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THREE KIRCHHOFFIAN INDICES OF THE CACTUS GRAPHS*†

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Abstract

In this paper we give six explicit formulae to compute the Kirchhoff index, the multiplicative degree-Kirchhoff index and the additive degree-Kirchhoff index of the k-cactus chain and the cactus graph which can be obtained from a k-cactus chain by expanding each of the cut-vertices to a cut edge.

Keywords polyphenyl chain; cactus graph; Kirchhoff index; multiplicative degree-Kirchhoff index; additive degree-Kirchhoff index

2000 Mathematics Subject Classification 05C12

1 Introduction

The objects nowadays known as cactus appeared in the scientific literature more than half a century ago. Motivated by papers of Husimi [28] and Riddell [41], [44] dealt with cluster integrals in the theory of condensation in statistical mechanics. Besides statistical mechanics, where cacti and their generalizations serve as simplified models of real lattices [36, 42], the concept has also found applications in the theory of electrical and communication networks [56] and in chemistry [25,55]. Many topological indices have been studied for these structures, including the matching and independence polynomials [4, 16], the Hosoya indices [1],π-electron energy [52], the Hosoya polynomials [32], and the subtree numbers [50].

A *cactus graph* G is a connected graph in which each edge lies on at most one cycle. Therefore, each block in G is either an edge or a cycle. A *k-cactus* is a cactus in which each block is a k-cycle. A *k-cactus chain* is a k-cactus in which each block contains at most two cut-vertices and each cut-vertex lies in exactly two blocks. The number of blocks in a k-cactus chain is the length of the chain. A 6-cactus chain is also called spiro hexagonal chain, and a polyphenyl chain is a cactus graph which

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can be obtained from a 6-cactus chain by expanding each of the cut-vertices to an cut edge. For example, see the first graph in Figure 1.

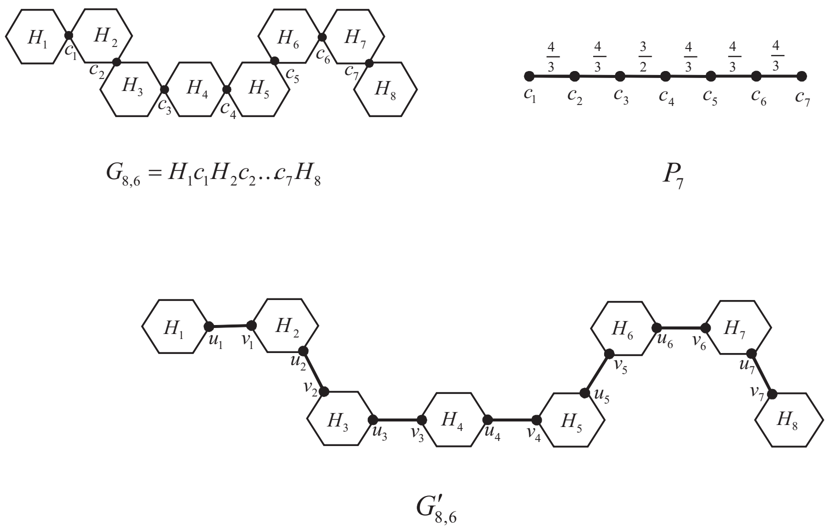

Figure 1: A spiro hexagonal chain, its corresponding weighted path and polyphenyl chain

Let G be a connected graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Based on the theory of electrical networks, Klein and Randić [30] introduced a new distance function named resistance distance. The resistance distance between a pair of vertices u and v in G, denoted by $r_G(u, v)$ or $r(u, v)$, is the effective resistance between them in the electrical network N constructed from G by replacing each edge with a unit resistor. This new intrinsic graph metric has being recognized as having more nice purely mathematical, chemical and physical interpretations [7, 12, 29–31].

Analogous to distance-based graph invariants, various graph invariants based on resistance distance have been defined and studied. Among these invariants, the most famous one is the Kirchhoff index [30], also known as the total effective resistance [21] or the effective graph resistance [18]. Like many topological indices, the Kirchhoff index is a structure descriptor and has been found very useful in purely mathematical, physical and chemical interpretations [30, 31, 54]. If the ordinary distance is replaced by the resistance distance in the expression for the Wiener index [47], one arrives at the Kirchhoff index [30].

Definition 1.1 The *Kirchhoff index* of a graph G is denoted by $Kf(G)$ and defined as follows:

$$Kf(G) = \sum_{\{u,v\} \subset V(G)} r_G(u, v).$$

Recently, two modifications of the Kirchhoff index, which takes the degrees of the graph into account, have been considered. One is the multiplicative degree-Kirchhoff index introduced by Chen and Zhang [11]. If the ordinary distance is replaced by resistance distance in the expression for the Gutman index [22], then one arrives at the multiplicative degree-Kirchhoff index.

Definition 1.2 The *multiplicative degree-Kirchhoff index* of a graph G is denoted by $Kf^*(G)$ and defined as follows:

$$Kf^*(G) = \sum_{\{u,v\}\subset V(G)} [\deg_G(u)\deg_G(v) r_G(u,v)],$$

where $\deg_G(u)$ is the degree of u in G.

The other one is the degree resistance distance introduced by Gutman et al. [24]. If the ordinary distance is replaced by resistance distance in the expression for the degree distance [15, 22], then one arrives at the degree resistance distance.

Definition 1.3 The *degree resistance distance* of a graph G is denoted by $Kf^+(G)$ and defined as follows:

$$Kf^+(G) = \sum_{u,v\subset V(G)} [(\deg_G(u) + \deg_G(v)) r_G(u,v)].$$

Palacios [38] named the same graph invariant "*additive degree-Kirchhoff index*".

The three Kirchhoffian indices have received much attention in recent years. Much work has been done to compute the three indices of some classes of graphs, or give some bounds for the three indices of graphs and characterize extremal graphs. In particular, Yang and Klein [49] gave formulae for the three indices of iterated subdivisions and triangulations of graphs. Huang et al. [27] gave some relations among the three indices of a connected graph and extended some results in [49]. In [17,34], the authors determined the first three minimum additive degree-Kirchhoff indices among all the cacti possessing fixed number of the vertices and cycles and characterized the corresponding extremal graphs. Deng et al. [14] obtained the explicit formulae to compute the Kirchhoff indices of spiro and polyphenyl hexagonal chains, determined the extremal values and characterized the extremal graph with respect to the Kirchhoff index among all spiro and polyphenyl hexagonal chains with h hexagons. Palacios [40] reviewed some known facts of the three indices, and found new relations among them. Motivated by the above results, in this paper we study the three Kirchhoffian indices of the k-cactus chain.

In this paper we obtain six explicit formulae to compute the Kirchhoff index, the multiplicative degree-Kirchhoff index and the additive degree-Kirchhoff index of the k-cactus chain and the cactus graph which can be obtained from a k-cactus chain by expanding each of the cut-vertices to a cut edge. Our results extend the main results in [14]. Moreover it reduces the problems on the three Kirchhoffian indices of the the graphs in the above two classes to the Winer index of the weighted-path, which make it trivial to determine the extremal problems on the three indices of the cactus graphs in the above two classes. The rest of the paper is organized as follows. We present our results in Section 2 and their proofs in Section 3.

2 Main Results

It is not difficult to see that beginning from a C_k, a k-cactus chain can be obtained by stepwise additions of a new terminal C_k. Denote by $G_{h,k} = H_1c_1H_2c_2\cdots c_{h-1}H_h$, a k-cactus chain with h k-cycles, where H_i is the ith k-cycle of $G_{h,k}$ and c_i is the common cut vertex of H_i and H_{i+1}, $i = 1, 2, \cdots, h-1$. Denote by $G'_{h,k}$ the corresponding cactus graph obtained by expanding each of the cut-vertices c_i of $G_{h,k}$ to a cut edge (u_i, v_i) with $u_i \in H_i$ and $v_i \in H_{i+1}$. For an example see Figure 1. Obviously, $G_{h,k}$ is determined completely by its cut vertex sequence $c_1, c_2, \cdots, c_{h-1}$.

For a k-cactus chain $G_{h,k} = H_1c_1H_2c_2\cdots c_{h-1}H_h$, we denote by (P_{h-1}, ω_r) the corresponding edge-weighted path $P_{h-1} = c_1c_2\cdots c_{h-1}$ with the weight function $\omega_r : E(P_{h-1}) \to \mathbb{R}^+$ such that $\omega_r(c_i, c_{i+1}) = r_{G_{h,k}}(c_i, c_{i+1})$ $(1 \le i \le h-1)$.

Recall that for an edge-weighted graph (G, ω), the length of a path in (G, ω) is the sum of the weights of all the edges in the path and the distance between u and v, and denote by $d_G(u, v)$ the minimum length of all the (u, v)-path. The Wiener index of an edge-weighted graph (G, ω) is defined as

$$W(G, \omega) = \sum_{\{u,v\}\subset V(G)} d_G(u, v).$$

Therefore, for a k-cactus chain $G_{h,k} = H_1c_1H_2c_2\cdots c_{h-1}H_h$,

$$W(P_{h-1}, \omega_r) = \sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G_{h,k}}(c_i, c_j) = \sum_{i=1}^{h-2}[i(h-1-i)r_{G_{h,k}}(c_i, c_{i+1})].$$

Theorem 2.1 *Let $G_{h,k}$ be a k-cactus chain with h k-cycles, $G'_{h,k}$ be the corresponding cactus graph obtained by expanding each of the cut-vertices c_i of $G_{h,k}$ to a cut edge and (P_{h-1}, ω_r) be the corresponding edge-weighted path. Then*

(1)

$$Kf(G_{h,k}) = (k-1)^2W(P_{h-1}, \omega_r) + \frac{1}{6}(k-1)(k^2-1)h^2 + \frac{1}{12}(2-k)(k^2-1)h,$$

(2)

$$Kf^+(G_{h,k}) = 4k(k-1)W(P_{h-1}, \omega_r) + \frac{1}{3}(2k-1)(k^2-1)h^2 - \frac{1}{3}(k-1)(k^2-1)h,$$

(3)
$$Kf^*(G_{h,k}) = 4k^2W(P_{h-1},\omega_r) + \frac{2}{3}(k^3-k)h^2 - \frac{1}{3}(k^3-k)h,$$
(4)
$$Kf(G'_{h,k}) = k^2W(P_{h-1},\omega_r) + \frac{1}{6}k^2h^3 + \frac{1}{6}(k^3-k)h^2 + \frac{1}{12}(-k^3-2k^2+k)h,$$
(5)
$$Kf^+(G'_{h,k}) = (4k^2+4k)W(P_{h-1},\omega_r) + \frac{2}{3}(k^2+k)h^3 + \frac{1}{3}(2k^3+k^2-5k-1)h^2 + \frac{1}{3}(-k^3-3k^2+2k+1)h,$$
(6)
$$Kf^*(G'_{h,k}) = 4(k+1)^2W(P_{h-1},\omega_r) + \frac{2}{3}(k+1)^2h^3 + \frac{2}{3}(k^3+k^2-4k-4)h^2 + \frac{1}{3}(-k^3+k^2-5k+9)h - 1,$$
which imply that

(7)
$$Kf(G'_{h,k}) = \frac{k^2}{(k-1)^2}Kf(G_{h,k}) + \frac{1}{6}k^2h^3 - \frac{1}{6}(k^2+k)h^2 + \frac{-2k^3+k^2-k}{12k-12}h,$$
(8)
$$Kf^+(G'_{h,k}) = \frac{k+1}{k-1}Kf^+(G_{h,k}) + \frac{2}{3}(k^2+k)h^3 - \frac{1}{3}(2k^2+5k)h^2 - \frac{1}{3}(2k^2-k)h,$$
(9)
$$Kf^*(G'_{h,k}) = \frac{(k+1)^2}{k^2}Kf^*(G_{h,k}) + \frac{2}{3}(k+1)^2h^3 + \frac{2}{3k}(k^3+4k^2+2k-1)h^2 - \frac{1}{3k}(2k^4-5k^3+15k^2-25k+1)h] - 1.$$

Let $k=6$. Then we have the following corollary.

Corollary 2.1 *Suppose that PPC_h is a polyphenyl chain with h ($h \geq 2$) hexagons, SPC_h is the corresponding 6-cactus chain obtained from PPC_h by squeezing off its cut edges and (P_{h-1},ω_r) is the corresponding edge-weighted path to SPC_h. Then*

(1)[14] $Kf(SPC_h) = 25W(P_{h-1},\omega_r) + \frac{175}{6}h^2 - \frac{35}{3}h,$

(2) $Kf^+(SPC_h) = 120W(P_{h-1}, \omega_r) + \frac{385}{3}h^2 - \frac{175}{3}h,$

(3) $Kf^*(SPC_h) = 144W(P_{h-1}, \omega_r) + 140h^2 - 70h,$

(4)[14] $Kf(PPC_h) = 36W(P_{h-1}, \omega_r) + 6h^3 + 35h^2 - \frac{47}{2}h,$

(5) $Kf^+(PPC_h) = 168W(P_{h-1}, \omega_r) + 28h^3 + \frac{437}{3}h^2 - \frac{311}{3}h,$

(6) $Kf^*(PPC_h) = 196W(P_{h-1}, \omega_r) + \frac{98}{3}h^3 + \frac{448}{3}h^2 - 67h - 1.$

Our main results reduce the problems on the Kirchhoff index (or multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index) of the k-cactus chain and the corresponding cactus graph, which can be obtained from a k-cactus graph by expanding each of the cut-vertices to a cut edge to the corresponding problems on the Wiener index of the weighted path, in which the weight of each edge is in the set $\{r_{C_k}(u,v)|u,v \in V(C_k)\} = \{\frac{i(k-i)}{k},\ 1 \leq i \leq \lfloor k/2 \rfloor\}$.

For example, if $k = 6$, then extremal problems on the polyphenyl and spiro chains with h hexagons are determined on the extremal problems on the Wiener of the weighted path P_{h-1}, in which the weight of each edge is $\frac{5}{6}$, $\frac{4}{3}$ or $\frac{3}{2}$.

Corollary 2.2 *Suppose $G_{h,k} = H_1c_1H_2c_2 \cdots c_{h-1}H_h$ to be a k-cactus chain with h k-cycles. Then the Kirchhoff index (or multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index) of $G_{h,k}$ receives the maximum value when*

$$r_{G_{h,k}}(c_i, c_{i+1}) = \frac{\lfloor k/2 \rfloor (k - \lfloor k/2 \rfloor)}{k} \quad for\ all\ 1 \leq h-1,$$

and the minimum value when

$$r_{G_{h,k}}(c_i, c_{i+1}) = \frac{k-1}{k} \quad for\ all\ 1 \leq h-1.$$

Remark 2.1 From the proof of Theorem 2.1, it can be found that, the above results also hold if we replace "$G_{h,k}$ a cactus chain" by "$G_{h,k}$ is a k-cactus graph, in which any cut vertex is at most in two k-cycles".

3 The Proof of Main Result

The resistance distance of a vertex $v \in V(G)$, denoted by $R(v|G)$, is the sum of distances between v and all other vertices of G, that is, $R(v|G) = \sum_{u \in V(G)} r_G(u,v)$. The degree resistance distance of a vertex $v \in V(G)$ is denoted by $R_D(v|G) = \sum_{u \in V(G)} \deg_G(u) r_G(u,v)$. Gutman et al. [24] presented the following explicit formulae for the cycles.

Lemma 3.1[24] *Let C_k be the cycle with k vertices, and $v \in V(C_k)$. Then*

$$R(v|C_k) = \frac{1}{6}(k^2-1), \quad R_D(v|C_k) = \frac{1}{3}(k^2-1),$$
$$Kf(C_k) = \frac{1}{12}(k^3-k), \quad Kf^*(C_k) = Kf^+(C_k) = \frac{1}{3}(k^3-k).$$

Lemma 3.2[46] *Suppose that G_1 and G_2 are two connected graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$ $(i = 1, 2)$. If we identify any vertex, say x_1, of G_1 with any other vertex, say x_2, of G_2 as a new common vertex x, and obtain a new graph G, then*

(1) $r_G(a, b) = r_{G_1}(a, x_1) + r_{G_2}(x_2, b),\ for\ any\ a \in V(G_1),\ b \in V(G_2)$,

(2) $Kf(G) = Kf(G_1) + Kf(G_2) + (n_1-1)R(x_2|G_2) + (n_2-1)R(x_1|G_1)$,

(3) $Kf^+(G) = Kf^+(G_1) + Kf^+(G_2) + (n_2-1)R_D(x_1|G_1) + (n_1-1)R_D(x_2|G_2)$
$+2m_2R(x_1|G_1) + 2m_1R(x_2|G_2)$.

Lemma 3.3 *Suppose that G_1 and G_2 are two connected graphs with $|V(G_i)| = n_i$, and $|E(G_i)| = m_i$ $(i = 1, 2)$. If we identify any vertex, say x_1, of G_1 with any other vertex, say x_2, of G_2 as a new common vertex x, and obtain a new graph G, then*

$$Kf^*(G) = Kf^*(G_1) + Kf^*(G_2) + 2m_1R_D(x_2|G_2) + 2m_2R_D(x_1|G_1).$$

Proof By the definition of the multiplicative degree-Kirchhoff index of a graph G, we have

$$\begin{aligned}
Kf^*(G) = &\sum_{\{y,z\}\subset V(G_1)} [\deg_G(y)\deg_G(z)r_G(y,z)] \\
&+ \sum_{\{y,z\}\subset V(G_2)} [\deg_G(y)\deg_G(z)r_G(y,z)] \\
&+ \sum_{y\in V(G_1)-\{x\}}\ \sum_{z\in V(G_2)-\{x\}} [\deg_G(y)\deg_G(z)r_G(y,z)] \\
= &Kf^*(G_1) + \sum_{y\in V(G_1)} [\deg_G(y)\deg_{G_2}(x)r_G(y,x)] \\
&+Kf^*(G_2) + \sum_{z\in V(G_2)} [\deg_G(z)\deg_{G_1}(x)r_G(z,x)] \\
&+ \sum_{y\in V(G_1)-\{x\}}\ \sum_{z\in V(G_2)-\{x\}} [(\deg_G(y)\deg_G(z)(r_G(y,x)+r_G(x,z))] \\
= &Kf^*(G_1) + \deg_{G_2}(x)\sum_{y\in V(G_1)} [\deg_G(y)r_G(y,x)] \\
&+Kf^*(G_2) + \deg_{G_1}(x)\sum_{z\in V(G_2)} [\deg_G(z)r_G(z,x)]
\end{aligned}$$

$$
\begin{aligned}
&+\left[\sum_{z\in V(G_2)-\{x\}} \deg_G(z)\right] \sum_{y\in V(G_1)-\{x\}} [\deg_G(y) r_G(y,x)] \\
&+\left[\sum_{y\in V(G_1)-\{x\}} \deg_G(y)\right] \sum_{z\in V(G_2)-\{x\}} [\deg_G(z) r_G(x,z)] \\
=\ & Kf^*(G_1)+Kf^*(G_2)+2m_1R_D(x_2|G_2)+2m_2R_D(x_1|G_1).
\end{aligned}
$$

The proof is completed.

Lemma 3.4 *Let $e=(x_1,x_2)$ be a cut edge of G. Suppose that G_1 and G_2 are two connected components of $G-e$ with $x_i\in V(G_i)$, $|V(G_i)|=n_i$ and $|E(G_i)|=m_i$ $(i=1,2)$. Then*

(1) $Kf(G)=Kf(G_1)+Kf(G_2)+n_2R(x_1|G_1)+n_1R(x_2|G_2)+n_1n_2$,

(2) $Kf^+(G)=Kf^+(G_1)+Kf^+(G_2)+n_2R_D(x_1|G_1)+n_1R_D(x_2|G_2)$
$+(2m_2+2)R(x_1|G_1)+(2m_1+2)R(x_2|G_2)$
$+n_1(2m_2+1)+n_2(2m_1+1)$,

(3) $Kf^*(G)=Kf^*(G_1)+Kf^*(G_2)+(2m_2+2)R_D(x_1|G_1)+(2m_1+2)R_D(x_2|G_2)$
$+(2m_1+1)(2m_2+1)$.

Proof By the definition of the Kirchhoff index and Lemma 3.2, we have

$$
\begin{aligned}
Kf(G) &= Kf(G_1)+Kf(G_2)+\sum_{y\in V(G_1),z\in V(G_2)} r_G(y,z) \\
&= Kf(G_1)+Kf(G_2)+\sum_{y\in V(G_1),z\in V(G_2)} [r_{G_1}(y,x_1)+r_{G_2}(x_2,z)+1] \\
&= Kf(G_1)+Kf(G_2)+n_2R(x_1|G_1)+n_1R(x_2|G_2)+n_1n_2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Kf^*(G) =\ & \sum_{\{y,z\}\subset V(G_1)} [\deg_G(y)\deg_G(z) r_G(y,z)] \\
&+\sum_{\{y,z\}\subset V(G_2)} [\deg_G(y)\deg_G(z) r_G(y,z)] \\
&+\sum_{y\in V(G_1),z\in V(G_2)} [\deg_G(y)\deg_G(z) r_G(y,z)] \\
=\ & Kf^*(G_1)+\sum_{y\in V(G_1)} [\deg_G(y) r_G(y,x_1)] \\
&+Kf^*(G_2)+\sum_{z\in V(G_2)} [\deg_G(z) r_G(z,x_2)] \\
&+\sum_{y\in V(G_1)}\sum_{z\in V(G_2)} [\deg_G(y)\deg_G(z)(r_G(y,x_1)+r_G(x_2,z)+1)]
\end{aligned}
$$

$$
\begin{aligned}
&= Kf^*(G_1)+R_D(x_1|G_1)+Kf^*(G_2)+R_D(x_2|G_2)\\
&\quad +R_D(x_1|G_1)\sum_{z\in V(G_2)}\deg_G(z)+R_D(x_2|G_2)\sum_{y\in V(G_1)}\deg_G(y)\\
&\quad +\Bigg[\sum_{y\in V(G_1)}\deg_G(y)\Bigg]\Bigg[\sum_{z\in V(G_2)}\deg_G(z)\Bigg]\\
&= Kf^*(G_1)+Kf^*(G_2)+(2m_2+2)R_D(x_1|G_1)\\
&\quad +(2m_1+2)R_D(x_2|G_2)+(2m_1+1)(2m_2+1),
\end{aligned}
$$

$$
\begin{aligned}
Kf^+(G) &= \sum_{\{y,z\}\subset V(G_1)}[(\deg_G(y)+\deg_G(z))r_G(y,z)]\\
&\quad +\sum_{\{y,z\}\subset V(G_2)}[(\deg_G(y)+\deg_G(z))r_G(y,z)]\\
&\quad +\sum_{y\in V(G_1),z\in V(G_2)}[(\deg_G(y)+\deg_G(z))r_G(y,z)]\\
&= Kf^+(G_1)+\sum_{y\in V(G_1)}r_G(y,x_1)+Kf^+(G_2)+\sum_{z\in V(G_2)}r_G(z,x_2)\\
&\quad +\sum_{y\in V(G_1),z\in V(G_2)}[(\deg_G(y)+\deg_G(z))(r_G(y,x_1)+r_G(x_2,z)+1)]\\
&= Kf^+(G_1)+\sum_{y\in V(G_1)}r_G(y,x_1)+Kf^+(G_2)+\sum_{z\in V(G_2)}r_G(z,x_2)\\
&\quad +n_2\sum_{y\in V(G_1)}[\deg_G(y)(r_G(y,x_1)]+\Bigg[\sum_{y\in V(G_1)}\deg_G(y)\Bigg]\Bigg[\sum_{z\in V(G_2)}r_G(x_2,z)\Bigg]\\
&\quad +\Bigg[\sum_{z\in V(G_2)}\deg_G(z)\Bigg]\Bigg[\sum_{y\in V(G_1)}r_G(y,x_1)\Bigg]+n_1\sum_{z\in V(G_2)}[\deg_G(z))r_G(x_2,z)]\\
&\quad +n_2\sum_{y\in V(G_1)}\deg_G(y)+n_1\sum_{z\in V(G_2)}\deg_G(z)\\
&= Kf^+(G_1)+Kf^+(G_2)+n_2R_D(x_1|G_1)+n_1R_D(x_2|G_2)\\
&\quad +(2m_2+2)R(x_1|G_1)+(2m_1+2)R(x_2|G_2)\\
&\quad +n_2(2m_1+1)+n_1(2m_2+1).
\end{aligned}
$$

The proof is completed.

Lemma 3.5 *Let $G_{h,k}=H_1c_1H_2c_2\cdots c_{h-1}H_h$ be a k-cactus chain with h k-cycles and $G'_{h,k}$ be the corresponding cactus graph obtained by expanding each of the cut-vertices c_i of $G_{h,k}$ to a cut edge (u_i,v_i) with $u_i\in H_i$ and $v_i\in H_{i+1}$, as showed* *in Figure* 1*. Denote by* $G_{h-1,k} = H_1c_1H_2c_2\cdots c_{h-2}H_{h-1}$ *the corresponding subgraph of* $G_{h,k}$*. Then*

$$R(c_{h-1}|G_{h-1,k}) = (k-1)\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{6}(k^2-1)(h-1),$$

$$R_D(c_{h-1}|G_{h-1,k}) = 2k\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{3}(k^2-1)(h-1),$$

$$R(u_{h-1}|G'_{h-1,k}) = k\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j, u_{h-1}) + \frac{1}{6}(k^2-1)(h-1),$$

$$R_D(u_{h-1}|G'_{h-1,k}) = 2(k+1)\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j, u_{h-1}) + \frac{1}{3}(k^2-4)(h-1)+1.$$

Proof By Lemma 3.1 $R(c_{h-1}|H_{h-1}) = \frac{1}{6}(k^2-1)$, which implies that in $G_{h,k}$ the sum of the resistance distances between c_{h-1} and all the vertices in $V(H_j)-\{c_j\}$ is

$$(k-1)r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{6}(k^2-1), \quad j = 1, 2, \cdots, h-1.$$

Then

$$\begin{aligned} R(c_{h-1}|G_{h-1,k}) &= \sum_{j=1}^{h-1}\Big[(k-1)r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{6}(k^2-1)\Big] \\ &= (k-1)\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{6}(k^2-1)(h-1), \\ R_D(c_{h-1}|G_{h-1,k}) &= 2R(c_{h-1}|G_{h-1,k}) + 2\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) \\ &= 2k\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{3}(k^2-1)(h-1). \end{aligned}$$

Similarly,

$$\begin{aligned} R(u_{h-1}|G'_{h-1,k}) &= \sum_{j=1}^{h-1}\Big[kr_{G'_{h,k}}(u_j, u_{h-1}) + \frac{1}{6}(k^2-1)\Big] \\ &= k\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j, u_{h-1}) + \frac{1}{6}(k^2-1)(h-1), \end{aligned}$$

$$\begin{aligned}R_D(u_{h-1}|G'_{h-1,k}) &= 2R(u_{h-1}|G'_{h-1,k}) + \sum_{j=1}^{h-2}[r_{G'_{h,k}}(u_j,u_{h-1}) + r_{G'_{h,k}}(v_j,u_{h-1})]\\ &= 2R(u_{h-1}|G'_{h-1,k}) + \sum_{j=1}^{h-2}[2r_{G'_{h,k}}(u_j,u_{h-1}) - 1]\\ &= 2(k+1)\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j,u_{h-1}) + \frac{1}{3}(k^2-4)(h-1)+1.\end{aligned}$$

The proof is completed.

Proof of Theorem 2.1 Suppose $G_{h,k} = H_1c_1H_2c_2\cdots c_{h-1}H_h$. Define by $G_{j,k} = H_1c_1H_2c_2\cdots c_{j-1}H_j$ $(1 \leq j \leq h-1)$ the corresponding subgraphs of $G_{h,k}$. Obviously,

$$\begin{aligned}&|V(G_{h-1,k})| = (k-1)(h-1)+1, \quad |E(G_{h-1,k})| = k(h-1),\\ &|V(G'_{h-1,k})| = k(h-1), \quad |E(G'_{h-1,k})| = (k+1)(h-1)-1.\end{aligned}$$

Recall that $R(v|C_k) = \frac{1}{6}(k^2-1)$ for any vertex $v \in V(C_k)$, $Kf(C_k) = \frac{1}{12}(k^3-k)$ and by Lemma 3.5

$$R(c_{h-1}|G_{h-1,k}) = (k-1)\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j,c_{h-1}) + \frac{1}{6}(k^2-1)(h-1).$$

Then by Lemma 3.2,

$$\begin{aligned}Kf(G_{h,k}) &= Kf(G_{h-1,k}) + Kf(H_h) + (k-1)R(c_{h-1}|G_{h-1,k})\\ &\quad +(k-1)(h-1)R(c_{h-1}|H_h)\\ &= Kf(G_{h-1,k}) + (k-1)\left[(k-1)\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j,c_{h-1}) + \frac{1}{6}(k^2-1)(h-1)\right]\\ &\quad +\frac{1}{12}(k^3-k) + \frac{1}{6}(k-1)(k^2-1)(h-1)\\ &= Kf(G_{h-1,k}) + (k-1)^2\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j,c_{h-1}) + \frac{1}{3}(k-1)(k^2-1)(h-1)\\ &\quad +\frac{1}{12}(k^3-k)\end{aligned}$$

$$
\begin{aligned}
&= Kf(G_{1,k}) + \sum_{i=1}^{h-1}\left[(k-1)^2\sum_{j=1}^{i} r_{G_{h,k}}(c_j,c_i) + \frac{1}{3}(k-1)(k^2-1)i + \frac{1}{12}(k^3-k)\right]\\
&= (k-1)^2\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G_{h,k}}(c_j,c_i) + \frac{1}{6}(k-1)(k^2-1)(h^2-h) + \frac{1}{12}(k^3-k)h\\
&= (k-1)^2 W(P_{h-1},\omega_r) + \frac{1}{6}(k-1)(k^2-1)h^2 + \frac{1}{12}(2-k)(k^2-1)h.
\end{aligned}
$$

By the same way

$$
\begin{aligned}
Kf^+(G_{h,k}) &= Kf^+(G_{h-1,k}) + Kf^+(C_k) + (k-1)R_D(c_{h-1}|G_{h-1,k})\\
&\quad +(k-1)(h-1)R_D(c_{h-1}|H_h) + 2kR(c_{h-1}|G_{h-1,k})\\
&\quad +2k(h-1)R(c_{h-1}|H_h)\\
&= Kf^+(G_{h-1,k}) + \frac{1}{3}(k^3-k) + \frac{1}{3}(k-1)(k^2-1)(h-1)\\
&\quad +\frac{1}{3}k(k^2-1)(h-1)\\
&\quad +(k-1)\left[2k\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j,c_{h-1}) + \frac{1}{3}(k^2-1)(h-1)\right]\\
&\quad +2k\left[(k-1)\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j,c_{h-1}) + \frac{1}{6}(k^2-1)(h-1)\right]\\
&= Kf^+(G_{h-1,k}) + 4k(k-1)\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j,c_{h-1})\\
&\quad +\frac{1}{3}(4k-2)(k^2-1)(h-1) + \frac{1}{3}(k^3-k)\\
&= Kf^+(G_{1,k})\\
&\quad +\sum_{i=1}^{h-1}\left[4k(k-1)\sum_{j=1}^{i} r_{G_{h,k}}(c_j,c_i) + \frac{1}{3}(4k-2)(k^2-1)i + \frac{1}{3}(k^3-k)\right]\\
&= 4k(k-1)\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G_{h,k}}(c_j,c_i) + \frac{1}{3}(2k-1)(k^2-1)h^2\\
&\quad -\frac{1}{3}(k-1)(k^2-1)h\\
&= 4k(k-1)W(P_{h-1},\omega_r) + \frac{1}{3}(2k-1)(k^2-1)h^2 - \frac{1}{3}(k-1)(k^2-1)h,
\end{aligned}
$$

$$\begin{aligned}
Kf^*(G_{h,k}) &= Kf^*(G_{h-1,k}) + Kf^*(C_k) + 2kR_D(c_{h-1}|G_{h-1,k}) \\
&\quad +2k(h-1)R_D(c_{h-1}|H_h) \\
&= Kf^*(G_{h-1,k}) + \frac{1}{3}(k^3-k) \\
&\quad +2k\left[2k\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) + \frac{1}{3}(k^2-1)(h-1)\right] + \frac{2}{3}k(k^2-1)(h-1) \\
&= Kf^*(G_{h-1,k}) + 4k^2\sum_{j=1}^{h-1} r_{G_{h,k}}(c_j, c_{h-1}) + \frac{4}{3}(k^3-k)(h-1) + \frac{1}{3}(k^3-k) \\
&= Kf^*(G_{1,k}) + \sum_{i=1}^{h-1}\left[4k^2\sum_{j=1}^{i} r_{G_{h,k}}(c_j, c_i) + \frac{4}{3}(k^3-k)i + \frac{1}{3}(k^3-k)\right] \\
&= 4k^2\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G_{h,k}}(c_j, c_i) + \frac{2}{3}(k^3-k)h^2 - \frac{1}{3}(k^3-k)h \\
&= 4k^2 W(P_{h-1}, \omega_r) + \frac{2}{3}(k^3-k)h^2 - \frac{1}{3}(k^3-k)h.
\end{aligned}$$

Recall that $|V(G'_{h-1,k})| = k(h-1)$ and $|E(G'_{h-1,k})| = (k+1)(h-1)-1$, by Lemmas 3.4 and 3.5, then we obtain

$$\begin{aligned}
Kf(G'_{h,k}) &= Kf(G'_{h-1,k}) + Kf(H_h) + kR(u_{h-1}|G'_{h,k}) \\
&\quad +k(h-1)R(v_{h-1}|H_h) + k^2(h-1) \\
&= Kf(G'_{h-1,k}) + \frac{1}{12}(k^3-k) + k\left[k\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j, u_{h-1}) + \frac{1}{6}(k^2-1)(h-1)\right] \\
&\quad +\frac{1}{6}k(k^2-1)(h-1) + k^2(h-1) \\
&= Kf(G'_{h-1,k}) + k^2\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j, u_{h-1}) \\
&\quad +\frac{1}{3}(k^3+3k^2-k)(h-1) + \frac{1}{12}(k^3-k) \\
&= Kf(G'_{1,k}) + \sum_{i=1}^{h-1}\left[k^2\sum_{j=1}^{i} r_{G'_{h,k}}(u_j, u_i) + \frac{1}{3}(k^3+3k^2-k)i + \frac{1}{12}(k^3-k)\right] \\
&= k^2\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G'_{h,k}}(u_j, u_i) + \frac{1}{6}(k^3+3k^2-k)h^2 + \frac{1}{12}(-k^3-6k^2+k)h.
\end{aligned}$$

It is not difficult to see that $r_{G'_{h,k}}(u_j, u_i) = r_{G_{h,k}}(c_j, c_i) + (i-j)$. So

$$\begin{aligned}\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G'_{h,k}}(u_j,u_i) &= \sum_{i=1}^{h-1}\sum_{j=1}^{i}[r_{G_{h,k}}(c_j,c_i)+(i-j)]\\ &= \sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G_{h,k}}(c_j,c_i)+\frac{1}{6}h(h-1)(h-2)\\ &= W(P_{h-1},\omega_r)+\frac{1}{6}h(h-1)(h-2).\end{aligned}$$

Then

$$\begin{aligned}Kf(G'_{h,k}) &= k^2\Big[W(P_{h-1},\omega_r)+\frac{1}{6}h(h-1)(h-2)\Big]\\ &\quad+\frac{1}{6}(k^3+3k^2-k)h^2+\frac{1}{12}(-k^3-6k^2+k)h\\ &= k^2W(P_{h-1},\omega_r)+\frac{1}{6}k^2h^3+\frac{1}{6}(k^3-k)h^2+\frac{1}{12}(-k^3-2k^2+k)h.\end{aligned}$$

Similarly, by Lemmas 3.4 and 3.5

$$\begin{aligned}Kf^+(G'_{h,k}) &= Kf^+(G'_{h-1,k})+Kf^+(H_h)+kR_D(u_{h-1}|G'_{h-1,k})\\ &\quad+k(h-1)R_D(v_{h-1}|H_h)+(2k+2)R(u_{h-1}|G'_{h-1,k})\\ &\quad+(2k+2)(h-1)R(v_{h-1}|H_h)+k(h-1)(2k+1)\\ &\quad+k[(2k+2)(h-1)-1]\\ &= Kf^+(G'_{h-1,k})+\frac{1}{3}(k^3-k)\\ &\quad+k\left[(2k+2)\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j,u_{h-1})+\frac{1}{3}(k^2-4)(h-1)+1\right]\\ &\quad+\frac{1}{3}k(k^2-1)(h-1)+(2k+2)\Bigg[k\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j,u_{h-1})\\ &\quad+\frac{1}{6}(k^2-1)(h-1)\Bigg]+\frac{1}{3}(k+1)(k^2-1)(h-1)\\ &\quad+k(h-1)(2k+1)+k[(2k+2)(h-1)-1]\\ &= Kf^+(G'_{h-1,k})+(4k^2+4k)\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j,u_{h-1})\\ &\quad+\frac{2}{3}(2k^3+7k^2+k-1)(h-1)+\frac{1}{3}(k^3-k)\end{aligned}$$

$$
\begin{aligned}
&= Kf^{+}(G'_{1,k}) + \sum_{i=1}^{h-1}\Bigg[(4k^2+4k)\sum_{j=1}^{i} r_{G'_{h,k}}(u_j,u_i) \\
&\quad + \frac{2}{3}(2k^3+7k^2+k-1)i + \frac{1}{3}(k^3-k)\Bigg] \\
&= (4k^2+4k)\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G'_{h,k}}(u_j,u_i) + \frac{1}{3}(2k^3+7k^2+k-1)h^2 \\
&\quad + \frac{1}{3}(-k^3-7k^2-2k+1)h \\
&= (4k^2+4k)\Big[W(P_{h-1},\omega_r) + \frac{1}{6}h(h-1)(h-2)\Big] \\
&\quad + \frac{1}{3}(2k^3+7k^2+k-1)h^2 + \frac{1}{3}(-k^3-7k^2-2k+1)h \\
&= (4k^2+4k)W(P_{h-1},\omega_r) + \frac{2}{3}(k^2+k)h^3 + \frac{1}{3}(2k^3+k^2-5k-1)h^2 \\
&\quad + \frac{1}{3}(-k^3-3k^2+2k+1)h, \\
Kf^{*}(G'_{h,k}) &= Kf^{*}(G'_{h-1,k}) + Kf^{*}(H_h) + (2k+2)R_D(u_{h-1}|G'_{h-1,k}) \\
&\quad + (2k+2)(h-1)R_D(v_{h-1}|H_h) + (2k+1)[(2k+2)(h-1)-1] \\
&= Kf^{*}(G'_{h-1,k}) + \frac{1}{3}(k^3-k) \\
&\quad + (2k+2)\Bigg[2(k+1)\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j,u_{h-1}) + \frac{(k^2-4)(h-1)}{3} + 1\Bigg] \\
&\quad + \frac{1}{3}(2k+2)(h-1)(k^2-1) + (2k+1)[(2k+2)(h-1)-1] \\
&= Kf^{*}(G'_{h-1,k}) + 4(k+1)^2\sum_{j=1}^{h-1} r_{G'_{h,k}}(u_j,u_{h-1}) \\
&\quad + \frac{4}{3}(k^3+4k^2+2k-1)(h-1) + \frac{1}{3}(k^3-k) + 1 \\
&= Kf^{*}(G'_{1,k}) + \sum_{i=1}^{h-1}\Bigg[4(k+1)^2\sum_{j=1}^{i} r_{G'_{h,k}}(u_j,u_i) \\
&\quad + \frac{4}{3}(k^3+4k^2+2k-1)i + \frac{1}{3}(k^3-k) + 1\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= 4(k+1)^2\sum_{i=1}^{h-1}\sum_{j=1}^{i} r_{G'_{h,k}}(u_j,u_i) + \frac{2}{3}(k^3+4k^2+2k-1)h^2 \\
&\quad + \frac{1}{3}(-k^3-8k^2-5k+5)h-1 \\
&= 4(k+1)^2\Big[W(P_{h-1},\omega_r)+\frac{1}{6}h(h-1)(h-2)\Big] + \frac{2}{3}(k^3+4k^2+2k-1)h^2 \\
&\quad + \frac{1}{3}(-k^3-8k^2-5k+5)h-1 \\
&= 4(k+1)^2W(P_{h-1},\omega_r) + \frac{2}{3}(k+1)^2h^3 + \frac{2}{3}(k^3+k^2-4k-4)h^2 \\
&\quad + \frac{1}{3}(-k^3+k^2-5k+9)h-1.
\end{aligned}
$$

The proof is completed.

References

[1] Y. Bai, B. Zhao, P. Zhao, Extremal Merriffield-Simmons index and Hosoya index of polyphenyl chains, *MATCH Commun. Math. Comput. Chem.*, **62**(2009),649-656.

[2] E. Bendito, A. Carmona, A.M. Encinas, J.M. Gesto, A formula for the Kirchhoff index, *Int. J. Quantum Chem.*, **108**:6(2008),1200-1206.

[3] H. Bian, F. Zhang, Tree-like polyphenyl systems with extremal Wiener indices, *MATCH Commun. Math. Comput. Chem.*, **61**(2009),631-642.

[4] H. Bian, F.J. Zhang, G.P. Wang, H.Z. Yu, Matching polynomials for chains of cycles, *Discrete. Math.*, **311**(2011),319-323.

[5] M. Bianchi, A. Cornaro, J.L. Palacios, A. Torriero, Bounds for the Kirkhhoff index via majorization techniques, *J. Math. Chem.*, **51**:2(2013),569-587.

[6] M. Bianchi, A. Cornaro, J.L. Palacios, A. Torriero, New upper and lower bounds for the additive degree-Kirchhoff index, *Croat. Chem. Acta*, **86**(2013),363-370.

[7] D. Bonchev, E.J. Markel, A.H. Dekmezian, Long chain branch polymer chain dimensions: Application of topology to the Zimm-Stockmayer model, *Polymer*, **43**(2002),203-222.

[8] N. Boumal, X. Cheng, Concentration of the Kirchhoff index for Erdös-Rényi graphs, *Syst. Control. Lett.*, **74**(2014),74-80.

[9] C. Bu, B. Yan, X. Zhou, J. Zhou, Resistance distance in subdivision-vertex join and subdivision-edge join of graphs, *Linear Algebra Appl.*, **458**(2014),454-462.

[10] A. Chen, X. Xiong, F. Lin, Distance-based topological indices of the tree-like polyphenyl systems, *Appl. Math. Comput.*, **281**(2016),233-242.

[11] H. Chen, F. Zhang, Resistance distance and the normalized Laplacian spectrum, *Discr. Appl. Math.*, **155**(2007),654-661.

[12] H. Chen, F. Zhang, Random walks and the effective resistance sum rules, *Discrete Appl. Math.*, **158**(2010),1691-1700.

[13] H. Deng, Wiener indices of spiro and polyphenyl hexagonal chains, *Math. Comput. Model.*, **55**:3(2012),634-644.

[14] H. Deng, Z. Tang, Kirchhoff indices of spiro and polyphenyl hexagonal chains, *Util. Math.*, **95**(2014),113-128.

[15] A.A. Dobrynin, A.A. Degree distance of a graph: a degree analogue of the Wiener index, *J. Chem. Inf. Comput. Sci.*, **34**(1994),1082-1086.

[16] T. Došlić, F. Måløy, Chain hexagonal cacti: matchings and independent sets, *Discrete Math.*, **310**:12(2010),1676-1690.

[17] J. Du, G. Su, J. Tu, I. Gutman, The degree resistance distance of cacti, *Discrete Appl. Math.*, **188**(2015),16-24.

[18] W. Ellens, F.M. Spieksma, P. Van Mieghem, A. Jamakovic, R.E. Kooij, Effective graph resistance, *Linear Algebra Appl.*, **435**(2011),2491-2506.

[19] L.H. Feng, I. Gutman, G. Yu, Degree Kirchhoff index of unicyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **69**(2013),629-648.

[20] L.H. Feng, G. Yu, On the hyper-Wiener index of cacti, *Util. Math.*, **93**(2014),57-64.

[21] A. Ghosh, S. Boyd, A. Saberi, Minimizing effective resistance of a graph, *SIAM Rev.*, **50**:1(2008),37-66.

[22] I. Gutman, Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.*, **34**(1994),1087-1089.

[23] I. Gutman, B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, *J. Chem. Inf. Comput. Sci.*, **36**(1996),982-985.

[24] I. Gutman, L. Feng, G. Yu, On the degree resistance distance of unicyclic graphs, *Trans. Comb.*, **1**:2(2012),27-40.

[25] H. Hosoya, K. Balasubramanian, Exact dimer statistics and characteristic polynomials of cacti lattices, *Theor. Chem. Acc.*, **76**(1989),315-329.

[26] G. Huang, M. Kuang, H. Deng, The expected values of Kirchhoff indices in the random polyphenyl and spiro chains, *Ars. Math. Contemp.*, **9**(2015),197-207.

[27] S. Huang, J. Zhou, C. Bu, Some Results on Kirchhoff Index and Degree-Kirchhoff Index, *MATCH Commun. Math. Comput. Chem.*, **75**(2016),207-222.

[28] K. Husimi, Note on Mayers theory of cluster integrals, *J. Chem. Phys.*, **18**(1950),682-684.

[29] J. Kigami, Harmonic analysis for resistance forms, *J. Funct. Anal.*, **204**(2003),399-444.

[30] D.J. Klein, M. Randić, Resistance distance, *J. Math. Chem.*, **12**(1993),81-95.

[31] D.J. Klein, Graph geometry, graph metrics and Wiener, *MATCH Commun. Math. Comput. Chem.*, **35**(1997),7-27.

[32] X. Li, G. Wang, H. Bian, R. Hu, The Hosoya polynomial decomposition for polyphenyl chains, *MATCH Commun. Math. Comput. Chem.*, **67**:2(2012),357-368.

[33] H. Liu, M. Lu, A unified approach to extremal cacti for different indices, *MATCH Commun. Math. Comput. Chem.*, **58**(2007),193-204.

[34] J.B. Liu, W.R. Wang, Y.M. Zhang, X.F. Pan, On degree resistance distance of cacti, *Discrete Appl. Math.*, **203**(2015),217-225.

[35] J.B. Liu, X.F. Pan, The maximum degree resistance distance of cacti, *J. Am. Chem. Soc.*, **93**:24(2015).

[36] J.L. Monroe, The bilayer ising model and a generalized Husimi tree approximation, *Physica A*, **335**(2004),563-576.

[37] J.L. Palacios, J.M. Renom, Another look at the degree-Kirchhoff index, *Int. J. Quantum Chem.*, **111**:14(2011),3453-3455.

[38] J.L. Palacios, Upper and lower bounds for the additive degree-Kirchhoff index, *MATCH Commun. Math. Comput. Chem.*, **70**(2013),651-655.

[39] J.L. Palacios, On the Kirchhoff index of graphs with diameter 2, *Discrete Appl. Math.*, **184**(2015),196-201.

[40] J.L. Palacios, Some interplay of the three Kirchhoffian indices, *MATCH Commun. Math. Comput. Chem.*, **75**(2016),199-206.

[41] R.J. Riddell, Contributions to the theory of condensation, Ph.D. Thesis, Univ. of Michigan, Ann Arbor, 1951.

[42] P. Serra, J.F. Stilck, W.L. Cavalcanti, K.D. Machado, Polymers with attractive interaction on the Husimi lattice, *J. Phys. A: Math. Gen.*, **37**(2004),8811-8821.

[43] J. Tu, J. Du, G. Su, The unicyclic graphs with maximum degree resistance distance, *Appl. Math. Comput.*, **268**(2015),859-864.

[44] G.E. Uhlenbeck, G.W. Ford, Lectures in Statistical Mechanics, AMS, Providence, RI, 1956.

[45] D. Wang, S. Tan, The maximum hyper-Wiener index of cacti, *J. Appl. Math. Comput.*, **47**(2015),91-102.

[46] H. Wang, H. Hua, D. Wang, Cacti with minimum, second-minimum, and third-minimum Kirchhoff indices, *Math. Commun.*, **15**:2(2010),347-358.

[47] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.*, **69**(1947),17-20.

[48] W. Yang, F. Zhang, Wiener index in random polyphenyl chains, *MATCH Commun. Math. Comput. Chem.*, **68**(2012),371-376.

[49] Y. Yang, D.J. Klein, Resistance distance-based graph invariants of subdivisions and triangulations of graphs, *Discrete Appl. Math.*, **181**(2015),260-274.

[50] Y. Yang, H. Liu, H. Wang, H. Fu, Subtrees of spiro and polyphenyl hexagonal chains, *Appl. Math. Comput.*, **268**(2015),547-560.

[51] Y. Yang, H. Zhang, Kirchhoff index of linear hexagonal chains, *Int. J. Quantum Chem.*, **108**(2008),503-512.

[52] P. Zhao, B. Zhao, X. Chen, Y. Bai, Two classes of chains with maximal and minimal total π-electron energy, *MATCH Commun. Math. Comput. Chem.*, **62**:3(2009),525-536.

[53] H. Zhang, Y. Yang, Resistance distance and Kirchhoff index in circulant graphs, *Int. J. Quantum Chem.*, **107**(2007),330-339.

[54] B. Zhou, N. Trinajstic, A note on Kirchhoff index, *Chem. Phys. Lett.*, **455**(2008),120-123.

[55] B. Zmazek, J. Zerovnik, Computing the weighted Wiener and Szeged number on weighted cactus graphs in linear time, *Croat. Chem. Acta*, **76**(2003),137-143.

[56] B. Zmazek, J. Zerovnik, Estimating the traffic on weighted cactus networks in linear time, in: Ninth International Conference on Information Visualization, IV05, London, 2005, pp.536-541.

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