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EXISTENCE OF PERIODIC SOLUTION FOR A KIND OF THIRD-ORDER GENERALIZED NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH VARIABLE PARAMETER*

A.M. Mahmoud†
(*Dept. of Math., Faculty of Science, New Valley Branch, Assiut University, New Valley, El-Khargah 72111, Egypt*)
E.S. Farghaly
(*Dept. of Math., Faculty of Science, Assiut University, Assiut 71516, Egypt*)

Abstract

In this paper, we investigate a third-order generalized neutral functional differential equation with variable parameter. Based on Mawhin's coincidence degree theory and some analysis skills, we obtain sufficient conditions for the existence of periodic solution for the equation. An example is also provided.

Keywords existence of periodic solution; third-order neutral functional differential equation; variable parameter; Mawhin's continuation theorem; coincidence degree

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1 Introduction

Neutral differential equations are widely used in many fields including biology, chemistry, physics, medicine, population dynamics, mechanics, economics, and so on (see [6,8,10,27]). For example, in population dynamics, since a growing population consumes more (or less) food than a matured one, depending on individual species, this leads to neutral equations [10]. These equations also arise in classical cobweb models in economics where current demand depends on price, but supply depends on the previous periodic [6]. In recent years, the problem of the existence of periodic solutions for neutral differential equations has been extensively studied in the literature. We refer the reader to [1-5,11-14,17-19,21-24] and the references cited therein for more details.

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†Corresponding author. E-mail: math_ayman27@yahoo.com

In this paper, we consider the generalized neutral functional differential equation with variable parameter

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\big(x(t)-c(t)x(t-\delta(t))\big)+f\big(t,\ddot{x}(t)\big)+g\big(t,\dot{x}(t)\big)+h\big(t,x\big(t-\tau(t)\big)\big)=e(t), \quad (1)$$

where $|c(t)|\neq 1,\ c,\delta\in C^2(\mathbb{R},\mathbb{R})$ and c,δ are ω-periodic functions for some $\omega>0$, $\tau,e\in C[0,\omega]$ and $\int_0^{\omega}e(t)\mathrm{d}t=0$; f,g and h are continuous functions defined on $\mathbb{R}^2$ and periodic in t with $f(t,\cdot)=f(t+\omega,\cdot)$, $g(t,\cdot)=g(t+\omega,\cdot)$, $h(t,\cdot)=h(t+\omega,\cdot)$, and $f(t,0)=g(t,0)=0$.

In recent years, when $c(t)$ is a constant c or $\delta(t)$ is a constant δ or both of them are constants, many researchers have extensively studied such types of neutral functional differential equations. We refer the reader [9,15-17,20,26] and their references therein. But the work to study the existence of periodic solutions for neutral functional differential equations with variable parameter has rarely appeared. There are two reasons for this. The first reason is that the criterion of L-compact of nonlinear operator N on the set $\overline{\Omega}$ is difficult to establish when $c(t)$ is not a constant. The second reason is that the linear operator $A:C_T\to C_T$, $[Ax](t)=x(t)-c(t)x(t-\tau)$, for all $t\in[0,T]$, has continuous inverse A^{-1}, which is far away from the answer.

For example, Du et al. [5] investigated the second-order neutral equation

$$\big(x(t)-c(t)x(t-\delta)\big)''+f\big(x(t)\big)x'(t)+g\big(x\big(t-\gamma(t)\big)\big)=e(t), \quad (2)$$

by using Mawhin's continuous theorem, the authors obtained the existence of periodic solution for (2).

Afterwards, in [19], Ren et al. considered the following neutral differential equation with deviating arguments:

$$\big(x(t)-cx\big(t-\delta(t)\big)\big)''=f(t,x'(t))+g\big(t,x\big(t-\tau(t)\big)\big)+e(t),$$

by the continuation theorem and some analysis techniques, some new results on the existence of periodic solutions were obtained.

Recently, Xin and Zhao [25] studied the neutral equation with variable delay

$$\big(x(t)-c(t)x\big(t-\delta(t)\big)\big)''+f(t,x'(t))+g\big(t,x\big(t-\tau(t)\big)\big)=e(t), \quad (3)$$

by coincidence degree theory and some analysis skills, the authors obtained sufficient conditions for the existence of periodic solution for (3).

Motivated by [5,19,25], in this paper, we consider the generalized neutral equation (1). Notice that here the neutral operator A is a natural generalization of the familiar operator $A_1=x(t)-cx(t-\delta)$, $A_2=x(t)-c(t)x(t-\delta)$, $A_3=x(t)-cx(t-\delta(t))$. But A possesses a more complicated nonlinearity than A_i, $i=1,2,3$. For example, the neutral operator A_1 is homogeneous in the following sense $\frac{\mathrm{d}}{\mathrm{d}t}(A_1x)(t)=(A_1\dot{x})(t)$, whereas the neutral operator A in general is inhomogeneous. As a consequence, many of the new results for differential equations with the neutral operator A will not be a direct extension of known theorems for neutral differential equations.

The paper is organized as follows. In Section 2, we first analyze qualitative properties of the generalized neutral operator A, which will be helpful for further studies of differential equations with this neutral operator; in Section 3, by Mawhin's continuation theorem, we obtain the existence of periodic solution for the generalized neutral equation with variable parameter. An illustrative example is given in Section 4.

2 Analysis of the Generalized Neutral Operator with Variable Parameter

Let

$$c_\infty = \max_{t\in[0,\omega]} |c(t)|, \quad c_0 = \min_{t\in[0,\omega]} |c(t)|.$$

Let $X = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ with the norm $\|x\| = \max_{t\in[0,\omega]} |x(t)|$, then $(X, \|\cdot\|)$ is a Banach space. Moreover, define operators $A, B : C_\omega \to C_\omega$ by

$$(Ax)(t) = x(t) - c(t)x\big(t - \delta(t)\big), \quad (Bx)(t) = c(t)x\big(t - \delta(t)\big).$$

Lemma 2.1[25] *If $|c(t)| \neq 1$, then the operator A has a continuous inverse A^{-1} on C_ω satisfying*

(1)

$$\big(A^{-1}f\big)(t) = \begin{cases} f(t) + \displaystyle\sum_{j=1}^{\infty}\prod_{i=1}^{j} c(D_i)x\Big(t - \sum_{i=1}^{j}\delta(D_i)\Big), & \text{for } |c(t)|<1 \text{ and } f\in C_\omega, \\ -\dfrac{f(t+\delta(t))}{c(t+\delta(t))} - \displaystyle\sum_{j=1}^{\infty} \dfrac{f\Big(t+\delta(t)+\sum\limits_{i=1}^{j}\delta(D_i')\Big)}{c(t+\delta(t))\prod\limits_{i=1}^{j}c(D_i')}, & \text{for } |c(t)|>1 \text{ and } f\in C_\omega. \end{cases}$$

(2)

$$\big|\big(A^{-1}f\big)(t)\big| \le \begin{cases} \dfrac{\|f\|}{1-c_\infty}, & \text{for } c_\infty < 1 \text{ and } f \in C_\omega, \\ \dfrac{\|f\|}{c_0 - 1}, & \text{for } c_0 > 1 \text{ and } f \in C_\omega. \end{cases}$$

(3)

$$\int_0^\omega \big|\big(A^{-1}f\big)(t)\big|\mathrm{d}t \le \begin{cases} \dfrac{1}{1-c_\infty}\displaystyle\int_0^\omega |f(t)|\mathrm{d}t, & \text{for } c_\infty < 1 \text{ and } f \in C_\omega, \\ \dfrac{1}{c_0-1}\displaystyle\int_0^\omega |f(t)|\mathrm{d}t, & \text{for } c_0 > 1 \text{ and } f \in C_\omega, \end{cases}$$

where $D_1 = t,\ D_i = t - \sum\limits_{k=1}^{i} \delta(D_k),\ k = 1, 2, \cdots,$ *and* $D'_1 = t,\ D'_i = t + \sum\limits_{k=1}^{i} \delta(D'_k),$ $k = 1, 2, \cdots.$

3 Existence of Periodic Solution for (1)

We first recall Mawhin's continuation theorem, which our study is based upon. Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\mathrm{Im}L$ is closed in Y and $\dim \mathrm{Ker}\, L = \dim(Y/\,\mathrm{Im}\, L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y, respectively such that $X = \mathrm{Ker}\, L \oplus X_1$, $Y = \mathrm{Im}\, L \oplus Y_1$. Let $P_1 : X \to \mathrm{Ker}\, L$ and $Q_1 : Y \to Y_1$ denote the natural projections. Clearly, $\mathrm{Ker}\, L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_{P_1} := L|_{D(L)\cap X_1}$ is invertible. Let $L_{P_1}^{-1}$ denote the inverse of L_{P_1}.

Let Ω be an open bounded subset of X with $D(L)\cap\Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $Q_1 N(\overline{\Omega})$ is bounded and the operator $L_{P_1}^{-1}(I - Q_1)N : \overline{\Omega} \to X$ is compact.

Lemma 3.1[7] *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx$, *for any* $x \in \partial\Omega \cap D(L),\ \lambda \in (0, 1)$;

(2) $Nx \notin \mathrm{Im}\, L$, *for any* $x \in \partial\Omega \cap \mathrm{Ker}\, L$;

(3) $\deg\{JQ_1N, \Omega \cap \mathrm{Ker}\, L, 0\} \neq 0$, *where* $J : \mathrm{Im}\, Q_1 \to \mathrm{Ker}\, L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of ω-periodic solutions for (1), we rewrite (1) in the following form:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t) = x_2(t), \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t) = \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -f(t, \ddot{x}_1(t)) - g(t, \dot{x}_1(t)) - h(t, x_1(t - \tau(t))) + e(t). \end{cases} \tag{4}$$

Clearly, if $x(t) = (x_1(t), x_2(t), x_3(t))^\top$ is an ω-periodic solution to (4), then $x_1(t)$ must be an ω-periodic solution to (1). Thus the problem of finding an ω-periodic solution for (1) reduces to that of finding one for (4). Recall that $C_\omega = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + \omega) \equiv \phi(t)\}$ with the norm $\|\phi\| = \max\limits_{t\in[0,\omega]} |\phi(t)|$. Define $X = Y = C_\omega \times C_\omega = \{x = (x_1(\cdot), x_2(\cdot), x_3(\cdot)) \in C(\mathbb{R}, \mathbb{R}^3) : x(t) = x(t + \omega), t \in \mathbb{R}\}$ with the norm $\|x\| = \max\{\|x_1\|, \|x_2\|, \|x_3\|\}$. Clearly, X and Y are Banach spaces. Moreover define

$$L : D(L) = \left\{x \in C^1\left(\mathbb{R}, \mathbb{R}^3\right) : x(t+\omega) = x(t), t \in \mathbb{R}\right\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t) \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

Also define $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} x_2(t) \\ x_3(t) \\ -f(t, \ddot{x}_1(t)) - g(t, \dot{x}_1(t)) - h(t, x_1(t-\tau(t))) + e(t) \end{pmatrix}. \tag{5}$$

Then (4) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^3, \quad \operatorname{Im} L = \left\{ y \in Y : \int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} \mathrm{d}s = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to \operatorname{Im} Q_1 \subset \mathbb{R}^3$ be defined by

$$P_1 x = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}; \quad Q_1 y = \frac{1}{\omega} \int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} \mathrm{d}s.$$

Then $\operatorname{Im} P_1 = \operatorname{Ker} L$, $\operatorname{Ker} Q_1 = \operatorname{Im} L$. Set $L_{P_1} = L|_{D(L) \cap \operatorname{Ker} P_1}$ and let $L_{P_1}^{-1} : \operatorname{Im} L \to D(L)$ denote the inverse of L_{P_1}, then it follows that

$$\begin{aligned} &\left[L_{P_1}^{-1} y\right](t) = \begin{pmatrix} (A^{-1}Fy_1)(t) \\ (Fy_2)(t) \\ (Fy_3)(t) \end{pmatrix}, \\ &[Fy_1](t) = \int_0^t y_1(s)\mathrm{d}s, \quad [Fy_2](t) = \int_0^t y_2(s)\mathrm{d}s, \quad [Fy_3](t) = \int_0^t y_3(s)\mathrm{d}s. \end{aligned} \tag{6}$$

From (5) and (6), it is clear that $Q_1 N$ and $L_{P_1}^{-1}(I - Q_1)N$ are continuous, and $Q_1 N(\overline{\Omega})$ is bounded, and then $L_{P_1}^{-1}(I - Q_1)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\overline{\Omega}$. For convenience, we list the following assumptions, which will be used repeatedly in the sequel:

(H1) There exists a positive constant K_1 such that $|f(t,u)| \le K_1$ for $(t,u) \in \mathbb{R}\times\mathbb{R}$;

(H2) there exists a positive constant K_2 such that $|g(t,u)| \le K_2$ for $(t,u) \in \mathbb{R}\times\mathbb{R}$;

(H3) there exists a positive constant D such that $|h(t,x)| > K_1+K_2$ and $x[f(t,u)+g(t,v)+h(t,x)] \neq 0$ for $t,u,v,x \in \mathbb{R}$ and $|x| > D$;

(H4) there exists a positive constant m_o such that $|h(t,x_1)-h(t,x_2)| \le m_o|x_1 - x_2|$, for all $t, x_1, x_2 \in \mathbb{R}$.

Now we give our main results on periodic solutions for (1).

Theorem 3.1 *Assume that conditions* (H1)-(H4) *hold. Suppose that one of the following conditions is satisfied:*

(i) *If* $c_\infty < 1$ *and* $1 - c_\infty - c_\infty\delta_1(\delta_1 - 2) - M_6 > 0$;

(ii) *if* $c_0 > 1$ *and* $c_0 - 1 - c_\infty\delta_1(\delta_1-2) - M_6 > 0$,

where

$$M_6 = \frac{1}{2}\Big(\sqrt{M_5\omega} + \frac{1}{2}c_2\omega^2 + 2c_1\omega - c_\infty\delta_2\omega\Big), \quad M_5 = \frac{1}{2}m_o\omega^2 M_1,$$
$$M_1 = 1 + \frac{1}{2}c_1\omega + c_\infty + c_\infty\delta_1, \quad c_1 = \max_{t\in[0,\omega]}|\dot{c}(t)|,$$
$$c_2 = \max_{t\in[0,\omega]}|\ddot{c}(t)|, \quad \delta_1 = \max_{t\in[0,\omega]}|\dot{\delta}(t)|, \quad \delta_2 = \max_{t\in[0,\omega]}|\ddot{\delta}(t)|.$$

Then equation (1) *has at least one* ω*-periodic solution.*

Proof By construction, (4) has an ω-periodic solution, if and only if, the following operator equation

$$Lx = Nx$$

has an ω-periodic solution. From (5) we see that N is L-compact on $\overline{\Omega}$, where Ω is any open, bounded subset of C_ω. For $\lambda \in (0,1]$, define $\Omega_1 = \{x \in C_\omega : Lx = \lambda Nx\}$. Then $x = (x_1,x_2,x_3)^\top \in \Omega_1$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t) = \lambda x_2(t), \\ \dot{x}_2(t) = \lambda x_3(t), \\ \dot{x}_3(t) = -\lambda f(t,\ddot{x}_1(t)) - \lambda g(t,\dot{x}_1(t)) - \lambda h(t, x_1(t-\tau(t))) + \lambda e(t). \end{cases} \tag{7}$$

Substituting $x_3(t) = \frac{1}{\lambda}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)$ into the third equation of (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{\lambda}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big] = -\lambda f(t,\ddot{x}_1(t)) - \lambda g(t,\dot{x}_1(t)) - \lambda h(t,x_1(t-\tau(t))) + \lambda e(t).$$

Therefore we find

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\big(Ax_1(t)\big) = -\lambda^2 f(t,\ddot{x}_1(t)) - \lambda^2 g(t,\dot{x}_1(t)) - \lambda^2 h(t,x_1(t-\tau(t))) + \lambda^2 e(t). \tag{8}$$

Integrating both sides of (8) over $[0,\omega]$, we have

$$\int_0^\omega \big[f(t,\ddot{x}_1(t))+g(t,\dot{x}_1(t))+h(t,x_1(t-\tau(t)))\big]\mathrm{d}t=0, \tag{9}$$

which yields that there exists at least one point t_1 such that

$$f(t_1,\ddot{x}_1(t_1))+g(t_1,\dot{x}_1(t_1))+h(t_1,x_1(t_1-\tau(t_1)))=0.$$

Thus by (H1) and (H2) we have

$$\big|h(t_1,x_1(t_1-\tau(t_1)))\big|=\big|-f(t_1,\ddot{x}_1(t_1))\big|+\big|-g(t_1,\dot{x}_1(t_1))\big|\le K_1+K_2:=K.$$

In view of (H3) we get that $|x_1(t_1-\tau(t_1))|\le D$. Since $x_1(t)$ is periodic with periodic ω. So $t_1-\tau(t_1)=n\omega+\xi$, $\xi\in[0,\omega]$, where n is some integer, then $|x_1(\xi)|\le D$. Therefore we have

$$|x_1(t)|=\left|x_1(\xi)+\int_\xi^t \dot{x}_1(s)\mathrm{d}s\right|\le D+\int_\xi^t\big|\dot{x}_1(s)\big|\mathrm{d}s,\quad t\in[\xi,\xi+\omega].$$

And

$$|x_1(t)|=|x_1(t-\omega)|=\left|x_1(\xi)-\int_{t-\omega}^\xi \dot{x}_1(s)\mathrm{d}s\right|\le D+\int_{t-\omega}^\xi\big|\dot{x}_1(s)\big|\mathrm{d}s,\quad t\in[\xi,\xi+\omega].$$

Combining the above two inequalities, we obtain

$$\begin{aligned}\|x_1\|_\infty&=\max_{t\in[0,\omega]}\big|x_1(t)\big|=\max_{t\in[\xi,\xi+\omega]}\big|x_1(t)\big|\\&\le\max_{t\in[\xi,\xi+\omega]}\left\{D+\frac{1}{2}\left(\int_\xi^t\big|\dot{x}_1(s)\big|\mathrm{d}s+\int_{t-\omega}^\xi\big|\dot{x}_1(s)\big|\mathrm{d}s\right)\right\}\\&\le D+\frac{1}{2}\int_0^\omega\big|\dot{x}_1(s)\big|\mathrm{d}s\le D+\frac{1}{2}\omega\|\dot{x}_1\|_\infty.\end{aligned}\tag{10}$$

Since $x_1(0)=x_1(\omega)$, there exists a constant $\eta\in[0,\omega]$ such that $\dot{x}_1(\eta)=0$. Hence

$$|\dot{x}_1(t)|=\left|\dot{x}_1(\eta)+\int_\eta^t\ddot{x}_1(s)\mathrm{d}s\right|\le\int_\eta^t|\ddot{x}_1(s)|\mathrm{d}s,\quad t\in[\eta,\omega+\eta]. \tag{11}$$

Also

$$\begin{aligned}|\dot{x}_1(t)|&=\left|\dot{x}_1(\eta+\omega)+\int_{\eta+\omega}^t\ddot{x}_1(s)\mathrm{d}s\right|\\&\le|\dot{x}_1(\eta+\omega)|+\int_t^{\eta+\omega}|\ddot{x}_1(s)|\mathrm{d}s=\int_t^{\eta+\omega}|\ddot{x}_1(s)|\mathrm{d}s,\quad t\in[0,\omega].\end{aligned}\tag{12}$$

From the above inequalities we have

$$\|\dot{x}_1\|_\infty=\max_{t\in[0,\omega]}\big|\dot{x}_1(t)\big|\le\frac{1}{2}\int_0^\omega\big|\ddot{x}_1(s)\big|\mathrm{d}s,\quad t\in[0,\omega]. \tag{13}$$

From the definition of the operator A, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(Ax_1(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(x_1(t)-c(t)x_1\big(t-\delta(t)\big)\big)\\ &= \dot{x}_1(t)-\dot{c}(t)x_1\big(t-\delta(t)\big)-c(t)\dot{x}_1\big(t-\delta(t)\big)\big(1-\dot{\delta}(t)\big).\end{aligned}$$

Then from (10) and condition (ii) of Theorem 3.1, we have

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\big((Ax_1)(t)\big)\Big| &\le \big|\dot{x}_1(t)\big|+\big|\dot{c}(t)\big|\big|x_1\big(t-\delta(t)\big)\big|+\big|c(t)\big|\big|\dot{x}_1\big(t-\delta(t)\big)\big|\big|1-\dot{\delta}(t)\big|\\ &\le \|\dot{x}_1\|_\infty+c_1\|x_1\|_\infty+c_\infty\|\dot{x}_1\|_\infty(1+\delta_1)\\ &\le \|\dot{x}_1\|_\infty+c_1D+\frac{1}{2}\|\dot{x}_1\|_\infty c_1\omega+c_\infty\|\dot{x}_1\|_\infty(1+\delta_1) \qquad (14)\\ &= c_1D+\Big(1+\frac{1}{2}c_1\omega+c_\infty+c_\infty\delta_1\Big)\|\dot{x}_1\|_\infty\\ &= c_1D+M_1\|\dot{x}_1\|_\infty,\end{aligned}$$

where $c_1=\max\limits_{t\in[0,\omega]}|\dot{c}(t)|$, $\delta_1=\max\limits_{t\in[0,\omega]}|\dot{\delta}(t)|$ and $M_1=1+\frac{1}{2}c_1\omega+c_\infty+c_\infty\delta_1$. Thus we can obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big) &= \ddot{x}_1(t)-\ddot{c}(t)x_1\big(t-\delta(t)\big)-\dot{c}(t)\dot{x}_1\big(t-\delta(t)\big)\big(1-\dot{\delta}(t)\big)\\ &\quad -\dot{c}(t)\dot{x}_1\big(t-\delta(t)\big)-c(t)\ddot{x}_1\big(t-\delta(t)\big)\big(1-\dot{\delta}(t)\big)\\ &\quad +\dot{c}(t)\dot{x}_1\big(t-\delta(t)\big)\dot{\delta}(t)+c(t)\ddot{x}_1\big(t-\delta(t)\big)\big(1-\dot{\delta}(t)\big)\dot{\delta}(t)\\ &\quad +c(t)\dot{x}_1\big(t-\delta(t)\big)\ddot{\delta}(t)\\ &= \big(A\ddot{x}_1\big)(t)-\ddot{c}(t)x_1\big(t-\delta(t)\big)-2\dot{c}(t)\dot{x}_1\big(t-\delta(t)\big)\\ &\quad +2c(t)\ddot{x}_1\big(t-\delta(t)\big)\dot{\delta}(t)-c(t)\ddot{x}_1\big(t-\delta(t)\big)\dot{\delta}^2(t)\\ &\quad +c(t)\dot{x}_1\big(t-\delta(t)\big)\ddot{\delta}(t)\\ &= \big(A\ddot{x}_1\big)(t)-\ddot{c}(t)x_1\big(t-\delta(t)\big)-\big[2\dot{c}(t)-c(t)\ddot{\delta}(t)\big]\dot{x}_1\big(t-\delta(t)\big)\\ &\quad -\big[\dot{\delta}(t)-2\big]c(t)\ddot{x}_1\big(t-\delta(t)\big)\dot{\delta}(t).\end{aligned}$$

Therefore we get

$$\begin{aligned}\big(A\ddot{x}_1\big)(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)+\ddot{c}(t)x_1\big(t-\delta(t)\big)+\big[2\dot{c}(t)-c(t)\ddot{\delta}(t)\big]\dot{x}_1\big(t-\delta(t)\big)\\ &\quad +\big[\dot{\delta}(t)-2\big]c(t)\ddot{x}_1\big(t-\delta(t)\big)\dot{\delta}(t). \qquad (15)\end{aligned}$$

On the other hand, multiplying both sides of (8) by $\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)$ and integrating it over $[0,\omega]$, we get

$$\begin{aligned}\int_0^\omega \frac{\mathrm{d}^3}{\mathrm{d}t^3}\big(Ax_1(t)\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(Ax_1(t)\big)\mathrm{d}t &= -\int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big|^2\mathrm{d}t\\ &= -\lambda^2\int_0^\omega f\big(t,\ddot{x}_1(t)\big)\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\mathrm{d}t\\ &\quad -\lambda^2\int_0^\omega g\big(t,\dot{x}_1(t)\big)\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\mathrm{d}t\\ &\quad -\lambda^2\int_0^\omega h\big(t,x_1\big(t-\tau(t)\big)\big)\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\mathrm{d}t\\ &\quad +\lambda^2\int_0^\omega e(t)\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\mathrm{d}t.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\int_0^\omega\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big|^2\mathrm{d}t \le{}& \int_0^\omega\big|f\big(t,\ddot{x}_1(t)\big)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t+\int_0^\omega\big|g\big(t,\dot{x}_1(t)\big)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t\\ &+\int_0^\omega\big|h\big(t,x_1\big(t-\tau(t)\big)\big)-h(t,0)+h(t,0)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t\\ &+\int_0^\omega\big|e(t)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t.\end{aligned}$$

Therefore from (H4) we have

$$\begin{aligned}\int_0^\omega\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big|^2\mathrm{d}t \le{}& \int_0^\omega\big|f\big(t,\ddot{x}_1(t)\big)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t+\int_0^\omega\big|g\big(t,\dot{x}_1(t)\big)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t\\ &+\int_0^\omega\big[m_o\big|x_1\big(t-\tau(t)\big)\big|+\big|h(t,0)\big|\big]\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t\\ &+\int_0^\omega\big|e(t)\big|\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\Big|\mathrm{d}t.\end{aligned}$$

Using (H1), (H2) and (14) we get

$$\begin{aligned}\int_0^\omega\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big|^2\mathrm{d}t \le{}& \big(K_1+K_2+m_o\|x_1\|_\infty\big)\big(c_1D+M_1\|\dot{x}_1\|_\infty\big)\omega\\ &+\big(\max\{|h(t,0)|:0\le t\le\omega\}+\|e\|_\infty\big)\big(c_1D+M_1\|\dot{x}_1\|_\infty\big)\omega.\end{aligned}$$

Hence from (10), we obtain

$$\int_0^\omega\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big|^2\mathrm{d}t \le c_1DM_2+\Big(M_1M_2+\frac{1}{2}m_o\omega^2c_1D\Big)\|\dot{x}_1\|_\infty+\frac{1}{2}m_o\omega^2M_1\|\dot{x}_1\|_\infty^2,$$

where $M_2=\big(K_1+K_2+m_oD+\max\{|h(t,0)|:0\le t\le\omega\}+\|e\|_\infty\big)\omega$. Thus we have

$$\int_0^\omega\Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\Big|^2\mathrm{d}t \le M_3+M_4\|\dot{x}_1\|_\infty+M_5|\dot{x}_1\|_\infty^2, \tag{16}$$

where

$$M_3 = c_1 D M_2, \quad M_4 = M_1 M_2 + \frac{1}{2} m_o \omega^2 c_1 D, \quad M_5 = \frac{1}{2} m_o \omega^2 M_1.$$

Case (i) If $c_\infty < 1$, by applying Lemma 2.1 (3), we obtain

$$\int_0^\omega \big|\ddot{x}_1(t)\big| \mathrm{d}t = \int_0^\omega \big|\big(A^{-1} A\ddot{x}_1\big)(t)\big| \mathrm{d}t \le \frac{\int_0^\omega |(A\ddot{x}_1)(t)| \mathrm{d}t}{1 - c_\infty}.$$

Substituting from (15) and using condition (ii) of Theorem 3.1 we have

$$\begin{aligned}
\int_0^\omega \big|\ddot{x}_1(t)\big| \mathrm{d}t \le{}& \frac{1}{1-c_\infty}\Big[\int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)\Big| \mathrm{d}t + \int_0^\omega \big|\ddot{c}(t) x_1\big(t-\delta(t)\big)\big| \mathrm{d}t\Big] \\
&+ \frac{1}{1-c_\infty}\Big[\int_0^\omega \big|\big\{2\dot{c}(t) - c(t)\ddot{\delta}(t)\big\}\dot{x}_1\big(t-\delta(t)\big)\big| \mathrm{d}t\Big] \\
&+ \frac{1}{1-c_\infty}\Big\{\int_0^\omega \big|\big\{\dot{\delta}(t) - 2\big\} c(t)\ddot{x}_1\big(t-\delta(t)\big)\dot{\delta}(t)\big| \mathrm{d}t\Big\} \\
\le{}& \frac{1}{1-c_\infty}\Big[\int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)\Big| \mathrm{d}t + c_2\omega\|x_1\|_\infty + (2c_1 - c_\infty\delta_2)\omega\|\dot{x}_1\|_\infty\Big] \\
&+ \frac{1}{1-c_\infty}\Big[c_\infty\delta_1(\delta_1 - 2)\int_0^\omega \big|\ddot{x}_1(t)\big| \mathrm{d}t\Big],
\end{aligned}$$

where

$$c_1 = \max_{t\in[0,\omega]} |\dot{c}(t)|, \quad c_2 = \max_{t\in[0,\omega]} |\ddot{c}(t)|, \quad \delta_1 = \max_{t\in[0,\omega]} |\dot{\delta}(t)|, \quad \delta_2 = \max_{t\in[0,\omega]} |\ddot{\delta}(t)|.$$

From (10) and by Schwarz inequality, we have

$$\begin{aligned}
\Big[1 - \frac{c_\infty\delta_1(\delta_1-2)}{1-c_\infty}\Big]\int_0^\omega \big|\ddot{x}_1(t)\big| \mathrm{d}t \le{}& \frac{1}{1-c_\infty}\Big[\omega^{\frac{1}{2}}\Big(\int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big] \\
&+ \frac{1}{1-c_\infty}\Big[c_2\omega\Big(D + \frac{1}{2}\|\dot{x}_1\|_\infty\omega\Big)\Big] \\
&+ \frac{1}{1-c_\infty}\big[(2c_1 - c_\infty\delta_2)\omega\|\dot{x}_1\|_\infty\big].
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\big[1 - c_\infty - c_\infty\delta_1(\delta_1 - 2)\big]\int_0^\omega \big|\ddot{x}_1(t)\big| \mathrm{d}t \le{}& \omega^{\frac{1}{2}}\Big(\int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \\
&+ c_2\omega\Big(D + \frac{1}{2}\|\dot{x}_1\|_\infty\omega\Big) + (2c_1 - c_\infty\delta_2)\omega\|\dot{x}_1\|_\infty.
\end{aligned}$$

Applying the inequality $(a+b)^k \le a^k + b^k$ for all $a, b > 0$, $0 < k < 1$, it follows from (16) that

$$\begin{aligned}\big[1 - c_\infty - c_\infty\delta_1(\delta_1 - 2)\big]\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t &\le \sqrt{\omega}\big[\sqrt{M_3} + \sqrt{M_4}\big(\|\dot{x}_1\|_\infty\big)^{\frac{1}{2}} + \sqrt{M_5}\|\dot{x}_1\|_\infty\big] \\ &\quad + c_2\omega\Big(D + \frac{1}{2}\|\dot{x}_1\|_\infty\omega\Big) + (2c_1 - c_\infty\delta_2)\omega\|\dot{x}_1\|_\infty \\ &\le \sqrt{M_3\omega} + c_2\omega D + \sqrt{M_4\omega}\big(\|\dot{x}_1\|_\infty\big)^{\frac{1}{2}} \\ &\quad + \Big(\sqrt{M_5\omega} + \frac{1}{2}c_2\omega^2 + 2c_1\omega - c_\infty\delta_2\omega\Big)\|\dot{x}_1\|_\infty.\end{aligned}$$

Substituting from (13), we get

$$\begin{aligned}\big[1 - c_\infty - c_\infty\delta_1(\delta_1 - 2)\big]\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t &\le \sqrt{M_3\omega} + c_2\omega D + \sqrt{M_4\omega}\sqrt{\frac{1}{2}}\Big(\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\quad + M_6\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t,\end{aligned}$$

where $M_6 = \frac{1}{2}\big(\sqrt{M_5\omega} + \frac{1}{2}c_2\omega^2 + 2c_1\omega - c_\infty\delta_2\omega\big)$.

Therefore we obtain

$$\big[1 - c_\infty - c_\infty\delta_1(\delta_1 - 2) - M_6\big]\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t \le \sqrt{M_3\omega} + c_2\omega D + \sqrt{\frac{1}{2}M_4\omega}\Big(\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{17}$$

Since $1 - c_\infty - c_\infty\delta_1(\delta_1 - 2) - M_6 > 0$, it is easy to see that there exists a constant $\mathcal{M} > 0$ (independent of λ) such that

$$\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t \le \mathcal{M}. \tag{18}$$

It follows from (13) that

$$\|\dot{x}_1\|_\infty \le \frac{1}{2}\mathcal{M}.$$

Thus, from (10) we obtain

$$\|x_1\|_\infty \le \mathcal{M}_1.$$

Case (ii) If $c_0 > 1$, by applying Lemma 2.1 (3), we have

$$\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t = \int_0^\omega \big|\big(A^{-1}A\ddot{x}_1\big)(t)\big|\mathrm{d}t \le \frac{1}{c_0 - 1}\int_0^\omega |(A\ddot{x}_1)(t)|\mathrm{d}t.$$

Following the same manner as in Case (i), we can get

$$\big[c_0 - 1 - c_\infty\delta_1(\delta_1 - 2) - M_6\big]\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t \le \sqrt{M_3\omega} + c_2\omega D + \sqrt{\frac{1}{2}M_4\omega}\Big(\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t\Big)^{\frac{1}{2}}.$$

Since $c_0 - 1 - c_\infty\delta_1(\delta_1 - 2) - M_6 > 0$, similarly, we can obtain

$$\|x_1\|_\infty \le \mathcal{M}_1.$$

By the first equation of system (7) we have

$$\int_0^\omega x_2(t)\mathrm{d}t = \int_0^\omega \frac{\mathrm{d}}{\mathrm{d}t}(Ax_1)(t)\mathrm{d}t = 0,$$

which implies that there is a constant $t_1 \in [0,\omega]$, such that $x_2(t_1) = 0$, hence from (16) we find

$$\begin{aligned}\|x_2\|_\infty &\le \int_0^\omega |\dot{x}_2(t)|\mathrm{d}t = \int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)\Big|\mathrm{d}t \le \omega^{\frac{1}{2}}\Big(\int_0^\omega \Big|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ax_1(t)\big)\Big|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\le \sqrt{\omega}\big[\sqrt{M_3} + \sqrt{M_4}\big(\|\dot{x}_1\|_\infty\big)^{\frac{1}{2}} + \sqrt{M_5}\|\dot{x}_1\|_\infty\big].\end{aligned}$$

In view of Cases (i) and (ii), it is easy to see that there exists a constant $\mathcal{M}_2 > 0$ (independent of λ) such that

$$\|x_2\|_\infty \le \mathcal{M}_2.$$

By the second equation of system (7), we obtain

$$\int_0^\omega x_3(t)\mathrm{d}t = \int_0^\omega \frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ax_1)(t)\mathrm{d}t = \int_0^\omega \dot{x}_2(t)\mathrm{d}t = 0,$$

which implies that there is a constant $t_2 \in [0,\omega]$ such that $x_3(t_2) = 0$, hence

$$\|x_3\|_\infty \le \int_0^\omega |\dot{x}_3(t)|\mathrm{d}t.$$

By the third equation of system (7), we have

$$\dot{x}_3(t) = -\lambda f(t, \ddot{x}_1(t)) - \lambda g(t, \dot{x}_1(t)) - \lambda h(t, x_1(t - \tau(t))) + \lambda e(t).$$

Using (H1), (H2) and (H4), we get

$$\begin{aligned}\|x_3\|_\infty &\le \int_0^\omega |\dot{x}_3(t)|\mathrm{d}t \\ &\le \int_0^\omega \big|f\big(t, \ddot{x}_1(t)\big)\big|\mathrm{d}t + \int_0^\omega \big|g\big(t, \dot{x}_1(t)\big)\big|\mathrm{d}t \\ &\quad + \int_0^\omega \big|h\big(t, x_1\big(t - \tau(t)\big)\big) - h(t,0) + h(t,0)\big|\mathrm{d}t + \int_0^\omega \big|e(t)\big|\mathrm{d}t \\ &\le \int_0^\omega \big|f\big(t, \ddot{x}_1(t)\big)\big|\mathrm{d}t + \int_0^\omega \big|g\big(t, \dot{x}_1(t)\big)\big|\mathrm{d}t \\ &\quad + \int_0^\omega \big[m_o\big|x_1\big(t - \tau(t)\big)\big| + \big|h(t,0)\big|\big]\mathrm{d}t + \int_0^\omega \big|e(t)\big|\mathrm{d}t \\ &\le \big(K_1 + K_2 + m_o\|x_1\|_\infty + \max\{|h(t,0)| : 0 \le t \le \omega\} + \|e\|_\infty\big)\omega := \mathcal{M}_3.\end{aligned}$$

To prove condition (1) of Lemma 3.1, we assume that for any $\lambda \in (0,1)$ and any $x = x(t)$ in the domain of L, which also belongs to $\partial\Omega$, we must have $Lx \neq \lambda Nx$. For otherwise in view of (7), we obtain

$$\|x_1\|_\infty \leq \mathcal{M}_1, \quad \|x_2\|_\infty \leq \mathcal{M}_2, \quad \|x_3\|_\infty \leq \mathcal{M}_3.$$

Let $\mathcal{M}_4 = \max\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}+1$, $\Omega = \{x = (x_1, x_2, x_3)^\top : \|x\| < \mathcal{M}_4\}$, then we see that x belongs to the interior of Ω, which contradicts the assumption that $x \in \partial\Omega$. Therefore condition (1) of Lemma 3.1 is satisfied. Now for any $x \in \partial\Omega \cap \operatorname{Ker} L$

$$Q_1 Nx = \frac{1}{\omega}\int_0^\omega \begin{pmatrix} x_2(t) \\ x_3(t) \\ -f(t, \ddot{x}_1(t)) - g(t, \dot{x}_1(t)) - h(t, x_1(t-\tau(t))) + e(t) \end{pmatrix} \mathrm{d}t.$$

If $Q_1 Nx = 0$, then $x_2(t) = 0$, $x_3(t) = 0$, $x_1 = \mathcal{M}_4$ or $-\mathcal{M}_4$. But if $x_1(t) = \mathcal{M}_4$, then we get

$$0 = \int_0^\omega h(t, \mathcal{M}_4)\mathrm{d}t,$$

from which there exists a point t_2 such that $h(t_2, \mathcal{M}_4) = 0$. From assumption (H3), we have $\mathcal{M}_4 \leq D$, which yields a contradiction. Similar analysis holds for $x_1 = -\mathcal{M}_4$. Therefore we have $Q_1 Nx \neq 0$, hence for all $x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so condition (2) of Lemma 3.1 is satisfied.

Define an isomorphism $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2, x_3)^\top = (-x_3, x_1, x_2)^\top.$$

Let $H(\mu, x) = \mu x + (1-\mu)JQ_1 Nx$, $(\mu, x) \in [0,1] \times \Omega$, then for any $(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} \mu x_1(t) + \dfrac{1-\mu}{\omega}\displaystyle\int_0^\omega [f(t, \ddot{x}_1(t)) + g(t, \dot{x}_1(t)) + h(t, x_1(t-\tau(t))) - e(t)]\mathrm{d}t \\ (\mu + (1-\mu))x_2(t) \\ (\mu + (1-\mu))x_3(t) \end{pmatrix}.$$

We have $\int_0^\omega e(t)\mathrm{d}t = 0$. So, we can get

$$H(\mu, x) = \begin{pmatrix} \mu x_1(t) + \dfrac{1-\mu}{\omega}\displaystyle\int_0^\omega [f(t, \ddot{x}_1(t)) + g(t, \dot{x}_1(t)) + h(t, x_1(t-\tau(t)))]\mathrm{d}t \\ (\mu + (1-\mu))x_2(t) \\ (\mu + (1-\mu))x_3(t) \end{pmatrix},$$

for all $(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$.

From (H3), it is obvious that $x^\top H(\mu,x)\neq 0$, for any $(\mu,x)\in(0,1)\times(\partial\Omega\cap\operatorname{Ker}L)$. Hence

$$\begin{aligned}\deg\{JQ_1N,\Omega\cap\operatorname{Ker}L,0\}&=\deg\big\{H(0,x),\Omega\cap\operatorname{Ker}L,0\big\}\\&=\deg\big\{H(1,x),\Omega\cap\operatorname{Ker}L,0\big\}\\&=\deg\{I,\Omega\cap\operatorname{Ker}L,0\}\neq 0.\end{aligned}$$

So condition (3) of Lemma 3.1 is satisfied. By applying Lemma 3.1, we conclude that equation $Lx=Nx$ has a solution $x=(x_1,x_2,x_3)^\top$ on $\overline{\Omega}\cap D(L)$, thus (1) has an ω-periodic solution $x(t)$.

Remark 3.1 If $\int_0^\omega e(t)\mathrm{d}t\neq 0$, $f(t,0)\neq 0$ and $g(t,0)\neq 0$, the problem of existence of an ω-periodic solution for (1) can be converted to the existence of an ω-periodic solution for the equation

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\big(x(t)-c(t)x\big(t-\delta(t)\big)\big)+f_1\big(t,\ddot{x}(t)\big)+g_1\big(t,\dot{x}(t)\big)+h_1\big(t,x\big(t-\tau(t)\big)\big)=e_1(t), \tag{19}$$

where $f_1(t,x)=f(t,x)-f(t,0)$, $g_1(t,x)=g(t,x)-g(t,0)$, $h_1(t,x)=h(t,x)+\int_0^\omega e(t)\mathrm{d}t+f(t,0)+g(t,0)$ and $e_1(t)=e(t)-\int_0^\omega e(t)\mathrm{d}t$. Clearly, $\int_0^\omega e_1(t)\mathrm{d}t=0$, $f_1(t,0)=0$ and $g_1(t,0)=0$. Therefore (19) can be discussed using Theorem 3.1.

4 Example

Example 4.1 Consider the following third-order neutral functional differential equation:

$$\begin{aligned}&\frac{\mathrm{d}^3}{\mathrm{d}t^3}\bigg(x(t)-\frac{1}{150}\sin 16t\cdot x\Big(t-\frac{1}{160}\sin 16t\Big)\bigg)+\cos 16t\sin\ddot{x}(t)\\&+\sin 16t\cos\dot{x}(t)+\frac{8}{\pi}x(t-\sin 16t)=\cos 16t.\end{aligned}\tag{20}$$

Comparing (20) to (1), we find $f(t,u)=\cos 16t\sin u$, $g(t,v)=\sin 16t\cos v$, $h(t,x)=\frac{8}{\pi}x$, $h(t,0)=0$, $m_o=\frac{8}{\pi}$, $c(t)=\frac{1}{150}\sin 16t$, $\delta(t)=\frac{1}{160}\sin 16t$, $\tau(t)=\sin 16t, e(t)=\cos 16t$ and let $\omega=\frac{\pi}{8}$.

Therefore we get

$$\begin{aligned}c_\infty&=\max_{t\in[0,\omega]}\big|c(t)\big|=\max_{t\in[0,\frac{\pi}{8}]}\Big|\frac{1}{150}\sin 16t\Big|=\frac{1}{150}<1,\\c_1&=\max_{t\in[0,\omega]}|\dot{c}(t)|=\max_{t\in[0,\frac{\pi}{8}]}\Big|\frac{16}{150}\cos 16t\Big|=\frac{8}{75},\\c_2&=\max_{t\in[0,\omega]}|\ddot{c}(t)|=\max_{t\in[0,\frac{\pi}{8}]}\Big|\frac{256}{150}\sin 16t\Big|=\frac{128}{75},\\\delta_1&=\max_{t\in[0,\omega]}|\dot{\delta}(t)|=\max_{t\in[0,\frac{\pi}{8}]}\Big|\frac{1}{10}\cos 16t\Big|=\frac{1}{10},\end{aligned}$$

$$\begin{aligned}
\delta_2 &= \max_{t\in[0,\omega]} |\ddot{\delta}(t)| = \max_{t\in[0,\frac{\pi}{8}]} \left|\frac{16}{10}\sin 16t\right| = \frac{8}{5},\\
M_1 &= 1+\frac{1}{2}c_1\omega + c_\infty + c_\infty\delta_1\\
&= 1+\frac{1}{2}\times\frac{8}{75}\times\frac{\pi}{8}+\frac{1}{150}+\frac{1}{150}\times\frac{1}{10}\simeq 1.0283,\\
M_5 &= \frac{1}{2}m_o\omega^2 M_1 = \frac{1}{2}\times\frac{8}{\pi}\times\left(\frac{\pi}{8}\right)^2\times 1.0283 \simeq 0.2019,\\
M_6 &= \frac{1}{2}\Big(\sqrt{M_5\omega}+\frac{1}{2}c_2\omega^2+2c_1\omega - c_\infty\delta_2\omega\Big)\\
&= \frac{1}{2}\Big[\Big(0.2019\times\frac{\pi}{8}\Big)^{\frac{1}{2}}+\frac{1}{2}\times\frac{128}{75}\times\Big(\frac{\pi}{8}\Big)^2\\
&\quad +2\times\frac{8}{75}\times\frac{\pi}{8}-\frac{1}{150}\times\frac{8}{5}\times\frac{\pi}{8}\Big]\\
&\simeq 0.2464.
\end{aligned}$$

We can choose $K_1 = 1$, $K_2 = 1$, $D > \frac{\pi}{8}$ and $m_o = \frac{8}{\pi}$ such that (H1)-(H4) hold. And

$$1 - c_\infty - c_\infty\delta_1(\delta_1 - 2) - M_6 = 0.7482 > 0.$$

To verify obtain (17), we calculate

$$\begin{aligned}
M_2 &= \big(K_1 + K_2 + m_oD + \max\{|h(t,0)| : 0 \le t \le \omega\} + \|e\|_\infty\big)\omega\\
&= (1+1+1+0+1)\times\frac{\pi}{8} = \frac{\pi}{2},\\
M_3 &= c_1DM_2 = \frac{8}{75}\times\frac{\pi}{8}\times\frac{\pi}{2} = 0.0658,\\
M_4 &= M_1M_2 + \frac{1}{2}m_o\omega^2c_1D = 1.0283\times\frac{\pi}{2}+\frac{1}{2}\times\frac{8}{\pi}\times\left(\frac{\pi}{8}\right)^2\times\frac{8}{75}\times\frac{\pi}{8}\\
&= 1.6152 + 0.0082 = 1.6234.
\end{aligned}$$

Then (17) becomes

$$0.7482\times\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t \le 0.4239 + 0.5646\Big(\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t\Big)^{\frac{1}{2}},$$

which can be considered as a quadratic inequality, whose roots are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{1}{2}(0.7546 \pm 1.6839).$$

From this, we obtain

$$\int_0^\omega \big|\ddot{x}_1(t)\big|\mathrm{d}t \le 1.4866.$$

The rest of the proof is clear. Hence, by Theorem 3.1, (20) has at least one $\frac{\pi}{8}$-periodic solution.

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