Ann. of Appl. Math.
34:3(2018), 319-330

ON THE CONDITIONAL EDGE CONNECTIVITY OF ENHANCED HYPERCUBE NETWORKS*†

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Abstract

Let $G = (V, E)$ be a connected graph and m be a positive integer, the conditional edge connectivity λ_δ^m is the minimum cardinality of a set of edges, if it exists, whose deletion disconnects G and leaves each remaining component with minimum degree δ no less than m. This study shows that $\lambda_\delta^1(Q_{n,k}) = 2n$, $\lambda_\delta^2(Q_{n,k}) = 4n - 4$ $(2 \leq k \leq n-1,\ n \geq 3)$ for n-dimensional enhanced hypercube $Q_{n,k}$. Meanwhile, another easy proof about $\lambda_\delta^2(Q_n) = 4n - 8$, for $n \geq 3$ is proposed. The results of enhanced hypercube include the cases of folded hypercube.

Keywords interconnected networks; connectivity; conditional edge connectivity; fault tolerance; enhanced hypercube

2000 Mathematics Subject Classification 05C40

1 Introduction

A multiprocessor system may contain numerous number of nodes, some of which may be faulty when the system is implemented. Reliability and fault tolerance are two of the most critical concerns of multiprocessor systems. Based on the definition proposed by Esfahanian [1], a multiprocessor system is fault tolerance if it can remain functional when failures occur. Two basic functionality criteria have received many attention. The first one is whether the network logically contains a certain topological structure. This problem occurs when embedding one architecture into another. This approach involves system wide redundancy and reconfiguration. The second functionality criterion considers a multiprocessor system function if a fault-free path exists between any two fault-free nodes. Hence, connectivity and edge connectivity are two important measurements of this criterion [2]. A vertex cut of a connected graph G is a set of vertices whose removal disconnects G. The connectivity $\kappa(G)$ of a connected graph G is the cardinality of a minimum vertex cut. An edge

*This project was supported by NSFC (11371162) and NSFC (11171172).
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cut of a connected graph G is a set of edges whose removal disconnects G. The edge connectivity $\lambda(G)$ of a connected graph G is the cardinality of a minimum edge cut [3]. However this two parameters may result in an isolated vertex. This is practically impossible in some network applications. To address this deficiency, many specific terms forbidden fault set and forbidden fault edge set are introduced such as conditional connectivity [4], extra connectivity and extra edge connectivity [5].

The n-dimensional hypercube Q_n is one of the most versatile and efficient interconnected networks because of its regular structure, small diameter, and good connection with a relative small node degree [2], all of which are very important for designing parallel systems. As the importance of hypercubes, many variants of Q_n have been proposed, among which, for instance, are crossed hypercube, argument hypercube, folded hypercube and enhanced hypercube. As an enhancement on the hypercube Q_n, the enhanced hypercube $Q_{n,k}$ proposed by Tzeng and Wei [6], not only retains some of the favorable properties of Q_n, but also improve the efficiency of the hypercube structure, since it possesses many properties superior to hypercube [7-9]. For example, the diameter of the enhanced hypercube is almost half of the hypercube. The hypercube is n-regular and n-connected, whereas the enhanced hypercube is $(n+1)$-regular and $(n+1)$-connected. Its special case of $k=1$ is the well-known Folded hypercube (denoted by FQ_n), which has been used as underlying topologies of several parallel systems, such as ATM switches [10,11] PM2I networks [12], and 3D-FoIHNOC networks [13] for high-speed cell-switching and reducing the diameter and traffic congestion of the hypercube with little hardware overhead.

The conditional edge connectivity λ_δ^m which is the generalization of edge connectivity, is defined as the cardinality of the minimum edge cut, if it exists, whose deletion disconnects G and leaves each component with minimum degree δ no less than m. Xu [2] provided $\kappa(Q_n) = \lambda(Qn) = n$, $\kappa(FQ_n) = \lambda(FQ_n) = n+1$ and $\kappa(Q_{n,k}) = \lambda(Q_{n,k}) = n+1$. Obviously, $\lambda_e^0(G) = \lambda(G)$. Paper [14] made a good job on the conditional edge connectivity of Q_n and proved that $\lambda_\delta^1(Q_n) = 2n-2$ for $n \geq 2$, $\lambda_\delta^2(Q_n) = 4n-8$ for $n \geq 3$ and $\lambda_\delta^2(FQ_n) = 4n-4$ for $n \geq 4$. However there is nothing about the conditional edge connectivity of enhanced hypercube. In this paper, we discuss the properties of enhanced hypercubes $Q_{n,k}$ and show that $\lambda_\delta^1(Q_{n,k}) = 2n$, $\lambda_\delta^2(Q_{n,k}) = 4n-4$ for $n \geq 3$ which includes the results of folded hypercube. Meanwhile, this paper proposes an easy proof of $\lambda_\delta^2(Q_n) = 4n-8$ for $n \geq 3$.

2 Preliminaries

The graph theoretical definitions and notations follow [2]. A network is usually modeled by a connected graph $G = (V, E)$, where V and E represent the vertex and edge sets of $G = (V, E)$ respectively. G is bipartite if V can be partitioned into two subsets V_1 and V_2, such that every edge in G joins a vertex of V_1 with a vertex in V_2. A bipartite graph is bipancyclic if it contains a cycle of every even length from 4 to $|V(G)|$, then $V_0 \cup V_1$ is a bipartition. A graph is said to be pancyclic if it contains a cycle of every length from 3 to $|V(G)|$. Two graphs G_1 and G_2 are isomorphic, denoted as $G_1 \cong G_2$, if there is a one to one mapping f from $V(G_1)$ to $V(G_2)$ such that $(u, v) \in E(G_1)$ if and only if $(f(u), f(v)) \in E(G_2)$. Two vertices which are incident with a common edge are adjacent. As are two edges which are incident with a common vertex. A path is a sequence of adjacent vertices, with the original vertex v_0 and end vertex v_m, represented as $P(v_0, v_m) = (v_0, v_1, v_2, \cdots, v_m)$ where all the vertices $v_0, v_1, v_2, \cdots, v_m$ are distinct except that possibly the path is a cycle when $v_0 = v_m$. A cycle is called a Hamiltonian cycle if it traverses every vertex of G exactly once. Two paths are internode disjoint if and only if they don't have any vertices in common. If $u, v \in V(G)$, $d(u, v)$ denotes the length of a shortest path between u and v. The diameter of G is defined as $\mathrm{diam}(G) = \max\{d(u, v) : u;, v \in V(G)\}$. Let $V_1 \subseteq V$ be a subset of V, $G[V_1] = (V_1, E_1)$ is an induced subgraph by V_1 such that for any edge $(x, y) \in E_1 \subseteq E$ if and only if $x, y \in V_1$. $G - V_1 = G[\overline{V}_1]$ is an induced subgraph by deleting both of all vertices in V_1 and all edges incident with vertices in V_1, where $\overline{V}_1 = V - V_1$. Let $F \subseteq E(G)$ be a subset of E, $G[F] = (V', F)$ is a subgraph induced by F satisfying $x \in V'$ if and only if x being incident with some edges of F, $G - F = G[\overline{F}]$ is a subgraph obtained by deleting all edges of F. For $X, Y \subset V$, denote by $[X, Y]$ the set of edges of $E(G)$ with one end in X and the other in Y, certainly, $[X, Y]$ is an edge cut of graph G.

An n-dimensional hypercube denoted by Q_n has 2^n vertices, and has vertex set $V(Q_n) = \{x_1x_2 \cdots x_n : x_i = 0 \text{ or } 1, 1 \le \mathrm{i} \le \mathrm{n}\}$, with two vertices $x_1x_2 \cdots x_n$ and $y_1y_2 \cdots y_n$ being adjacent if and only if they differ in exactly one bit. Let x and y be two vertices of hypercube Q_n, $d_{Q_n}(x, y)$ be the length of the shortest path between vertices x and y in hypercube Q_n. The Hamming distance between x and y, denoted by $h(x, y)$, is the number of different bits between the corresponding strings of x and y, that is the length of the distance of the shortest path between the x and y in Q_n. Obviously, by the definition of Hamming distance, we know that $h(x, y) = d_{Q_n}(x, y)$, where $d_{Q_n}(x, y)$ is the shortest path between the vertex x and node y in Q_n. The Hamming weight of a vertex x is defined as $hw(x) = \sum\limits_{i=1}^{n} x_i$, so whether a vertex is even or odd, bases on whether the hamming weight of the vertex is even or odd. In this paper, if we denote the node $x = x_1x_2 \cdots x_{i-1}x_ix_{i+1} \cdots x_n$ then $x^i = x_1x_2 \cdots x_{i-1}\bar{x}_ix_{i+1} \cdots x_n$, for some $i \in \{1, 2, \cdots, n\}$, in other words, the binary strings of the two vertices x and x^i is different exactly on the ith position. And the edge (x, x^i) represents the i-dimensional hypercube edge in Q_n. The set of

i-dimensional edges is denoted by $E_i = \{(x, x^i) |\, h(x, x^i) = 1, i \in \{1, 2, \cdots n\}\}$.

An n-dimensional enhanced hypercube $Q_{n,k}$ is obtained by adding some complementary edges from hypercube Q_n, and it can be defined as follows:

Definition 1 The enhanced hypercube $Q_{n,k} = (V, E)$ $(1 \leq k \leq n-1)$ is an undirected simple graph. It has the same vertices of Q_n, that is, $V = \{x_1x_2 \cdots x_n : x_i = 0 \text{ or } 1,\ 1 \leq i \leq n\}$. Two vertices $x = x_1x_2 \cdots x_n$ and y are connected by an edge of E if and only if y satisfies one of the following two conditions:

(1) $y = x_1x_2 \cdots x_{i-1}\bar{x}_i x_{i+1} \cdots x_n$, $1 \leq i \leq n$;

(2) $y = x_1x_2 \cdots x_{k-1}\bar{x}_k\bar{x}_{k+1} \cdots \bar{x}_n$.

From Definition 1, the enhanced hypercube $Q_{n,k}$ is the extension of the hypercube Q_n by adding the edges $(x_1x_2 \cdots x_n, x_1x_2 \cdots x_{k-1}\overline{x}_k\overline{x}_{k+1} \cdots x_n)$, which called complementary edges of enhanced hypercube, denoted by $E_c = \{(u, \overline{u}) \in E(Q_{n,k})|\, h(u, \overline{u}) = n - k + 1\}$, where $u = x_1x_2 \cdots x_n$, and $\overline{u} = x_1x_2 \cdots x_{k-1}\overline{x}_k\overline{x}_{k+1} \cdots x_n$. As mentioned above, $Q_{n,k}$ contains hypercube Q_n as its subgraph. In addition, the folded hypercube FQ_n, which is the extension of the hypercube, is regarded as the special case of the enhanced hypercube when $k = 1$. It has been showed that the enhanced hypercube $Q_{n,k}$ is $(n+1)$-regular, has 2n vertices and $(n+1)2^{n-1}$ edges, vertex-transitive and edge-transitive. Due to its good properties, the enhanced hypercube $Q_{n,k}$ have received substantial researches.

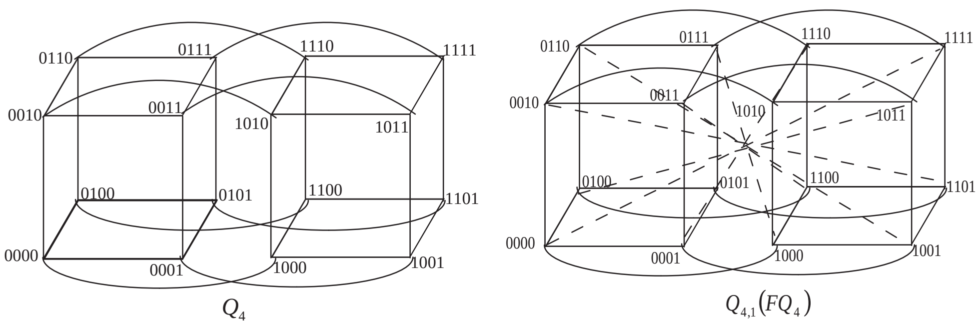

Figure 1: $Q_4, Q_{4,1}(FQ_4)$ (Dotted lines represent the complementary edges.)

Lemma 1[7] *The enhanced hypercube $Q_{n,k}$ can be partitioned into two subgraphs along some component i $(1 \leq i \leq n)$ such that $Q_{n,k} = Q^{i0}_{n-1,k} \cup Q^{i1}_{n-1,k}$. A vertex $x_1x_2 \cdots x_n$ belongs to $Q^{i0}_{n-1,k}$ if and only if the ith position $x_i = 0$; similarly, $x_1x_2 \cdots x_n$ belongs to $Q^{i1}_{n-1,k}$ if and only if the ith position $x_i = 1$. If $i < k$, $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ are two $(n-1)$-dimensional enhanced hypercubes; if $i \geq k$, $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ are two $(n-1)$-dimensional hypercubes.*

Guo made a good job in [14] and obtained that $\lambda^1_\delta(Q_n) = 2n - 2$ for $n \geq 2$, $\lambda^2_\delta(Q_n) = 4n-8$ for $n \geq 3$. Next, We will propose a simple proof for $\lambda^2_\delta(Q_n) = 4n-8$ for $n \geq 3$.

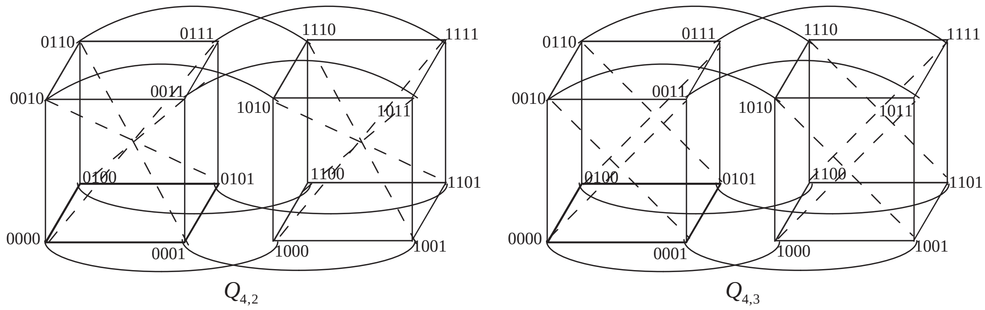

Figure 2: $Q_{4,2}, Q_{4,3}$ (Dotted lines represent the complementary edges.)

Lemma 2[14] (1) $\lambda_\delta^1(Q_n) = 2n-2$ *for* $n \geq 2$. (2) $\lambda_\delta^2(Q_n) = 4n-8$ *for* $n \geq 3$.

Proof Select a 4-cycle C_4, then $|[V(C_4), V - V(C_4)]| = 4n-8$, for any vertex $x \in C_4$, $d_{C_4}(x) = 2$ and $\delta(Q_n - C_4) \geq 2$, and $Q_n - C_4$ is connected. Therefore $\lambda_\delta^2(Q_n) \leq 4n-8$. In the following, we will prove $\lambda_\delta^2(Q_n) \geq 4n-8$ by induction.

When $n = 3$, it is easy to check that the conclusion is true. Suppose the conclusion is true for all positive integers less than n. We will consider the case of $n \geq 4$.

Let $F \subset E$ with $|F| \leq 4n-9$ such that each component of $Q_n - F$ has minimum degree greater than or equal to 2. We will prove $Q_n - F$ is connected.

Let $f_i = |E_i \cap F|$, $i = 1, 2, \cdots, n$, $f_{\min} = \min\{f_i : i = 1, 2, \cdots, n\}$. We will partition Q_n along dimension i into two $(n-1)$-dimensional hypercubes, denoted by Q_{n-1}^L and Q_{n-1}^R. Let $f_L = |E(Q_{n-1}^L) \cap F|$, $f_R = |E(Q_{n-1}^R) \cap F|$, then $|F| = f_L + f_R + f_{\min}$, without loss of generality, suppose $f_L \leq f_R$. Since $|F| \leq 4n-9$, $f_L \leq 2n-5 \leq 4n-12$ when $n \geq 4$. This leads to the conclusion that $Q_{n-1}^L - F$ is connected by induction. For convenience, letting $f_{\min} = f_1$, we partition Q_n along dimension 1. Since $|F| \leq 4n-9$, $f_{\min} \leq 3$, we will prove for any vertex $x \in Q_{n-1}^R - F$ that there exists a path connecting x and $Q_{n-1}^L - F$, which leads to $Q_n - F$ being connected.

Case 1 $f_{\min} = f_1 = 0$

For any vertex $x \in Q_{n-1}^R - F$, the edge $(x, x^1) \notin F$, so (x, x^1) is an edge joining x and $Q_{n-1}^L - F$.

Case 2 $f_{\min} = f_1 = 1$, $f_L + f_R \leq 4n-10$

For any vertex $x \in Q_{n-1}^R - F$, if edge $(x, x^1) \notin F$, (x, x^1) is an edge joining x and $Q_{n-1}^L - F$; if edge $(x, x^1) \in F$, since the minimum degree $\delta \geq 2$ for each component of $Q_n - F$, there are at least two edges, say $(x, x^2), (x, x^3) \in E(Q_{n-1}^R) - F$, that is $(x, x^2), (x, x^3) \notin F$, meanwhile $(x^2, x^{21}), (x^3, x^{31}) \notin F$, therefore $P(x, x^{21}) = xx^2x^{21}$ is a path connecting x and $Q_{n-1}^L - F$.

Case 3 $f_{\min} = f_1 = 2$

For any vertex $x \in Q_{n-1}^R - F$, if edge $(x, x^1) \notin F$, (x, x^1) is an edge joining x and $Q_{n-1}^L - F$; if edge $(x, x^1) \in F$, since $f_{\min} = f_1 = 2$, and $n > 4$, $\{(x^2, x^{21}), (x^3, x^{31}), \cdots, (x^n, x^{n1})\} \not\subset F$, say $(x^2, x^{21}) \in E(Q_{n-1}^R) - F$, that is $(x^2, x^{21}) \notin F$, therefore $P(x, x^{21}) = xx^2x^{21}$ is a required path connecting x and $Q_{n-1}^L - F$.

Case 4 $f_{\min} = f_1 = 3$

Since $n \geq 4$, $\{(x, x^1), (x^2, x^{21}), \cdots, (x^n, x^{n1})\} \subset E_1$, there is $\{(x, x^1), (x^2, x^{21}), \cdots, (x^n, x^{n1})\} \not\subset F$, that is, either (x, x^1) is an edge joining x and $Q_{n-1}^L - F$, or $P(x, x^{i1}) = xx^ix^{i1}$ $(2 \leq i \leq n)$ is a required path connecting x and $Q_{n-1}^L - F$.

The proof is complete.

3 Main Results

In this section, we demonstrate the conditional edge connectivity for enhanced hypercube networks.

Theorem 1 $\lambda_\delta^1(Q_{n,k}) = 2n,\ for\ 2 \leq k \leq n-1,\ n \geq 3.$

Proof Since the conditional edge connectivity λ_δ^k is the cardinality of the minimum edge cut such that each component after removing the edge cut has minimum degree no less than k. This theorem means that there exists an edge cut F with $|F| = 2n$ such that $Q_{n,k} - F$ is disconnected and each component of $Q_{n,k} - F$ has minimum degree $\delta \geq 1$.

If $n = 3$, then $k = 2$ or $k = 1$. It is easy to check $\lambda_\delta^1(Q_{3,2}) = 6$ and $\lambda_\delta^1(Q_{3,1}) = 6$. We just consider $n \geq 4$.

Take a path P_1 of length one, that is an edge in $Q_{n,k}$, then $[V(P_1), V(Q_{n,k} - P_1)]$ is an edge cut of $Q_{n,k}$ with $|[V(P_1), V(Q_{n,k} - P_1)]| = 2n$, and for any vertex $x \in V(P_1)$, $d_{P_1}(x) = 1$. The connectivity and the edge connectivity of $Q_{n,k}$ are $n+1$, that is $\lambda(Q_{n,k}) = \kappa(Q_{n,k}) = n+1$, so $Q_{n,k} - P_1$ is connected, therefore $\delta(Q_{n,k} - P_1) \geq 1$. This leads to $\lambda_\delta^1(Q_{n,k}) \leq 2n$.

On the other hand, the proof of $\lambda_\delta^1(Q_{n,k}) \geq 2n$ is needed. This will be proved by showing that for any edge subset $F \subseteq E(Q_{n,k})$ with $|F| \leq 2n-1$, $Q_{n,k} - F$ is connected.

According to Lemma 1, When $i < k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional enhanced hypercubes $Q_{n-1,k}^{i0}$ and $Q_{n-1,k}^{i1}$ along some components i. When $i \geq k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional hypercubes Q_{n-1}^{i0} and Q_{n-1}^{i1} along some components i. In the following, let $f_L = |F \cap E(Q_{n-1,k}^{i0})|$ and $f_R = |F \cap E(Q_{n-1,k}^{i1})|$, then $f_L + f_R = |F| - |[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})] \cap F|$. For convenience, suppose that $f_L \leq f_R$. There are several cases needed to be considered according to the distribution of edges in F.

Case 1 Assume that there exists some dimension $i \in \{1, 2, \cdots, n\}$ such that $f_L \leq f_R \leq 2n-5$. If $i \geq k$, $Q_{n,k} = Q_{n-1}^{i0} \cup Q_{n-1}^{i1}$, then $[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})] =$

$E_i \cup E_c$, $|[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})]| = |E_i \cup E_c| = 2^n > 2n-1$ for $n \geq 4$. If $i < k$, $Q_{n,k} = Q^{i0}_{n-1,k} \cup Q^{i1}_{n-1,k}$, then $[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})] = E_i$, $|[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})]| = |E_i| = 2^{n-1} > 2n-1$ for $n > 4$.

No matter whatever cases occur, since $f_L \leq f_R \leq 2n-5$, $Q^{i0}_{n-1,k} - F$ and $Q^{i1}_{n-1,k} - F$ are connected. Furthermore there exists some vertex $x \in Q^{i1}_{n-1,k} - F$ such that the edge $(x, x^i) \in E_i - F$. Noting that $x^i \in Q^{i0}_{n-1,k}$, $Q_{n,k} - F$ is connected, hence $\lambda^1_\delta(Q_{n,k}) \geq 2n$.

<u>Case 2</u> If there exists some dimension $i \in \{k, k+1, \cdots, n\}$ such that $|E_i \cap F| \geq 4$. Since $i \geq k$, $Q_{n,k} = Q^{i0}_{n-1} \cup Q^{i1}_{n-1}$. Now $f_L + f_R = |F| - |E_i \cap F| \leq (2n-1) - 4 = 2n-5$, then $f_L \leq f_R \leq 2n-5$. So $Q^{i0}_{n-1,k} - F$ and $Q^{i1}_{n-1,k} - F$ are connected by Lemma 1. Since $[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})] = E_i \cup E_c$, $|[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})]| = |E_i \cup E_c| = 2^n > 2n-1$ for $n \geq 4$. Then there exists some vertex $x \in Q^{i1}_{n-1,k} - F$ joined to $Q^{i0}_{n-1,k} - F$ by edge $(x, x^i) \in E_i - F$ or edge $(x, \overline{x}) \in E_c - F$. Thus $Q_{n,k} - F$ is connected and $\lambda^1_\delta(Q_{n,k}) \geq 2n$.

<u>Case 3</u> If there exists some dimension $i \in \{k, k+1, \cdots, n\}$ such that $|E_i \cap F| = 3$. Since $i \geq k$, $Q_{n,k} = Q^{i0}_{n-1,k} \cup Q^{i1}_{n-1,k}$, $Q^{i0}_{n-1,k} \cong Q^{i0}_{n-1}$, $Q^{i1}_{n-1,k} \cong Q^{i1}_{n-1}$, We have the following subcases:

<u>Case 3.1</u> If $f_L \leq f_R \leq 2n-5$, the proof is similar to Case 2.

<u>Case 3.2</u> If $f_R = 2n-4$, then $Q^{i0}_{n,k} - F$ is connected. For any vertex $x \in Q^{i1}_{n-1,k} - F$, when $\{(x, x^i), (x, \overline{x})\} \not\subset F$, say $(x, x^i) \notin F$, the vertex x is joined to $Q^{i0}_{n-1,k} - F$ by the edge $(x, x^i) \in E_i - F$.

When $\{(x, x^i), (x, \overline{x})\} \subset F$, since $d_{Q^{i0}_{n-1,k} - F}(x) \geq 1$, there exists at least one edge $(x, y) \in Q^{i1}_{n-1,k} - F$. Because $|E_i \cap F| = 3$, $\{(y, y^i), (y, \overline{y})\} \not\subset F$. Note that $y^i, \overline{y} \in Q^{i1}_{n-1,k} - F$, therefore either xyy^i or $xy\overline{y}$ is a path between x and $Q^{i0}_{n-1,k} - F$. Thus $Q_{n,k} - F$ is connected and $\lambda^1_\delta(Q_{n,k}) \geq 2n$.

<u>Case 4</u> If there exists some dimension $i \in \{k, k+1, \cdots, n\}$ such that $|E_i \cap F| = 2$. Since $i \geq k$, $Q_{n,k} = Q^{i0}_{n-1,k} \cup Q^{i1}_{n-1,k}$, $Q^{i0}_{n-1,k} \cong Q^{i0}_{n-1}$, $Q^{i1}_{n-1,k} \cong Q^{i1}_{n-1}$. We have the following subcases:

<u>Case 4.1</u> If $f_L \leq f_R \leq 2n-5$, similar to Case 2, the the conclusion is true.

<u>Case 4.2</u> If $f_L \leq 1, f_R \geq 2n-4$, then $Q^{i0}_{n,k} - F$ is connected. For any vertex $x \in Q^{i1}_{n-1,k} - F$, when $\{(x, x^i), (x, \overline{x})\} \not\subset F$, say $(x, x^i) \notin F$, the vertex x is joined to $Q^{i0}_{n-1,k} - F$ by the edge $(x, x^i) \in E_i - F$.

When $\{(x, x^i), (x, \overline{x})\} \subset F$, since $d_{Q^{i0}_{n,k} - F}(x) \geq 1$, there exists at least one edge $(x, y) \in Q^{i1}_{n-1,k} - F$. Because $|E_i \cap F| = 2$, $E_i \cap F = \{(x, x^i), (x, \overline{x})\}$, $(y, y^i), (y, \overline{y}) \notin F$. Note that $y^i, \overline{y} \in Q^{i1}_{n-1,k} - F$, so xyy^i and $xy\overline{y}$ are paths connecting x and $Q^{i0}_{n-1,k} - F$. Thus $Q_{n,k} - F$ is connected and $\lambda^1_\delta(Q_{n,k}) \geq 2n$.

<u>Case 5</u> If there exists some dimension $i \in \{1, 2, \cdots, k-1\}$ such that $|E_i \cap F| \geq 2$.

Since $i<k$, $Q_{n,k}=Q^{i0}_{n-1,k}\cup Q^{i1}_{n-1,k}$. We consider the following subcases:

Case 5.1 If $|E_i\cap F|\geq 3$, then $f_L\leq f_R\leq 2n-5$, similar to Case 1, the conclusion is true.

Case 5.2 If $|E_i\cap F|=2$, in this case, we consider the following subcases:

Case 5.2.1 If $f_L\leq f_R\leq 2n-5$, since $Q_{n,k}-E_c\cong Q_n$, by Lemma 2, $Q^{i0}_{n-1,k}-F$ and $Q^{i1}_{n-1,k}-F$ are connected. For any vertex $x\in Q^{i1}_{n-1,k}-F$, if $(x,x^i)\notin F$, then the vertex x is joined to $Q^{i0}_{n-1,k}-F$ by the edge (x,x^i). If $(x,x^i)\in F$, since $d_{Q^{i0}_{n-1,k}-F}(x)\geq 1$, there exist some edges, say $(x,y)\in Q^{i1}_{n-1,k}-F$. If $(y,y^i)\notin F$, then xyy^i is a path joining x and $Q^{i0}_{n-1,k}-F$. Noting $\{(x,x^j)\cup(y,y^j)\}\subset Q^{i1}_{n-1,k}$, $j=1,2,\cdots,n,\ j\neq i$, if $\{(x,x^j),(y,y^j)\}-(x,y)\subset F$, then $|\{(x,x^j),(y,y^j)\}|-1=2n-3>2n-5$, that is, there exists a $(y,y^i)\notin F$. Since $|E_i\cap F|=2$, then $(y^j,y^{ji})\notin F$. Therefore xyy^jy^{ji} is a path connecting x and $Q^{i0}_{n-1,k}-F$. Thus $Q_{n,k}-F$ is connected and $\lambda^1_\delta(Q_{n,k})\geq 2n$.

Case 5.2.2 If $f_L\leq 1, f_R\geq 2n-4$, then $Q^{i0}_{n,k}-F$ is connected. For convenience, let $i=1$, then $Q_{n,k}=Q^{10}_{n-1,k}\cup Q^{11}_{n-1,k}$. For any vertex $x\in Q^{11}_{n-1,k}-F$, when $(x,x^1)\notin F$, the vertex x is joined to $Q^{10}_{n-1,k}-F$ by the edge (x,x^1).

When $(x,x^1)\in F$, since $d_{Q^{11}_{n-1,k}-F}(x^{'})\geq 1$, there exists a $j\in\{2,3,\cdots,n\}$, say, $j=2$, $(x,x^2)\notin F$. If $(x^2,x^{21})\notin F$, x can be connected to $Q^{10}_{n-1,k}-F$ by the path xx^2x^{21}. If $(x^2,x^{21})\in F$, then $|E_i\cap F|=\{(x,x^1),(x^2,x^{21})\}$. Noting $\{(x,x^j)\}\cup\{(y,y^j)\}\cup\{(x,\overline{x}),(x^2,\overline{x^2})\}\subset Q^{11}_{n,k}$, $j=3,4,\cdots,n$, and the edges $(x^2,x^{22}),(x^2,x)$ are the same, if $\{(x,x^j)\}\cup\{(y,y^j)\}\cup\{(x,\overline{x}),(x^2,\overline{x^2})\}\subset E(Q^{11}_{n,k})\cap F$, then $f_R=|E(Q^{11}_{n-1,k})|\cap F>\{(x,x^j)\}\cup\{(y,y^j)\}\cup\{(x,\overline{x}),(x^2,\overline{x^2})\}=2n-2$. When $f_L=1$, $f_R=2n-4$; when $f_L=0$, $f_R=2n-3$, which implies that there exist some edges $(x^2,x^{2j})\notin F$ or $(x^2,\overline{x^2})\notin F$. Hence either $xx^2x^{2j}x^{2j1}$ or $xx^2\overline{x^2}$ is a path connecting x and $Q^{10}_{n-1,k}-F$. Thus $Q_{n,k}-F$ is connected and $\lambda^1_\delta(Q_{n,k})\geq 2n$.

By analyzing the above cases, the proof of the theorem is complete.

Then we will study $\lambda^2_\delta(Q_{n,k})$ and we have the following theorem.

Theorem 2 $\lambda^2_\delta(Q_{n,k})=4n-4,\ for\ 2\leq k\leq n-1,\ n\geq 3.$

Proof It is straightforward to prove the theorem when $n=3$ or 4. In the following, we just verify the conclusion $n\geq 5$ in $Q_{n,k}$.

Select a 4-cycle C_4 in $Q_{n,k}$, then $|[V(C_4),V(Q_{n,k}-C_4)]|=4n-4$. For any vertex $x\in C_4$, $d_{C_4}(x)=2$, and for any vertex $x\in Q_{n,k}-C_4$, $d_{Q_{n,k}-C_4}(x)\geq 2$, so $Q_{n,k}-C_4$ is connected. This leads to $\lambda^2_\delta(Q_{n,k})\leq 4n-4$.

On the other hand, we need to prove $\lambda^2_\delta(Q_{n,k})\geq 4n-4$. This will be proved by considering $F\subset E(Q_{n,k})$ with $|F|\leq 4n-5$, and $Q_{n-k}-F$ is connected.

According to Lemma 1, when $i<k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional enhanced hypercubes $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ along some components i;

when $i \geq k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional hypercubes Q_{n-1}^{i0} and Q_{n-1}^{i1} along some components i. In the following, let $f_L = |F \cap E(Q_{n-1,k}^{i0})|$ and $f_R = |F \cap E(Q_{n-1,k}^{i1})|$, then $f_L + f_R = |F| - |[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})] \cap F|$. For convenience, suppose that $f_L \leq f_R$. There are several cases needed to be considered according to the distribution of edges in F.

Case 1 If there exists some dimension $i \in \{1, 2, \cdots, n\}$ such that $f_L \leq f_R \leq 4n-9$, we consider the following subcases.

Case 1.1 When $i < k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional enhanced hypercubes along some components i, that is $Q_{n,k} = Q_{n-1,k}^{i0} \cup Q_{n-1,k}^{i1}$, then $[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})] = E_i$, $|[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})]| = |E_i| = 2^{n-1} > 4n-5$ for $n > 4$. Since $i \leq k-1$, for convenience, letting $i = 1$, for any vertex $x \in Q_{n-1,k}^{11} - F$, we will find a path joining x and $Q_{n-1,k}^{10} - F$.

Case 1.1.1 If the edge $(x, x^1) \notin F$, (x, x^1) is an edge joining x and $Q_{n-1,k}^{10} - F$.

Case 1.1.2 If the edge $(x, x^1) \in F$, $A = \{(x, x^j), (x^j, x^{j1}) : j = 2, 3, \cdots, n\} \cap F$. If $|A| < n-1$, then there exists some $j \in \{2, 3, \cdots, n\}$ such that (x, x^j) and (x^j, x^{j1}) are not in F. Therefore xx^jx^{j1} is a required path joining x and $Q_{n-1,k}^{10} - F$. We claim $(x, \overline{x})$ is not in F, then $x\overline{x}$ is a required path joining x and $Q_{n-1,k}^{10} - F$. Otherwise, since $d_{Q_{n,k}^{11}-F}(x) \geq 2$, that is (x, x^2) and (x, x^3) are not in F, but $(x^{21}, x^2), (x^3, x^{31}) \in F$ and $|\{(x, x^j), (x^j, x^{j1}) : j = 3, 4, \cdots, n\} \cap F| \geq n-3$. Let $B = \{(x^2, x^{2j}), (x^{2j}, x^{2j1}) : j = 4, 5, \cdots, n\} \cap F$, $C = \{(x^3, x^{3j}), (x^{3j}, x^{3j1}) : j = 4, 5, \cdots, n\} \cap F$. If $|B| < n-3$ or $|C| < n-3$, the edge $(x^2, \overline{x^2})$ and $(x^3, \overline{x^3})$ are not in F. For example, the edge $(x^2, x^{2j}), (x^{2j}, x^{2j1}) \notin F$, then $xx^2x^{2j}x^{2j1}$ is a required path joining x and $Q_{n-1,k}^{10} - F$. Otherwise, $|B| \geq n-3$, $|C| \geq n-3$, $\{(x^2, \overline{x^2}), (x^3, \overline{x^3})\} \subset F$. Since $d_{Q_{n,k}^{11}-F}(x) \geq 2$, $d_{Q_{n,k}^{11}-F}(x^{2j}) \geq 2$ and $d_{Q_{n,k}^{11}-F}(x^j) \geq 2$. Assume that the edge $(x^2, x^{2s}), (x^3, x^{3t}) \notin F$. Let $D = \{(x^{2s}, x^{2sj}), (x^{2sj}, x^{2sj1}) : j = 1, 2, s\} \cap F$, $H = \{(x^{3t}, x^{3tj}), (x^{3tj}, x^{3tj1}) : j = 1, 2, t\} \cap F$. If $|C| < n-3$ or $|D| < n-3$, the edge $(x^{2s}, \overline{x^{2s}})$ and $(x^{3t}, \overline{x^{3t}})$ are not in F, then there exists a path $xx^2x^{2s}x^{2sj}x^{2sj1}$ joining x and $Q_{n-1,k}^{10} - F$. Otherwise $|A \cup B \cup C \cup D \cup \{(x, x^1), (x, \overline{x}), (x^2, x^{21}), (x^2, \overline{x^2}), (x^3, x^{31}), (x^3, \overline{x^3}), (x^{2s}, \overline{x^{2s}}), (x^{3t}, \overline{x^{3t}})\}| \geq 4n-2 > |F|$, which is impossible. Hence, For any vertex $x \in Q_{n,k}^{11} - F$, there exists a path joining x and $Q_{n-1,k}^{10} - F$.

Case 1.2 When $i \geq k$, $Q_{n,k}$ can be partitioned into $(n-1)$-dimensional hypercubes along some components i, that is $Q_{n,k} = Q_{n-1,k}^{i0} \cup Q_{n-1,k}^{i1}$, $Q_{n-1,k}^{i0} \cong Q_{n-1}^{i0}$, $Q_{n-1,k}^{i1} \cong Q_{n-1}^{i1}$, then $[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})] = E_i \cup E_c$ and $|[V(Q_{n-1,k}^{i0}), V(Q_{n-1,k}^{i1})]| = |E_i \cup E_c| = 2^n > 4n-5$ for $n \geq 4$. Since $i \geq k$, for convenience, let $i = n$, then for any vertex $x \in Q_{n-1,k}^{n1} - F$, we will find a path joining x and $Q_{n-1,k}^{n0} - F$.

Case 1.2.1 If the edge (x, x^n) or $(x, \overline{x})$ not in F, the required path is (x, x^n) or $(x, \overline{x})$ joining x and $Q^{n0}_{n-1,k} - F$.

Case 1.2.2 If the edge $\{(x, x^n), (x, \overline{x})\} \subset F$, if there exist edges $(x, x^j), (x^j, x^{jn}) \notin F$ or $(x, x^j), (x^j, \overline{x^j}) \notin F$, then the required path is xx^jx^{jn} or $xx^j\overline{x^j}$. Otherwise, let $\{(x, x^1), (x, x^2)\} \not\subset F$, but $\{(x^1, x^{1n}), (x^1, \overline{x^1}), (x^2, x^{2n}), (x^2, \overline{x^2})\} \subset F$. Let $A = \{(x, x^j), (x^j, x^{jn}), (x^j, \overline{x^j}) : j \neq 1, 2\} \cap F$, then $|A| > 2$. Suppose $\{(x^1, x^{1s}), (x^2, x^{2t})\} \not\subset F$, then there exist both $(x^{1s}, x^{1sp}) \notin F$ and $(x^{1sp}, \overline{x^{1sp}}) \notin F$, or both $(x^{1s}, x^{1sp}) \notin F$ and $(x^{1sp}, x^{1spn}) \notin F$, or both $(x^{2t}, x^{2tq}) \notin F$ and $(x^{2tq}, \overline{x^{2tq}}) \notin F$, or both $(x^{2t}, x^{2tq}) \notin F$ and $(x^{2tq}, x^{2tqn}) \notin F$, for example, both $(x^{1s}, x^{1sp}) \notin F$ and $(x^{1sp}, \overline{x^{1sp}})$ $\notin F$, then $xx^1x^{1s}x^{1sp}\overline{x^{1sp}}$ is the required path joining x and $Q^{n0}_{n-1,k} - F$. Otherwise $\{(x^{1s}, x^{1sp}), (x^{1s}, \overline{x^{1s}}), (x^{2t}, x^{2tn}), (x^{2t}, \overline{x^{2t}})\} \subset F$, let $B = \{(x^{1s}, x^{1sj}), (x^{1sj}, x^{1sjn}), (x^{1sj}, \overline{x^{1sj}}) : j \neq 1, 2, s\} \cap F$ and $C = \{(x^{2t}, x^{2tj}), (x^{2tj}, x^{2tjn}), (x^{2tj}, \overline{x^{2tj}}) : j \neq 1, 2, t\} \cap F$, then $|B| \geq n-3$ and $|C| \geq n-3$. Since $d_{Q^{n1}_{n-1,k}-F}(x) \geq 2$, for any vertex $x \in Q^{n1}_{n-1,k}$, suppose edge $(x^{1s}, x^{1sp}) \notin F$ and $(x^{2t}, x^{2tq}) \notin F$. If $(x^{1sp}, x^{1spn}) \notin F$, then $xx^1x^{1s}x^{1sp}x^{1spn}$ is the required path joining x and $Q^{n0}_{n-1,k} - F$. If $(x^{1sp}, \overline{x^{1sp}}) \notin F$, then $xx^1x^{1s}x^{1sp}\overline{x^{1sp}}$ is the required path joining x and $Q^{n0}_{n-1,k} - F$. Otherwise, if there exists some j such that both $(x^{1sp}, x^{1spj}) \notin F$ and $(x^{1spj}, x^{1spj}) \notin F$, or both $(x^{1sp}, x^{1spj}) \notin F$ and $(x^{1spj}, \overline{x^{1spj}}) \notin F$, then $xx^1x^{1s}x^{1sp}x^{1spj}x^{1spjn}$ or $xx^1x^{1s}x^{1sp}x^{1spj}\overline{x^{1spj}}$ is the required path joining x and $Q^{n0}_{n-1,k} - F$.

Case 2 If there exists some dimension $i \in \{1, 2, \cdots, n\}$ such that $f_R \geq 4n - 8$, then $f_L + |[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})]| \cap F \leq 3$, so $Q^{i0}_{n-1,k} - F$ is connected. We consider the following two subcases:

Case 2.1 When $i \geq k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional hypercubes along some components i, that is $Q_{n,k} = Q^{i0}_{n-1,k} \cup Q^{i1}_{n-1,k}$, $Q^{i0}_{n-1,k} \cong Q^{i0}_{n-1}$ and $Q^{i1}_{n-1,k} \cong Q^{i1}_{n-1}$, then $[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})] = E_i \cup E_c$. For any vertex $x \in Q^{i1}_{n-1,k} - F$, if edges $\{(x, x^i), (x, \overline{x})\} \not\subset F$, the conclusion is true. If edges $\{(x, x^i), (x, \overline{x})\} \subset F$, since $d_{Q^{i1}_{n-1,k}-F}(x) \geq 2$, for any vertex $x \in Q^{i1}_{n-1,k}$, we can suppose edge $(x, x^s) \notin F$ and $(x, x^t) \notin F$. Because $f_L + |[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})]| \cap F \leq 3$, $\{(x^s, x^{si}), (x^s, \overline{x^s}), (x^t, x^{ti}), (x^t, \overline{x^t})\} \not\subset F$, say $(x^s, x^{si}) \notin F$, so xx^sx^{si} is the required path joining x and $Q^{i0}_{n-1,k} - F$.

Case 2.2 When $i < k$, $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional enhanced hypercubes along some components i, that is $Q_{n,k} = Q^{i0}_{n-1,k} \cup Q^{i1}_{n-1,k}$, then $[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})] = E_i$. For any vertex $x \in Q^{i1}_{n-1,k}$, if edge $(x, x^i) \notin F$, the conclusion is true. If edges $(x, x^i) \in F$, since $d_{Q^{i1}_{n-1,k}-F}(x) \geq 2$, for any vertex $x \in Q^{i1}_{n-1,k} - F$, then $|\{(x, x^j), (x, \overline{x}) : i \neq j\} - F| \geq 2$, say $(x, x^s), (x, x^t) \notin F$. If $\{(x^s, x^{si}), (x^t, x^{ti})\} \not\subset F$, let $(x^s, x^{si}) \notin F$, then xx^sx^{si} is the required path joining x and $Q^{i0}_{n-1,k} - F$. If $\{(x^s, x^{si}), (x^t, x^{ti})\} \subset F$, since $d_{Q^{i1}_{n-1,k}-F}(x^s) \geq 2$, say $(x^s, x^{sp}) \notin$

F. Because $f_L + |[V(Q^{i0}_{n-1,k}), V(Q^{i1}_{n-1,k})]| \cap F \le 3$ and $\{(x, x^i), (x^s, x^{si}), (x^t, x^{ti})\} \subset F$, we obtain $(x^{sp}, x^{spj}) \notin F$, and $xx^s x^{sp} x^{spj}$ is the required path joining x and $Q^{i0}_{n-1,k} - F$. So $\lambda^2_\delta(Q_{n,k}) \ge 4n-4$. Therefore $\lambda^2_\delta(Q_{n,k}) = 4n-4$, for $2 \le k \le n-1$, $n \ge 3$.

By analyzing the above cases, the proof of the theorem is complete.

4 Conclusion

Network topology is an important issue in the design of computer networks since it is crucial to many key properties such as the efficiency and fault tolerance. We consider the conditional edge connectivity for hypercube and enhanced hypercube networks. This parameter is the generalization of edge connectivity of graph, and can reflect the real cases more better. The results show that at least $2n$ edges must be removed to disconnect the n-dimensional enhanced hypercube networks, provided that the removal of these edges will preserve the each component having minimum degree one, and at least $4n-4$ edges must be removed to disconnect the n-dimensional enhanced hypercube networks, provided that the removal of these edges will preserve the each component having minimum degree two.

This paper reveals the conditional edge connectivity of enhanced hypercube. The study of other conditional connectivity in the enhanced hypercube is one of our further projects.

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(*edited by Liangwei Huang*)