partial_t \omega_h$ are bounded in $L^\infty([0,T]; H^{s-2}(\mathbb{R}^d))$. We can extract subsequences of a_h and ω_h converging to a_0' and ω_0', respectively in $L^\infty([0,T]; H^{s-2}(\mathbb{R}^d))$ and the limits are the unique solution of

$$\begin{aligned} &\partial_t \omega_0' + \omega_0' \cdot \nabla_x \omega_0' + f'(|a_0'|^2)\nabla_x |a_0'|^2 = 0, \\ &\partial_t a_0' + \omega_0' \cdot \nabla_x a_0' + \frac{1}{2} a_0' \nabla_x \omega_0' = 0 \end{aligned}$$

satisfying $a_0' = \lim\limits_{h\to 0} a^0(h)$, $\omega_0' = \nabla S^0$.

2. First order approximation

Let $v_h = u_h - u$, we can prove the following energy estimates

$$||v_h||_{H^{s-2}} \le hC(||u||_{L^\infty([0,T],H^s)}), \quad \text{for any } t \le T.$$

Set $\widetilde{v}_h = v_h/h$. Then $\widetilde{v}_h$ is bounded in $L^\infty([0,T], H^{s-2})$ and $\partial_t \widetilde{v}_h$ is bounded in $L^\infty([0,T], H^{s-4})$. If necessary, we can extract subsequences of $\widetilde{v}_h$ converging strongly to u_1' in $L^\infty([0,T], H^{s-4})$. Taking the limit equation of $\widetilde{v}_h$, one finds that u_1' satisfies the linear equation:

$$\partial_t u_1' + \sum_{i=1}^d A^i(u_0')\partial_i u_1' + \sum_{i=1}^d (\nabla A^i(u_0')u_1')\partial_i u_0' = L(u_0'),$$

and the initial condition

$$u_1' = \lim_{h\to 0} \frac{u_h(0) - u_0'}{h}.$$

The solution to this problem is unique. In fact, there exists a subsequence of $\widetilde{v}_h$ converge to u_1'.

3. Higher order approximation

Assume the N-th approximation to be

$$u_h = \sum_{j=0}^{N} u'_j h^j + o(h^N),$$

where the function $u'_j \in L^\infty([0,T], H^{s-2j}(\mathbb{R}^d))$. Set $\widetilde{u}_h = \sum\limits_{j=0}^{N} u'_j h^j$, $v_h = u_h - \sum\limits_{j=0}^{N} u'_j h^j$, one finds

$$\partial_t v_h + \sum_{i=1}^{d} A^i(\widetilde{u}_h + v_h)\partial_i v_h - \sum_{i=1}^{d}(A^i(\widetilde{u}_h) - A^i(\widetilde{u}_h + v_h))\partial_i \widetilde{u}_h = hL(v_h) + h^{N+1}B_h^N,$$

where B_h^N is a function of $\widetilde{u}_h$ and bounded with respect to h in $L^\infty([0,T], H^\sigma)$ for $\sigma = s-2N-2$. Assume the initial data $h^{-N-1}v_h(0)$ is bounded in H^s. Similar to the energy estimates in the former cases, one can obtain the boundedness of $h^{-N-1}v_h$ in $L^\infty([0,T], H^\sigma)$.

Let $\widetilde{v}_h = v_h/h^{N+1}$, then we obtain $\widetilde{v}_h \to u'_{N+1}$, $h \to 0$. u'_{N+1} can be obtained by solving the following equation

$$\partial_t(\widetilde{u}_h + h^{N+1}\omega_h) + \sum_{i=1}^{d} A^i(\widetilde{u}_h + h^{N+1}\omega_h)\partial_i(\widetilde{u}_h + h^{N+1}\omega_h) - hL(\widetilde{u}_h) = 0.$$

4. WKB expansion

We have expanded the formal solution a_h and S_h to any order. To return to the WKB expansion, one can write the following two formal series

$$\sum_{j=0}^{+\infty} a_j(x,t)h^j \exp\Big(\frac{iS(x,t)}{h}\Big) = \left(\sum_{k=0}^{+\infty} a'_k h^k\right) \exp\left(i\sum_{k=0}^{\infty} S'_k h^k\right).$$

For instance, $S = S'_0$, $a_0 = a'_0 e^{iS'_1}$, $a_1 = e^{iS'_1}(a'_1 + iS'_2 a'_0)$. We thus complete the proof of Theorem 2.3.

3 Semiclassical Limit to the Derivative Schrödinger equation

Now, let us consider the following derivative Schrödinger equation (GDNLS)

$$ih\psi_t + \frac{h^2}{2}\psi_{xx} + ih(f(|\psi|^2)\psi)_x = 0, \tag{3.1}$$

$$\psi^h(x,0) = \psi_0^h(x) = A_0^h \exp\Big(\frac{i}{h}S_0(x)\Big), \tag{3.2}$$

where $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$, $S_0 \in H^s(\mathbb{R})$ for s large enough, A_0^h is a polynomial in h with coefficients of Sobolev regularity in x. We consider the limit of (3.1)-(3.2) when $h \to 0$, for $-\infty < x < +\infty$ and $0 \le t \le T$ with T being finite.

When $f(x) = x$ in equation (3.1), the resulting equation is used to describe the nonlinear propagation of magnetosonic wave trains parallel to the magnetic field in a hot or collisionless ideal plasma with dispersion due to Hall currents [7, 13].

Suppose $\psi(x,t) = A(x,t)\exp\left(\frac{i}{h}S(x,t)\right)$, where A and S are real-valued functions representing the amplitude and the classical action, respectively. Substituting this to (3.1) leads to

$$\partial_t A + \frac{1}{2}(AS_{xx} + 2A_x S_x) + (f(A^2)A)_x = 0, \tag{3.3}$$

$$\partial_t S + \frac{1}{2}(S_x)^2 + f(A^2)S_x = \frac{h^2}{2}\frac{A_{xx}}{A}. \tag{3.4}$$

Introducing the new variables $\rho = A^2 = |\psi|^2$, $u = S_x$, we have

$$\partial_t \rho + \partial_x(\rho u + Q(\rho)) = 0, \tag{3.5}$$

$$\partial_t u + uu_x + \partial_x(f(\rho)u) = \frac{h^2}{2}\frac{\partial}{\partial_x}\left(\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{3.6}$$

where

$$Q(\rho) = 2\rho f(\rho) - \frac{1}{2}\Phi(\rho), \quad \Phi'(\rho) = 2f(\rho). \tag{3.7}$$

Thus, equation (3.5)-(3.6) can be regarded as a perturbation to the following compressible Euler equation

$$\partial_t \rho + \partial_x(\rho u + Q(\rho)) = 0, \tag{3.8}$$

$$\partial_t u + \partial_x\Big(\frac{1}{2}u^2 + f(\rho)u\Big) = 0. \tag{3.9}$$

Multiplying (3.6) by ρ and using (3.5) lead to

$$\partial_t \mu + \partial_x(\mu(u+f) + uP'\partial_x\rho) = \frac{h^2}{2}\partial_x(\rho\partial_x^2\log\rho), \tag{3.10}$$

where $\mu = \rho u$ is the momentum and $P'(\rho) = 2\rho f'(\rho) = \rho\Phi''(\rho)$. From (3.5), using $P(\rho) = \rho\Phi'(\rho) - \Phi(\rho)$, one finds

$$\Phi'(\partial_t\rho + u\partial_x P) + \rho\Phi'\partial_x u + \Phi' Q'\rho_x = 0, \tag{3.11}$$

or

$$\partial_t\Phi + \partial_x(\Phi u) + P\partial_x u + \pi'\rho_x = 0, \tag{3.12}$$

where

$$\pi' = \pi'(\rho) = \Phi'(\rho)Q'(\rho) = 2f(\rho)(2\rho f'(\rho) + f(\rho)) = \frac{\mathrm{d}}{\mathrm{d}\rho}[2\rho f^2(\rho)].$$

Thus, there holds

$$\pi(\rho) = 2\rho f^2(\rho). \tag{3.13}$$

Adding (3.10) and (3.12) leads to

$$\partial_t(\mu + \Phi) + \partial_x((\mu + \Phi)u) + \partial_x(\mu f + Pu) + \partial_x \pi = \frac{h^2}{4}\partial_x(\rho \partial_x^2 \log \rho).$$

As in [1], we denote by $M = \mu + \Phi(\rho)$ the noncanonical momentum. Using $Q(\rho) = 2\rho f(\rho) - \frac{1}{2}\Phi(\rho)$, equation (3.5)-(3.6) can be written as

$$\partial_t \rho + \partial_x(M + Q - \Phi) = 0, \tag{3.14}$$

$$\partial_t M + \partial_x\Big[\frac{M}{\rho}(M + P - \Phi + \rho f) + \frac{P}{\rho}(\rho f - \Phi)\Big] = \frac{h^2}{4}\partial_x(\rho \partial_x^2 \log \rho), \tag{3.15}$$

and this can be rewritten as the local conservation laws of ρ, M, Φ

$$\partial_t \rho + \partial_x\Big(M + \rho\Phi' - \frac{3}{2}\Phi\Big) = 0, \tag{3.16}$$

$$\begin{aligned}&\partial_t M + \partial_x\Big[\frac{M}{\rho}\Big(M + \rho\Phi' - \frac{3}{2}\Phi' - \frac{3}{2}\Phi + \frac{1}{2}(\rho\Phi' - \Phi)\Big)\Big]\\ &+\partial_x\Big[\frac{1}{\rho}(\rho\Phi' - \Phi)\Big(\frac{1}{2}\rho\Phi' - \Phi\Big)\Big] = \frac{h^2}{2}\partial_x(\rho\partial_x^2 \log \rho).\end{aligned} \tag{3.17}$$

Collecting the above arguments, we obtain the following theorem.

Theorem 3.1 *Equation* (3.1) *is equivalent to the dispersive perturbation of the quasilinear hyperbolic equation* (3.14)-(3.15) *or* (3.16)-(3.17). *The density* ρ *and the noncanonical momentum* M *are the conserved quantities of the GDNLS equation. In particular, when* $f(\rho) = \pm\rho^\gamma$, $\Phi(\rho) = \pm\frac{2}{\gamma+1}\rho^{\gamma+1}$ *and* $M = \mu \pm \frac{2}{\gamma+1}\rho^{\gamma+1}$, (3.16)-(3.17) *is equivalent to*

$$\partial_t \rho + \partial_x\Big(M \pm \frac{2\gamma - 1}{\gamma + 1}\rho^{\gamma+1}\Big) = 0, \tag{3.18}$$

$$\partial_t M + \partial_x\Big[\frac{M}{\rho}\Big(M \pm \frac{3\gamma - 1}{\gamma + 1}\rho^{\gamma+1}\Big) + \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}\rho^{2\gamma+1}\Big] = \frac{h^2}{4}\partial_x(\rho\partial_x^2 \log \rho). \tag{3.19}$$

In addition, we will prove the following theorem.

Theorem 3.2 *Let* $*$ *be the complex conjugate. The GDNLS equation admits the following conserved quantities*

$$\int_{-\infty}^{+\infty} \rho \mathrm{d}x = const. = C_1, \tag{3.20}$$

$$\int_{-\infty}^{+\infty} \widetilde{u} \mathrm{d}x = const. = C_2, \tag{3.21}$$

$$\int_{-\infty}^{+\infty} \widetilde{M} \mathrm{d}x = const. = C_3, \tag{3.22}$$

where the fluid dynamical variables $\rho, \widetilde{u}, \widetilde{M}$ *can be represented by the wave function* ψ *as*

$$\rho = |\psi|^2 = |A|^2, \tag{3.23}$$

$$\widetilde{u} = \frac{\widetilde{\mu}}{\rho} = \frac{ih}{2}\Big(\frac{\psi_x^*}{\psi^*} - \frac{\psi_x}{\psi}\Big) = S_x + \frac{ih}{2}\Big(\log\frac{A^*}{A}\Big)_x, \tag{3.24}$$

$$\widetilde{M} = \widetilde{\mu} + \Phi(\rho) = \frac{ih}{2}(\psi\psi_x^* - \psi^*\psi_x) + \Phi(|\psi|^2). \tag{3.25}$$

Proof Obviously, the theorem can be deduced directly by (3.14)-(3.15). Here, we will derive it from (3.1) and thus we can better understand the relation between the classical mechanics and the quantum mechanics.

Note that we have the following equality

$$\frac{\partial}{\partial t}(\psi\psi^*) + \frac{\partial}{\partial x}\Big[\frac{ih}{2}(\psi\psi_x^* - \psi^*\psi_x)\Big] + [f(\psi\psi^*) + 2\psi\psi^* f'(\psi\psi^*)]\frac{\partial}{\partial x}(\psi\psi^*) = 0. \tag{3.26}$$

Because the third term in (3.26) only depends on $\psi\psi^* = |\psi|^2$, we can integrate it with respect to x to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\rho\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}|\psi|^2\mathrm{d}x = 0. \tag{3.27}$$

This proves (3.20).

Multiplying (3.1) by ψ_x^* and ψ_x, respectively, and then adding together, one finds

$$i(\psi_t\psi_x^* - \psi_t^*\psi_x) + \frac{h}{2}\frac{\partial}{\partial x}(\psi_x\psi^*) + i[(\psi\psi^*)_x f'(|\psi(x)|^2)(\psi_x^*\psi - \psi_x\psi^*)] = 0. \tag{3.28}$$

Also, one easily deduces

$$i(\psi^*\psi_{xt} - \psi\psi_{xt}^*) + \frac{h}{2}\frac{\partial}{\partial x}((\psi\psi^*)_{xx} - 3\psi_x\psi_x^*) + i\frac{\partial}{\partial x}[f(\psi^*\psi_x - \psi\psi_x^*)] = 0. \tag{3.29}$$

Subtracting (3.28) from (3.29), we obtain

$$\partial_t\widetilde{\mu} + \partial_x(f\widetilde{\mu}) + 2f'\rho_x\widetilde{\mu}\widetilde{\mu} = \frac{h^2}{2}\partial_x[(\psi\psi^*)_{xx} - 4\psi_x\psi_x^*]. \tag{3.30}$$

Next, note that the following equality

$$\partial_t\widetilde{u} + \partial_x\Big(\frac{1}{2}\widetilde{u}^2 + f\widetilde{u}\Big) = \frac{h^2}{2}\partial_x\Big(\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}}\Big) \tag{3.31}$$

is equivalent to (3.5)-(3.6) with

$$\frac{h^2}{4}\frac{1}{|\psi|^2}\partial_x[(\psi\psi^*)_{xx} - 4\psi_x\psi_x^*] = \frac{h^2}{2}\partial_x\Big(\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}}\Big). \tag{3.32}$$

Integrating (3.31) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\widetilde{u}\mathrm{d}x=0. \tag{3.33}$$

Multiplying (3.28) by $\Phi'(\rho)$ leads to

$$\partial_t\Phi+\Phi'\partial_x\widetilde{\mu}+\Phi'(f+\rho f')\rho_x=0. \tag{3.34}$$

Adding (3.30) and (3.34) and using

$$\partial_x(\Phi'\widetilde{\mu})=\Phi''\rho_x\widetilde{\mu}+\Phi'\widetilde{\mu}_x=2f'\rho_x\widetilde{\mu}+\Phi'\widetilde{\mu}_x, \tag{3.35}$$

we obtain

$$\partial_t(\widetilde{\mu}+\Phi)+\partial_x[(f+\Phi')\widetilde{\mu}]+\Phi'(f+\rho f')\rho_x=\frac{h^2}{2}\partial_x\Big(\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}}\Big), \tag{3.36}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\widetilde{M}\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}(\widetilde{\mu}+\Phi(\rho))\mathrm{d}x=0. \tag{3.37}$$

This complete the proof.

For the GDNLS equation

$$i\psi_t^h+\frac{h}{2}\psi_{xx}^h+i(f(|\psi^h|^2)\psi^h)_x=0 \tag{3.38}$$

with the initial condition

$$\psi^h(x,0)=\psi_0^h(x)=A_0(x)\exp\Big(\frac{i}{h}S_0(x)\Big), \tag{3.39}$$

where the amplitude $A_0(x)$ is nonnegative, the phase $S_0(x)$ is real-valued and smooth and is independent of h. One can take

$$\rho^h(0,x)=|A_0(x)|^2,\quad M^h(0,x)=|A_0(x)|^2\partial_xS_0(x)+\Phi(|A_0(x)|^2). \tag{3.40}$$

We can prove that ψ^h is a dispersive perturbation with $O(h^2)$ error as $h\to 0$ to the following deformed Euler equation

$$\partial_t\rho+\partial_x\Big(M+\rho\Phi'-\frac{3}{2}\Phi\Big)=0, \tag{3.41}$$

$$\partial_tM+\partial_x\Big[\frac{M}{\rho}\Big(M+\rho\Phi'-\frac{3}{2}\Phi'-\frac{3}{2}\Phi+\frac{1}{2}(\rho\Phi'-\Phi)\Big)\Big]+\partial_x\Big[\frac{1}{\rho}(\rho\Phi'-\Phi)\Big(\frac{1}{2}\rho\Phi'-\Phi\Big)\Big]=0 \tag{3.42}$$

with the initial condition

$$\rho(0,x)=|A_0(x)|^2,\quad M(0,x)=|A_0(x)|^2\partial_x S_0(x)+\Phi(|A_0(x)|^2). \tag{3.43}$$

Consider

$$f(\rho)=\pm\rho,\quad \Phi(\rho)=\pm\rho^2.\quad M=\mu\pm\rho^2. \tag{3.44}$$

Then (3.18)-(3.19) can be transformed to

$$\partial_t\rho+\partial_x\Big(M\pm\frac{1}{2}\rho^2\Big)=0, \tag{3.45}$$

$$\partial_t M+\partial_x\Big[\frac{M^2}{\rho}\pm\rho M\Big]=\frac{h^2}{2}\partial_x(\rho\partial_x^2\log\rho). \tag{3.46}$$

From

$$-\frac{M^2}{2\rho^2}\Big[\partial_t\rho+\partial_x\Big(M\pm\frac{\rho^2}{2}\Big)\Big]=0, \tag{3.47}$$

$$\frac{M}{\rho}\Big[\partial_t M+\partial_x\Big(\frac{M^2}{\rho}\pm\rho M\Big)\Big]=\frac{h^2}{4}\frac{M}{\rho}\partial_x(\rho\partial_x^2\log\rho), \tag{3.48}$$

we have

$$\partial_t\Big(\frac{M^2}{2\rho}\Big)+\partial_x\Big(\frac{M^2}{2\rho}\frac{M}{\rho}\pm\frac{M^2}{2}\Big)\pm\frac{M^2}{2\rho}\rho_x=\frac{h^2}{2}\frac{M}{\rho}\partial_x(\rho\partial_x^2\log\rho). \tag{3.49}$$

On the other hand

$$\frac{1}{2}\Big[\partial_t(\rho M)+\partial_x\Big(\frac{3M^2}{2}\pm\rho^2 M\Big)-\frac{M^2}{\rho}\partial_x\rho\Big]=\frac{h^2}{8}\rho\partial_x(\rho\partial_x^2\log\rho), \tag{3.50}$$

$$\begin{aligned}&\partial_t\Big(\frac{h^2}{8}\frac{\rho_x^2}{\rho}\Big)+\frac{h^2}{2}\partial_x\Big(\frac{\rho_x M_x}{\rho}-\frac{\rho_{xx}}{\rho}\Big)+\frac{h^2}{8}\partial_x\Big[\frac{M\rho_x^2}{\rho^2}\pm\Big(\frac{1}{2}\rho_x^2-\rho_{xx}\Big)\Big]\\ =&-\frac{h^2}{4}\frac{M}{\rho}\partial_x(\rho\partial_x^2\log\rho)-\frac{h^2}{8}\rho\partial_x(\rho\partial_x^2\log\rho).\end{aligned} \tag{3.51}$$

Consequently, one obtains the energy equation

$$\partial_t E_\pm^h+\partial_x\Big[\frac{M}{\rho}\Big(E_\pm^h+\frac{\rho^3}{2}\pm\frac{3}{2}\rho M\Big)\Big]+\frac{h^2}{8}\partial_x\Big[2\Big(\frac{\rho_x M_x}{\rho}-\frac{\rho_{xx}}{\rho}\Big)\pm\frac{1}{2}\rho_x^2-\rho_{xx}\Big]=0 \tag{3.52}$$

with

$$E_\pm^h=\frac{M^2}{2\rho}\pm\frac{1}{2}\rho M+\frac{h^2}{8}\frac{\rho_x^2}{\rho}=\frac{M^2}{2\rho}\pm\frac{3}{2}\rho M+\rho^3+\frac{h^2}{8}\frac{\rho_x^2}{\rho}. \tag{3.53}$$

Assume $f(\rho)=\rho$, then the Euler equation derived by the semiclassical limit is

$$\partial_t\rho + \partial_x\Big(M + \frac{1}{2}\rho^2\Big) = 0, \tag{3.54}$$

$$\partial_t M + \partial_x\Big(\frac{M^2}{\rho} + \rho M\Big) = 0 \tag{3.55}$$

with initial data

$$\rho(0,x) = A^2(x), \quad M(0,x) = A^2(x)S_x + A^4(x). \tag{3.56}$$

Energy equation (3.52) is then

$$\partial_t\Big(\frac{M^2}{2\rho} + \frac{\rho M}{2}\Big) + \partial_x\Big(\frac{M^2}{2\rho}\frac{M}{\rho} + \frac{\rho^2 M}{2} + \frac{5M^2}{4}\Big) = 0. \tag{3.57}$$

One can rewritten (3.54)-(3.55) as

$$V_t + BV_x = 0, \quad V = (\rho, M)^t \tag{3.58}$$

with

$$B = \begin{pmatrix} \rho & 1 \\ -\dfrac{M^2}{\rho^2} + M & \dfrac{2M}{\rho} + \rho \end{pmatrix}. \tag{3.59}$$

The eigenvalues of B are the roots of

$$\lambda^2 - 2\Big(\rho + \frac{M}{\rho}\Big)\lambda + \rho^2 + M + \frac{M^2}{\rho^2} = 0 \tag{3.60}$$

or

$$\lambda = \lambda_\pm = \rho + \frac{M}{\rho} \pm \sqrt{M}, \quad M \geq 0, \tag{3.61}$$

the corresponding right and left eigenvector are respectively

$$r_\pm = \Big(1, \frac{M}{\rho} \pm \sqrt{M}\Big)^t, \quad l_\pm = \Big(-\frac{M}{\rho} \pm M, 1\Big)^t. \tag{3.62}$$

The Riemann invariants is

$$R_\pm = \sqrt{\frac{M}{\rho}} \pm \sqrt{\rho}. \tag{3.63}$$

The eigenvalues λ_+, λ_- can be represented by the Riemann invariants as

$$\lambda_+ = \frac{3}{4}R_+^2 + \frac{1}{4}R_-^2, \quad \lambda_- = \frac{1}{4}R_+^2 + \frac{3}{4}R_-^2. \tag{3.64}$$

The equation can be rewritten by the Riemann invariants as

$$\partial_t R_+ + \Big(\frac{3}{4}R_+^2 + \frac{1}{4}R_-^2\Big)\partial_x R_+ = 0, \tag{3.65}$$

$$\partial_t R_- + \Big(\frac{1}{4}R_+^2 + \frac{3}{4}R_-^2\Big)\partial_x R_- = 0 \tag{3.66}$$

with initial data

$$R_\pm(x) = \sqrt{S_x + A(x)^2} \pm A(x). \tag{3.67}$$

Theorem 3.3 *The blowup time t_b can be estimated as*

$$t_b = \min\{t_{+,b}, t_{-,b}\}$$

with

$$t_{\pm,b} \le \inf_{x_0\in\Omega_\pm}\{t : G_\pm(t, x_0) = 0\}, \quad \Omega_\pm = \{x_0 : \partial_x R_\pm(x_0) \le 0\},$$

where

$$G_\pm = 1 + \frac{3}{2}\frac{\partial_x R_\pm(x)}{\sqrt{R_+^2(x) - R_-^2(x)}}\int_0^t R_\pm(\tau, x_\pm(\tau))\sqrt{R_+^2(\tau, x_\pm(\tau)) - R_-^2(\tau, x_\pm(\tau))}\mathrm{d}\tau$$

and $x_\pm(t)$ satisfying

$$\frac{\mathrm{d}x_\pm}{\mathrm{d}t} = \frac{1}{2}R_\pm^2(t, x_\pm) + \frac{1}{4}[R_+^2(t, x_\pm) + R_-^2(t, x_\pm) - 1], \quad x_\pm(0) = x.$$

Proof The break-time t_b can be estimated by Laxs recipe [8, 14].

From (3.65), we obtain

$$R_+' = \partial_t R_+ + \Big(\frac{3}{4}R_+^2 + \frac{1}{4}R_-^2\Big)\partial_x R_+ = 0, \tag{3.68}$$

where "$'$" indicates the differentiation along the characteristic direction, then

$$\frac{\mathrm{d}x_+(t)}{\mathrm{d}t} = \lambda_+(t, x_+(t)) = \Big(\frac{3}{4}R_+^2 + \frac{1}{4}R_-^2\Big)(t, x_+(t)).$$

Similarly,

$$\dot{R}_- = \partial_t R_- + \Big(\frac{1}{4}R_+^2 + \frac{3}{4}R_-^2\Big)\partial_x R_- = 0, \tag{3.69}$$

where "$\cdot$" indicates the differentiation is along the characteristic direction, then

$$\frac{\mathrm{d}x_-(t)}{\mathrm{d}t} = \lambda_-(t, x_-(t)) = \Big(\frac{1}{4}R_+^2 + \frac{3}{4}R_-^2\Big)(t, x_-(t)).$$

Differentiating (3.68) with respect to x and setting $Z_+ = \partial_x R_+$, we obtain

$$\partial_t Z_+ = \Big(\frac{3}{2}Z_+^2 R_+ + \frac{1}{2}Z_+ R_- \partial_x R_-\Big) + \Big(\frac{3}{4}R_+^2 + \frac{3}{4}R_-^2\Big)\partial_x Z_+ = 0. \tag{3.70}$$

(3.69) leads to

$$R'_- = \partial_t R_- + \Big(\frac{3}{4}R_+^2 + \frac{1}{4}R_-^2\Big)\partial_x R_-. \tag{3.71}$$

Therefore

$$\partial_x R_- = \frac{2R'_-}{R_+^2 - R_-^2}. \tag{3.72}$$

Substituting (3.72) into (3.70) yields

$$Z'_+ + \frac{R_- R'_-}{R_+^2 - R_-^2} Z_+ + \frac{3}{2} R_+ Z_+^2 = 0. \tag{3.73}$$

Let $h = h(R_+, R_-)$ satisfy $h' = \frac{R_- - R'_-}{R_+^2 - R_-^2}$ and $h = \frac{1}{2}\log\frac{1}{R_+^2 - R_-^2}$. Multiplying (3.73) by $e^h = \frac{1}{\sqrt{R_+^2 - R_-^2}}$ and defining

$$q_+ = \mathrm{e}^h Z_+ = \frac{\partial_x R_+}{\sqrt{R_+^2 - R_-^2}}, \quad k_+ = \frac{3}{2} R_+ \mathrm{e}^{-h} = \frac{3}{2} R_+ \sqrt{R_+^2 - R_-^2}, \tag{3.74}$$

one can derive the standard Raccati equation

$$q'_+ + k_+ q_+^2 = 0. \tag{3.75}$$

The solution to (3.75) is

$$q_+(x,t) = \frac{q_+^0}{1 + q_+^0 K_+(t)}, \quad q_+^0 = q_+(0, x(0)), \tag{3.76}$$

where

$$K_+(t) = \int_0^t k_+(\tau, x_+(\tau))\mathrm{d}\tau = \frac{3}{2}\int_0^t R_+(\tau, x_+(\tau))\sqrt{R_+^2(\tau, x_+(\tau)) - R_-^2(\tau, x_+(\tau))}\mathrm{d}\tau.$$

The integration is along the characteristic of λ_+ and the sign of $q_0 K(t)$ impact its singularity essentially. If the initial data satisfies $\partial_x R_+(0, x(0)) = \partial_x R_+(x(0)) < 0$, namely, $q_0 < 0$, then $q_+(x,t)$ will tend to infinity in finite time, which implies that $q_+(x,t)$ certainly will blowup and $1 + q_0 K(t) = 0$. Therefore, the blowup time t_b can be estimated as follows.

Let $t_{+,b}$ satisfy

$$t_{+,b} \le \inf_{x_0 \in \Omega}\{t : G_+(t, x_0) = 0\}, \quad \Omega = \{x_0 : \partial_x R_+(x_0) \le 0\}$$

with

$$G_+ = 1 + \frac{3}{2}\frac{\partial_x R_+(x)}{\sqrt{R_+^2(x) - R_-^2(x)}}\int_0^t R_+(\tau, x_+(\tau))\sqrt{R_+^2(\tau, x_+(\tau)) - R_-^2(\tau, x_+(\tau))}\mathrm{d}\tau.$$

The particle trajectory $x = x_+(t)$ satisfies

$$\frac{\mathrm{d}x_+(t)}{\mathrm{d}t} = \frac{1}{2}R_+^2(t, x_+(t)) + \frac{1}{4}[R_+^2(t, x_+(t)) + R_+^2(t, x_-(t))], \quad x_+(0) = x.$$

Similarly, to estimate $t_{-,b}$, we consider the characteristic of λ_-. When $f(\rho) = -\rho$, we have

$$\partial_t \rho + \partial_x \Big(M - \frac{\rho^2}{2} \Big) = 0, \tag{3.77}$$

$$\partial_t M + \partial_x \Big(\frac{M^2}{\rho} - \rho M \Big) = 0, \tag{3.78}$$

and the energy equation is

$$\partial_t \Big(\frac{M^2}{2\rho} - \frac{\rho M}{2} \Big) + \partial_x \Big(\frac{M^2}{2} \cdot \frac{M}{\rho} + \frac{\rho^2 M}{2} - \frac{5}{4} M^2 \Big) = 0. \tag{3.79}$$

(3.77), (3.78) can be write as a matrix form as follows

$$\widetilde{V}_t + \widetilde{B}\widetilde{V}_x = 0, \quad \widetilde{V} = (\rho, M)^t, \tag{3.80}$$

where

$$\widetilde{B} \equiv \begin{pmatrix} -\rho & 1 \\ -\dfrac{M^2}{\rho^2} - M & \dfrac{2M}{\rho} - \rho \end{pmatrix}, \tag{3.81}$$

of which the eigenvalues are the roots of

$$\lambda^2 - 2\Big(\frac{M}{\rho} - \rho \Big)\lambda + \rho^2 + \frac{M^2}{\rho^2} - M = 0, \tag{3.82}$$

that is,

$$\lambda = \lambda_\pm = \frac{M}{\rho} - \rho \pm \sqrt{-M}, \quad M \le 0, \tag{3.83}$$

then the corresponding eigenvector are

$$\widetilde{r}_\pm = \Big(1, \frac{M}{\rho} \pm \sqrt{-M} \Big)^t, \quad \widetilde{l}_\pm = \Big(-\frac{M}{\rho} \pm \sqrt{-M}, 1 \Big)^t. \tag{3.84}$$

The Riemann invariants are

$$\widetilde{R}_\pm = \sqrt{\frac{-M}{\rho}} \mp \sqrt{\rho}. \tag{3.85}$$

The two eigenvalues $\widetilde{\lambda}_+, \widetilde{\lambda}_-$ can be represented by the Riemann invariants as

$$\widetilde{\lambda}_+ = -\frac{3}{4}\widetilde{R}_+^2 - \frac{1}{4}\widetilde{R}_-^2, \quad \widetilde{\lambda}_- = -\frac{1}{4}\widetilde{R}_+^2 - \frac{3}{4}\widetilde{R}_-^2. \tag{3.86}$$

The equation can be written as

$$\partial_t \widetilde{R}_+ - \Big(\frac{3}{4}\widetilde{R}_+^2 + \frac{1}{4}\widetilde{R}_-^2\Big)\partial_x \widetilde{R}_+ = 0, \tag{3.87}$$

$$\partial_t \widetilde{R}_- - \Big(\frac{1}{4}\widetilde{R}_+^2 + \frac{3}{4}\widetilde{R}_-^2\Big)\partial_x \widetilde{R}_- = 0, \tag{3.88}$$

with the initial data

$$\widetilde{R}_\pm(x) = \sqrt{-S_x + A(x)^2} \mp A(x). \tag{3.89}$$

This completes the proof.

Similarly, we can derive the estimate of the blowup time $\widetilde{t}_b$.

Theorem 3.4 *The blowup time $\widetilde{t}_b$ can be estimated as*

$$\widetilde{t}_b = \min\{\widetilde{t}_{+,b}, \widetilde{t}_{-,b}\},$$

where

$$\widetilde{t}_{\pm,b} = \inf_{x_0 \in \widetilde{\Omega}_\pm}\{t : \widetilde{G}_\pm(t, x_0) = 0\}, \quad \widetilde{\Omega}_\pm = \{x_0 : \partial_x \widetilde{R}_\pm(x_0) \le 0\},$$

and

$$\widetilde{G}_\pm = 1 - \frac{3}{2}\frac{\partial_x \widetilde{R}_\pm(x)}{\sqrt{\widetilde{R}_+^2(x) - \widetilde{R}_-^2(x)}}\int_0^t \widetilde{R}_\pm(\tau, \widetilde{x}_\pm(\tau))\sqrt{\widetilde{R}_-^2(\tau, \widetilde{x}_\pm(\tau)) - \widetilde{R}_+^2(\tau, \widetilde{x}_\pm(\tau))}\mathrm{d}\tau,$$

with $\widetilde{x}_\pm(t)$ satisfying

$$\frac{\mathrm{d}\widetilde{x}_\pm}{\mathrm{d}t} = -\frac{1}{2}\widetilde{R}_\pm^2(t, \widetilde{x}_\pm) - \frac{1}{4}[\widetilde{R}_+^2(t, \widetilde{x}_\pm) + \widetilde{R}_-^2(t, \widetilde{x}_\pm)], \quad \widetilde{x}_\pm(0) = x.$$

4 Semiclassical Limit to the Generalized NLS: Subsonic, Supersonic, Transonic

Consider the generalized NLS

$$i\epsilon\phi_t + \frac{1}{2}\epsilon^2\frac{\partial^2\phi}{\partial t^2} + |\phi|^2\phi + i\alpha\epsilon\frac{\partial}{\partial x}(|\phi|^2\phi) = 0, \quad \alpha, \epsilon > 0. \tag{4.1}$$

Setting

$$\rho_\epsilon(x,t) = |\phi_\epsilon(x,t)|^2, \quad u_\epsilon(x,t) = \epsilon\mathrm{Im}\Big\{\frac{\partial}{\partial x}\log(\phi_\epsilon(x,t))\Big\}, \tag{4.2}$$

$$\phi_\epsilon(x,0) = A_0(x)\exp\Big(\frac{iS_0(x)}{\epsilon}\Big), \tag{4.3}$$

one obtains

$$\begin{aligned}
&\frac{\partial \rho_\epsilon}{\partial t}+\frac{\partial}{\partial x}\Big(\rho_\epsilon u_\epsilon+\frac{3}{2}\alpha\rho_\epsilon^2\Big)=0,\\
&\frac{\partial u_\epsilon}{\partial t}+\frac{\partial}{\partial x}\Big(\frac{1}{2}u_\epsilon^2-\rho_\epsilon+\alpha\rho_\epsilon u_\epsilon\Big)=\frac{1}{2}\epsilon^2\frac{\partial F[\rho_\epsilon]}{\partial x},
\end{aligned}\tag{4.4}$$

where

$$F[\rho_\epsilon]=\frac{1}{2\rho_\epsilon}\frac{\partial^2\rho_\epsilon}{\partial x^2}-\Big(\frac{1}{2\rho_\epsilon}\frac{\partial\rho_\epsilon}{\partial x}\Big)^2.\tag{4.5}$$

When $\epsilon=0$, the MNLS is

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\Big(\rho u+\frac{3}{2}\alpha\rho^2\Big)=0,\\
&\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\Big(\frac{1}{2}u^2-\rho+\alpha\rho u\Big)=0.
\end{aligned}\tag{4.6}$$

Suppose $\alpha\neq 0$, and equation (4.6) is of mixed type. If $Q>0$, the equation is of hyperbolic; if $Q<0$, the equation is of elliptic. When $Q(x)$ changes sign, the domain is of transonic. Here,

$$Q=\alpha^2\rho+\alpha u-1,\tag{4.7}$$

where we define the sonic

$$c(x,t)=\frac{1}{\alpha}-\alpha\rho(x,t).\tag{4.8}$$

We consider the inverse scattering method to MNLS.

To do so, we first consider the initial condition of (4.6),

$$\rho_\epsilon(x,0)=A_0(x)^2,\quad u_\epsilon(x,0)=u_0(x)=S_0'(x).\tag{4.9}$$

Assume $\rho_0(x)$ and $u_0'(x)$ are real-valued Schwartz functions for $x\in\mathbb{R}$ and $\rho_0(x)\neq 0$. In addition, suppose $u_0(x)\to u_\pm$ for $x\to\pm\infty$. Then

$$\begin{aligned}
&S_0(x)=S_0(0)+\int_0^x u_0(y)\mathrm{d}y\Rightarrow S_0(x)=u_\pm x+S_\pm+o(1),\quad x\to\pm\infty,\\
&S_+=S_0(0)+\int_0^\infty[u_0(y)-u_+]\mathrm{d}y,\quad S_-=S_0(0)-\int_{-\infty}^0[u_0(y)-u_-]\mathrm{d}y.
\end{aligned}$$

In particular, choose $u_0(x)$ and $S_0(x)$ as

$$A_0(x)=\nu\mathrm{sech}(x),\quad S_0(x)=S_0(0)+\delta x+\mu\log(\cosh(x)),\tag{4.10}$$

where $S_0(0)$, δ and μ are real-valued parameters and $\nu \neq 0$. Therefore,

$$\rho_0(x) = \nu^2 \text{sech}^2(x), \quad u_0(x) = \delta + \mu \tanh(x). \tag{4.11}$$

Without loss of generality, let $\nu = 1$, $S_0(0) = 0$.

When $\nu = 1$, Q in the definition of (4.7) becomes ($t \neq 0$):

$$\begin{aligned} &Q(x) = \alpha^2 \text{sech}^2(x) + \alpha\delta + \alpha\mu \tanh(x) - 1 = -\alpha^2 T^2 + \alpha\mu T + \alpha^2 + \alpha\delta - 1, \\ &T = \tanh(x). \end{aligned} \tag{4.12}$$

In the transonic domain, $Q(x)$ will necessarily change sign. From (4.12), it is easy to see that $Q(x)$ is a quadratic equation of $T(x)$ and has two roots. For simplicity, suppose T has only one root for $T \in (-1, 1)$. For this, assume

$$\alpha|\mu| > |1 - \alpha\mu|, \tag{4.13}$$

$$\mu^2 > 4(1 - \alpha\delta) \tag{4.14}$$

and

$$\mu > 0. \tag{4.15}$$

According to inequalities (4.13)-(4.15), (4.12) has a unique solution in $(-1, 1)$,

$$T = T_c = 2B - \sqrt{4B^2 - 4A + 1}. \tag{4.16}$$

Consequently, when $t = 0$, $x < x_c$, $\alpha < 0$ and the domain is of elliptic, $\alpha > 0$, and the domain is of hyperbolic, $x > x_c = \text{arctanh}(T_c)$. Next we present our main result.

Theorem 4.1 *Let $\phi_\epsilon(x,t)$ be a solution of Cauchy problem to the MNLS equation* (4.1) *with the initial data* (4.3)*, where $A_0(x)$ and $S_0(x)$ are given by* (4.10)*. Then there exists a smooth curve $x = x_c(t)$, $t \geq 0$, $x_c(0) = x_c$, such that for all $t \geq 0$ and $x < x_c(t)$, there holds*

$$\phi_\epsilon(x,t) = A(x,t) \exp\Big(\frac{i}{\epsilon} S(x,t)\Big) + O(\epsilon), \quad \epsilon \to 0, \ \epsilon > 0. \tag{4.17}$$

Proof We use Riemann-Hilbert method to solve this problem. Assume the unknown matrix $M(k; x, t)$ admits a discontinuous matrix in the complex plane,

$$M(-k; x, t) = i^{\sigma_3} M(k; x, t) i^{-\sigma_3}, \quad M(-k^*; x, t)^* = \sigma_1 M(k; x, t) \sigma_1. \tag{4.18}$$

For instance, $\text{Im}(k^2) = 0$, $M_\pm(k; x, t)$ represent M with the boundary data from $\pm\text{Im}(k^2) < 0$.

$$M_+(k;x,t) = M_-(k;x,t) \exp\Big(\frac{i\theta(k^2;x,t)\sigma_3}{\epsilon}\Big) \begin{pmatrix} 1 \pm |r(k)|^2 & r(k) \\ \pm|r(k)|^2 & 1 \end{pmatrix} \exp\Big(\frac{-i\theta(k^2;x,t)\sigma_3}{\epsilon}\Big), \quad \pm k^2 > 0, \tag{4.19}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.20}$$

are the Pauli spin matrices.

$$a^{b\sigma_3} = \begin{pmatrix} a^b & 0 \\ 0 & a^{-b} \end{pmatrix}, \quad a, b \in \mathbb{C}, \tag{4.21}$$

$$\theta(z) = \theta(z; x, t) = -\frac{2}{\alpha}\Big(z - \frac{1}{4}\Big)x - \frac{4}{\alpha^2}\Big(z - \frac{1}{4}\Big)^2 t,$$

where $r(k) = -r(-k)$ is the reflection coefficient, and

$$\lim_{k\to\infty} M(k; x, t) = I. \tag{4.22}$$

Then the solution to the MNLS equation can be derived from $M(k; x, t)$

$$\phi_\epsilon(x, t) = \lim_{k\to\infty} \frac{2k}{\alpha} \frac{M_{12}(k; x, t)}{M_{22}(k; x, t)}. \tag{4.23}$$

We thus complete the proof.

Remark 4.1 Here, the error term holds uniformly in any compact subset of (x, t). $A(x, t)$ and $S(x, t)$ are smooth real-valued functions, independent of ϵ and satisfy $A(x, 0) = A_0(x)$ and $S(x, 0) = S_0(x)$. When $x < x_c(t)$, there holds $Q < 0$. And when $Q \to 0$, we have $x \to x_c(t)$, $x < x_c(t)$. At last, when $x < x_c(t)$ and $t > 0$,

$$\rho(x, t) = A(x, t)^2, \quad u(x, t) = \frac{\partial S}{\partial x}(x, t) \tag{4.24}$$

satisfy the MNLS equation (4.6).

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