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# SOME PROBLEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN FIELD THEORY*

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**Abstract**

This paper is a brief introduction to Yang-Mills-Higgs model, Maxwell-Higgs model, Einstein's vacuum model, Yang-Baxter model, Chern-Simons-Higgs model and a discussion of the associated partial differential equation problems.

**Keywords** field theory; nonlinear analysis; partial differential equations; topological invariants

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## 1 Introduction

There are many interesting and challenging problems in the area of classical field theory. Classical field theory offers all types of differential equation problems which come from the two basic sets of equations in physics describing fundamental interactions, namely, the Yang-Mills equations [1,2], governing electromagnetic, weak, and strong forces, reflecting internal symmetry, the Einstein equations governing gravity, and reflecting external symmetry.

It is well known that many important physical phenomena are the consequences of various levels of symmetry breakings, internal or external, or both. These phenomena are manifested through the presence of locally concentrated solutions of the corresponding governing equations, giving rise to physical entities such as electric point charges, gravitational blackholes, cosmic strings, superconducting vortices, monopoles, dyons, and instantons. The study of these types of solutions, commonly referred to as solitons due to their particle-like behavior in interactions.

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The main purpose of this paper is to provide a quick and self-contained mathematical introduction to field theory. In particular, we shall see the origins of some important physical quantities such as energy, momentum, charges, and currents. In Section 2, we introduce the Yang-Mills theory, and give the existence of solution in different conditions. In Section 3, we discuss the existence and instability of solution of the Maxwell-Higgs equation respectively. In Section 4, we consider the solution of the initial value problem to Einstein's vacuum equation. In Sections 5 and 6, we introduce the Yang-Baxter equation and Chern-Simons-Higgs equation briefly.

## 2 Yang-Mills-Higgs Equation

Yang-Mills theory is a gauge theory based on the $SU(N)$ group, or more generally any compact, reductive Lie algebra. From a mathematical point of view, the gauge field is equivalent to the connection between the principal and the slave, and the material field is equivalent to the cross section of the vector cluster. The self-dual Yang-Mills equation can be deduced the integrable system.

By introducing physical variable $(t,x)$=$(t,x_1,x_2,x_3,x_4)$, $r$=$|x|$, define derivative

$$\partial^0=\partial_t=\frac{\partial}{\partial t},\quad \partial^k=\frac{\partial}{\partial x_k},\quad \partial_r=\frac{\partial}{\partial r}=\sum_{k=1}^{3}\frac{x_k}{r}\partial^k.$$

Yang-Mills potential is $A^\mu=A^\mu(x,t)$ $(\mu=0,1,2,3)$, and the field functions are

$$\mathrm{E}^k=\partial^k A^0+\partial^0 A^k+A^k\times A^0,\quad H^k=\varepsilon_{ijk}\Big(\partial^j A^i+\frac{1}{2}A^j\times A^i\Big),\quad k,i,j=1,2,3,$$

where $\varepsilon_{ijk}$ is the convertible symbol and $\varepsilon_{123}=1$. The covariant derivative is

$$D^0=\partial^0=A^0x,\quad D^k=\partial^k+A^kx,\quad k=1,2,3.$$

For three-dimensional space vector $\phi=\phi(x,t)$ (Higgs field), set $\psi^\mu=D^\mu\phi$, then Lagrange density [3] is

$$\mathcal{L}=\frac{1}{2}\Bigg\{\sum_{k=1}^{3}|\mathrm{E}^k|^2-\sum_{k=1}^{3}|H^k|^2-m^2\Bigg(|A^0|^2-\sum_{k=1}^{3}|A^k|^2\Bigg)+|D^0\phi|^2-\sum_{k=1}^{3}|D^k\phi|^2\Bigg\}-V(\phi),\tag{2.1}$$

where $V(\phi)=V_0(|\phi|^2)$, $V_0$ is a real function of real variable. Let $V'(\phi)=2\phi V_0'(|\phi|^2)$, where $V_0'$ is the derivative of $V$, the motion equations of $\mathcal{L}$ has the following form:

$$\begin{cases} D^0\mathrm{E}^i=\varepsilon_{ijk}D^jH^k-m^2A^i+\psi^i\times\phi,\quad i=1,2,3,\\ \displaystyle\sum_{k=1}^{3}D^kH^k=\psi^0\times\phi+m^2A^0,\\ \displaystyle D^0\psi^0-\sum_{k=1}^{3}D^k\psi^k=-V'(\phi).\end{cases}\tag{2.2}$$

By the definitions of $\mathrm{E}^i$, $H^i$ and Jacobi identity, we can find the following equations

$$D^0H^0 = -\varepsilon_{ijk}D^j\mathrm{E}^k, \quad i,j,k = 1,2,3, \tag{2.3}$$

$$\sum_{k=1}^{3} D^kH^k = 0, \tag{2.4}$$

$$D^0\psi * -D^j\psi^0 = \mathrm{E}^j \times \phi, \quad i = 1,2,3, \tag{2.5}$$

$$\varepsilon_{ijk}D^i\psi^j = \phi \times H^k, \quad i,j,k = 1,2,3, \tag{2.6}$$

where $e^1 = (1,0,0)$, $e^2 = (0,1,0)$, $e^3 = (0,0,1)$, $w = \frac{x}{r}$, $v^k = e^k \times w$ $(k = 1,2,3)$, $\|\phi\|_2 = \big(\int_{\mathbb{R}^3} |\phi|^2 \mathrm{d}x\big)^2$.

There is the following result.

**Theorem 2.1** *Assume that*

(i) $V(\phi) = c_1|\phi|^2 + c_2|\phi|^4$, *where* $c_1$ *and* $c_2$ *are nonnegative constants,*

(ii) $\alpha_0, \beta_0 \in \widetilde{H}_1^r \cap L_r^4$, $\alpha_1, \beta_1 \in L_r^2$, $m = 0$, $c_2 = 0$,

*or*

(iii) $\alpha_0, \beta_0 \in H_1^r \cap L_r^4$, $\alpha_1, \beta_1 \in L_r^2$, $m > 0$, $c_2 > 0$,

*thus under conditions* (i)(ii) *or* (i)(iii), *Yang-Mills-Higgs equations* (2.2)-(2.6) *have a unique solution satisfying*

$$A^0(x,t) = 0, \quad A^k(x,t) = \alpha(r,t)v^k \ (k = 1,2), \quad \phi(x,t) = \beta(r,t)w, \tag{2.7}$$

$$A^k(x,0) = \alpha_0(r)v^k, \quad \partial_t A^k(x,0) = \alpha_1(r)v^k, \quad \phi(x,0) = \beta_0(r),$$

$$\partial_t\phi(x,0) = \beta_1(r)w, \tag{2.8}$$

$$\partial_t A^k, \partial_t\phi \in C(\mathbb{R}; L_r^2), \tag{2.9}$$

$$\textit{If } \text{(i), (ii)} \textit{ hold, } A^k, \phi \in C(\mathbb{R}; \widetilde{H}_r^1), \tag{2.10}$$

$$\textit{If } \text{(i), (iii)} \textit{ hold, } A^k, \phi \in C(\mathbb{R}; H_r^1), \tag{2.11}$$

*where* $\widetilde{H}_r^1$, $H_r^1$ *are radial functions of* $r = |x|$, *having compact support, with the complete space being equaled with the norms* $\|\nabla\phi\|_2$ *and* $\|\phi\|_2 + \|\nabla\phi\|_2$ *respectively, and* $L_r^p$ *denotes the radial symmetric function in* $L^p$ *integrable space.*

## 3 Maxwell-Higgs Equation

The Maxwell-Higgs equations (also known as Ginzburg-Landau equations) takes the following form

$$D_\mu F^{\mu\nu} = -j^\mu, \tag{3.1}$$

$$D_\mu D^\mu\phi - \frac{\lambda}{2}(|\phi|^2 - 1)\varphi = 0. \tag{3.2}$$

Denote the time variable $t$ by $x^0$ and the spatial variables by $x_j$ $(j = 1,2,3)$, $\partial_\mu = \partial_x\mu$. $x^\mu$ are the coordinates of Minkowski space $(\mathbb{R}^{1+3}, g_{\mu\nu})$; $\mu, \nu = 0$, $(g_{\mu\nu}) =$

diag$(-1,1,1,1)$, $x_\mu = g_{\mu\nu}x_\nu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $j^\nu = \mathrm{Im}(\varphi\overline{D^\nu}\varphi)$, $D_\mu = \partial_\mu + A_\mu$ is the covariant derivative for any space or time variables, $A_\mu$ is the electromagnetic potential, $\varphi$ is the complex function and an order parameter of Higgs field, $\lambda$ is Ginzburg-Landau constant, for simplicity, letting $\lambda = 1$.

The electromagnetic field $F_{\mu\nu}$ gives rise to the induced electric and magnetic fields as follows

$$E_i = F_{0i}, \quad H^i = *F_{0i} = \frac{1}{2}\varepsilon_{ijk}F^{jk},$$

where $*F$ is the Hodge dual of $F$, $*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$.

For given function $\varphi(x,t)$, we define its spatial gradient $\nabla\varphi = (\partial_i\varphi)_{i=1,2,3}$, where $\partial\varphi = (\partial_0\varphi, \nabla\varphi)$ is the full spatiotemporal gradient. Denote D'Alembert operator by $\Box$,

$$\Box = -\partial_t^2 + \Delta = -\partial_t^2 + \partial_1^2 + \partial_2^2 + \partial_3^2.$$

It is easy to see that equations (3.1) and (3.2) have total energy conservation

$$E(t) = E(0),$$

where

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left\{\mathrm{E}^2 + H^2 + |D_0\varphi|^2 + \sum_i |D_i\varphi|^2 + \frac{1}{4}(|\varphi|^2 - 1)^2\right\}\mathrm{d}x.$$

Ginzburg-Landau equations are gauge-invariant with respect to the appropriate smooth function $\chi$, defining

$$\psi = \varphi\mathrm{e}^{i\chi}, \quad B_\mu = A_\mu + \partial_\mu\chi.$$

It is easy to known that if $(\varphi, A)$ is the solution of equations (3.1) and (3.2), then $(\psi, B)$ is also the solution of them. In order to make definite solutions of well-posed problem, we must add gauge conditions

(i) Coulomb gauge

$$\nabla^i A_i = 0, \tag{3.3}$$

(ii) Temporal gauge

$$A_0 = 0. \tag{3.4}$$

In fact, if the finite energy solution exists under the coulomb gauge condition, then we can obtain the same result under the appropriate gauge transformation. Therefore, we only need to consider the coulomb gauge condition (3.3).

It's different from the method obtained by Eardley Monerief, from which we can obtain the existence of global smooth solution to Yang-Mills-Higgs equations under the temporal gauge condition. By the method obtained by Klainerman et al., the existence of finite energy global solution of equations (3.1),(3.2),(3.3) or (3.4) is obtained.

Under the coulomb gauge condition, the solution of equations (3.1)-(3.3) can be written as the following form

$$\begin{cases} \Box A_i = -PI_m(\varphi D\overline{\varphi}), \\ D^\mu D_\mu \varphi - \frac{1}{2}(|\varphi|^2 - 1)\varphi = 0, \\ \Delta A_0 = -I_m(\varphi : D_0\overline{\varphi}), \end{cases} \tag{3.5}$$

where $P$ denotes the projection operator on a divergence field, namely, for any field $B$,

$$PB = (-\Delta)^{-1}(\nabla \times (\nabla \times B)).$$

By $\nabla\partial^0 A_0 = 0$, (3.1) can be written as $(3.5)_1$. Consider the initial condition

$$A(x,0) = a_{(0)}(x), \quad \partial_t A(0,x) = a_{01}(x), \tag{3.6}$$

$$\varphi(0,x) = \varphi_{(0)}(x), \quad \partial_t\varphi(0,x) = \varphi_{01}(x), \tag{3.7}$$

$$\mathrm{div} a_0(x) = \mathrm{div} a_1(x) = 0. \tag{3.8}$$

It follows from $(3.5)_2$ that

$$\partial^\mu I_m(\varphi, D_\mu\overline{\varphi}). \tag{3.9}$$

By (3.9), it yields

$$\Box\partial^i A_i = 0. \tag{3.10}$$

We know from equation (3.10) that if the initial values $a_0$ and $a_1$ are the form of divergence free, then (3.3) is automatically satisfied for all time $t$.

Introduce energy module

$$\mathcal{F}(A,\varphi)(t) = \|\partial A\|_{L^2} + \|\varphi\|_{L^2} + \|\partial\varphi\|_{L^2} + \frac{1}{4}\int(|\varphi|^2 - 1)^2 \mathrm{d}x,$$

then there exists the following result.

**Theorem 3.1** *Consider the general initial value* $a_{(0)}, a_{(1)}, \varphi_{(0)}, \varphi_{(1)}$, *for equations* (3.6)-(3.8), *and let*

$$\mathcal{F}_0 = \|\nabla a_{(0)}\|_{L^2} + \|a_{(1)}\|_{L^2} + \|\nabla\varphi_{(0)}\|_{L^2} + \|\varphi_{(1)}\|_{L^2} + \frac{1}{4}\int(|\varphi_{(0)}^2 - 1)^2 \mathrm{d}x,$$

*then there exists a unique generalized solution*

$$\varphi \in C(0,T;H^1) \cap C^1(0,T;L^2), \quad A_\alpha \in C(0,T;\dot{H}^1), \quad \partial_\alpha A_\alpha \in C(0,T;L^2),$$

*where* $\dot{H}^1$ *denotes the homogeneous Sobolev space and satisfies the energy inequality* $E(A_0, A, \varphi) \le \mathcal{F}_0$. *Furthermore, we obtain*

(i)

$$\mathcal{F}_0(A,\varphi)(t) \le C(1+t)\mathcal{F}_0, \quad t \in [0,T],$$

(ii)

$$\int_0^T (\|\Box A(t,\cdot)\|_{L^2} + \|\Box\varphi(t,\cdot)\|_{L^2})\mathrm{d}t < \infty,$$

(iii) *if the initial value is smooth,* $\nabla a_{(0)} \in H^s(\mathbb{R}^3)$, $a_{(1)} \in H^s(\mathbb{R}^3)$, $\varphi_{(0)} \in H^{s+1}(\mathbb{R}^3)$, $\varphi_{(1)} \in H^s(\mathbb{R}^3)$, $s > 0$, *then for any* $t > 0$, *there is*

$$\begin{aligned} &A_0(t,\cdot), A(t,\cdot) \in H^{s+1}(\mathbb{R}^3),\\ &\partial_t A_0(t,\cdot), \partial_t A(t,\cdot) \in H^s(\mathbb{R}^3),\\ &\varphi(t,\cdot) \in H^{s+1}(\mathbb{R}^3), \quad \partial_t\varphi(t,\cdot) \in H^s(\mathbb{R}^3). \end{aligned}$$

Next consider the instability of Maxwell-Higgs equations with respect to symmetric vorticity.

Consider Maxwell-Higgs equations [4]

$$\begin{aligned} &\partial^\mu F_{\mu\nu} = -j_\nu,\\ &D_\nu D^\mu\phi + \frac{\lambda}{2}(|\phi|^2 - 1)\phi = 0, \end{aligned} \tag{3.11}$$

where $D_\mu = \partial_\mu - iA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu = 0,1,2$, $A_\mu(x)$ is the electromagnetic field potential, $\phi(x)$ is Higgs field, $F_{0j}, j = 1,2$ are the electric fields, $-F_{12}$ is the magnetic field,

$$j_\nu = \mathrm{Im}(\phi\overline{D_\nu\phi}) = -\frac{i}{2}(\phi\overline{D_\nu\phi} - \overline{\phi}D_\nu\phi), \quad \nu = 0,1,2. \tag{3.12}$$

The conservation of energy is

$$\frac{1}{2}\int_{\mathbb{R}^2}\Big(|F_{\mu\nu}|^2 + |D_\mu\phi|^2 + \frac{\lambda}{4}(|\phi|^2-1)^2\Big)\mathrm{d}x_1\mathrm{d}x_2. \tag{3.13}$$

For the stationary Ginzburg-Landau equations

$$\begin{aligned} &\mathrm{curl}^2 A + \frac{i}{2}[\overline{\phi}D\phi - \phi D\overline{\phi}] = 0,\\ &-D^2\phi + \frac{\lambda}{2}(|\phi|^2-1)\phi = 0, \end{aligned} \tag{3.14}$$

there is a vortex number as follows:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\nabla\times A\mathrm{d}x = \frac{1}{2\pi}\lim_{N\to\infty}\int_{|x|=N} A\mathrm{d}x, \tag{3.15}$$

which has a topological significance as the winding number of the Higgs field $\phi$. The numerical results indicate that for $\lambda \leq 1$ and all charges $n$, the vortices are stable. On the other hand, for $\lambda > 1$ and $|n| \geq 2$, the vortices are unstable. We can give its analytical proof.

Under the temporal gauge condition $A_0 = 0$, the Maxwell-Higgs system takes a form as follows:

$$[\partial_{tt} - \Delta]A_\nu - \partial_\mu \partial^\mu A_\mu = \frac{i}{2}(\phi\overline{D_\nu\phi} - \overline{\phi}D_\nu\phi), \tag{3.16}$$

$$[\partial_{tt} - \Delta]\phi - iA_\mu\partial^\mu\phi - A_\mu A^\mu\phi + \frac{\lambda}{2}(|\phi|^2 - 1)\phi = 0, \tag{3.17}$$

where $\nu, \mu = 0, 1, 2$. $A_\mu = A_0^2$, $\mu = 0$, $A_\mu A^\mu = -A_\mu^2$, $\mu \neq 0$. We can rewrite (3.16), (3.17) as the following form:

$$\partial_t(\partial_1 A_1 + \partial_2 A_2) = \frac{i}{2}(\phi\overline{\partial_t\phi} - \overline{\phi}\partial_t\phi), \tag{3.18}$$

$$\partial_{tt}(A_h) = \Delta A_h - \partial_h(\partial_1 A_1 + \partial_2 A_2) + \frac{i}{2}(\overline{\phi}D_h\overline{\phi} - \overline{\phi}D_h\phi), \tag{3.19}$$

$$\partial_{tt}\phi = \Delta\phi - iA_jA_j\phi - i\partial_j(A_j\phi) - A_j^2\phi - \frac{\lambda}{2}(|\phi|^2 - 1)\phi, \tag{3.20}$$

where $h = 1, 2,\ j = 1, 2$.

First we linearize the Maxwell-Higgs system around a given radial vortex $(a, \eta)$ in the temporal gauge $A_0 = 0$. Plug $A = a + \varepsilon\omega$ and $\phi = \eta + \varepsilon\psi$ into the Maxwell- Higgs system (2.2), (2.3), and (2.4), and take the derivative with respect to $\varepsilon$. Evaluating at $\varepsilon = 0$, we obtain the linearized Maxwell-Higgs system:

$$\frac{\mathrm{d}^2\nu}{\mathrm{d}\varepsilon^2} = L\nu = -\xi''|_{(a,\eta)}\nu, \tag{3.21}$$

where $\nu = (\omega_1, \omega_2, \psi_1, \psi_2)^{\mathrm{T}}$, $\psi_1 = \mathrm{Re}\psi = \mathrm{Im}\psi$.

The linear operator $L$ has the form

$$\begin{pmatrix} \partial_{22} - |\eta|^2 & -\partial_{12} & g_{13} - \eta_2\partial_1 & g_{14} + \eta_1\partial_1 \\ -\partial_{12} & \partial_{11} - |\eta|^2 & g_{23} - \eta_2\partial_2 & g_{24} + \eta_1\partial_2 \\ g_{31} + \partial_1(\eta_2 \times & g_{23} + \partial_2(\eta_2 \times & \Delta + g_{33} & 2a_j\partial_j - \lambda\eta_2\eta \\ g_{41} - \partial_1(\eta_1 \times & g_{42} + \partial_2(\eta_1 \times & -2\varepsilon_j\partial_j - \lambda\eta_2\eta_1 & \Delta + g_{44} \end{pmatrix},$$

where

$$g_{13} = g_{31} = \partial_1\eta_2 - 2a_1\eta_1, \quad g_{14} = g_{41} = -\partial_1\eta_1 - 2a_1\eta_2,$$
$$g_{23} = g_{32} = \partial_2\eta_2 - 2a_2\eta_1, \quad g_{24} = g_{42} = -\partial_2\eta_1 - 2a_2\eta_2,$$
$$g_{33} = -\frac{\lambda}{2}(2|\eta_1|^2 + |\eta|^2 - 1) - |a|^2, \quad g_{44} = -\frac{\lambda}{2}(2|\eta_2|^2 + |\eta|^2 - 1) - |a|^2.$$

It can be proved that for the vortex $(a,\eta)$ that the system is not only linear stability but also nonlinear instability in the norm $\|\nu\|_X = \|\nu\|_{H^1(\mathbb{R}^2)} + \|\nu\|_{L^\infty(\mathbb{R}^2)}$.

**Theorem 3.2** *Let $(a,\eta)$ be vortices such that*

$$\langle \xi''(\nu_1), \nu_1 \rangle < 0,$$

*for some $\nu_1 \in H^1(\mathbb{R}^2)$, then there exists an $\varepsilon_0 > 0, c > 0$, for any small $\delta > 0$ there exists a family of solution $\nu^\delta(t)$ of the Maxwell-Higgs equations such that the vortex number of $\omega^\delta(0)$ is zero and*

$$\|\nu^\delta(0)\|_X = \delta$$

*but*

$$\sup_{0\le t\le |\ln\delta|} \|\nu^\delta(0)\|_X \le \delta.$$

*Thus $(a,\eta)$ is unstable with the norm $X$.*

## 4 Einstein's Vacuum Equation

Einstein got the following famous Einstein equation in studying the gravitational field

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -kT_{\mu\nu}, \tag{4.1}$$

where $R^{\mu\nu}$ is Ricci tensor, the scalar curvature $R$ is

$$R = g^{\mu\nu}R_{\mu\nu},$$

$T_{\mu\nu}$ is the energy-momentum tensor. If $k = 8\pi G$, $G$ is the Newton's gravitational constant, then there is Einstein gravity field equation

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G T_{\mu\nu}. \tag{4.2}$$

If $k = 8\pi G + 4\Lambda,\ T = g^{\mu\nu}T_{\mu\nu}$, there is

$$R_{\mu\nu} - \wedge g_{\mu\nu} = -8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big). \tag{4.3}$$

If there is no matter, the Einstein's vacuum equation can be obtained,

$$R_{\mu\nu} = \wedge g_{\mu\nu}, \tag{4.4}$$

where $\Lambda \ge 0$ is the cosmical constant, $g_{\mu\nu}$ is the Riemann matrix.

Consider the following form of Einstein's vacuum equation

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = N_{\mu\nu}(g, \partial g), \tag{4.5}$$

where $N$ denotes the term with the square of the first derivative $\partial g$. Consider the initial value problem along the super-plane $\Sigma : t = x^0 = v$,

$$\nabla g_{\alpha\beta}(0) \in H^{s-1}(\Sigma), \quad \partial_t g_{\alpha\beta}(0) \in H^{s-1}(\Sigma), \tag{4.6}$$

where $\nabla$ represents the gradient with respect to the space coordinates $x^i,\ i = 1, 2, 3.$ $H^s$ is Sobolev space. Let $g_{\alpha\beta}(0)$ be continuous Lorentz matrix and

$$\sup_{|x|=r} |g_{\alpha\beta}(0) - m_{\alpha\beta}| \to 0, \quad r \to \infty, \tag{4.7}$$

where $|x| = \Big(\sum\limits_{i=1}^{3} |x_i|^2\Big)^{\frac{1}{2}}$, $m_{\alpha\beta}$ is Minkwski matrix.

**Theorem 4.1** *Assume that equation* (4.5) *satisfies the initial value conditions* (4.6), (4.7), *for* $s > \frac{5}{2}$, *then there exist temporal interval* $[0, T]$ *and unique solution* (*lorentz matrix*) $g \in C^0([0, T] \times \mathbb{R}^3)$, $\partial g_{\mu\nu} \in C^0([0, T], H^{s-1})$, $T$ *only depends on the norm* $\|\partial g_{\mu\nu}(0)\|_{H^{s-1}}$.

**Theorem 4.2** *Consider the classic solution of equation* (4.5), *then there exists a temporal interval* $T$ *only depending on* $\|\partial g_{\mu\nu}(0)\|_{H^{s-1}}$ *for any given* $s > 2$.

To show Theorem 4.2, rewrite (4.5) as

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = N(\phi, \partial\phi), \tag{4.8}$$

where $\phi = (g_{\mu\nu}),\ N = (N_{\mu\nu}),\ g^{\alpha\beta} = g^{\alpha\beta}(\phi)$.

In order to prove Theorem 4.2, the following estimates need to be established: Energy estimate. For the solutions of equation (4.5), there is

$$\|\partial\phi\|_{L^\infty([0,T];\dot{H}^{s-1})} \le \|\partial\phi(0)\|_{\dot{H}^{s-1}}, \tag{4.9}$$

where the constant $C$ only depends on $\|\phi\|_{L^\infty([0,T];L^\infty_x)}$ and $\|\partial\phi\|_{L^1([0,T];L^\infty_x)}$.

By Strichatz estimate,

$$\|\partial\phi\|_{L^1([0,T];L^\infty_x)} \le C\|\partial\phi(0)\|_{H^{s-1}}, \quad s > 2, \tag{4.10}$$

$$\|\partial\phi\|_{L^2([0,T];L^\infty_x)} \le C\|\partial\phi(0)\|_{H^{1+\gamma}}, \quad \gamma > 0, \tag{4.11}$$

and bootstrap hypothesis

$$\|\partial\phi\|_{L^2([0,T];H^{1+\gamma})} + \|\partial\phi\|_{L^2([0,T];L^\infty_x)} \le B_0, \tag{4.12}$$

we can make the better estimates

$$\|\partial\phi\|_{L^2([0,T];L^\infty_x)} \le C(B_0)T^\delta, \quad \delta > 0. \tag{4.13}$$

## 5 Yang-Baxter Equation

Yang-Baxter equation is obtained by Yang when studying the $S$ scattering matrix of multibody problem in 1967. In 1982, Baxter introduced it to the exactly solvable model in statistical mechanics, which plays an important role in the quantum backscattering method.

Yang-Baxter equation is the matrix equation of three space $\nu_1 \times \nu_2 \times \nu_3$

$$R^{\nu_1\nu_2}(u,\eta)R^{\nu_1\nu_3}(u+v,\eta) = R^{\nu_2\nu_3}(v,\eta) = R^{\nu_2\nu_3}(u,\eta)R^{\nu_1\nu_3}(u+v,\eta)R^{\nu_1\nu_2}(u,\eta), \tag{5.1}$$

where $k$ matrix acts only on two direct product spaces, and the action on the third space is equivalent to the identity transformation. Yang-Baxter equation contains two parameters, one is the quantum parameter $\eta$, the other is the spectral parameter $u$. Yang obtained Yang solution:

$$R(u,J) = \frac{u - iJQ}{u + iJ} = \frac{1}{u+iJ}\begin{pmatrix} u-iJ & 0 & 0 & 0 \\ 0 & u & -iJ & 0 \\ 0 & -iJ & u & 0 \\ 0 & 0 & 0 & u-iJ \end{pmatrix},$$

where $J$ is the coupling constant, $Q$ is the exchange operator of two spaces

$$Q : \nu_1 \times \nu_2 \to \nu_2 \times \nu_1.$$

In two-dimensional ice pole model, Quantum parameters and spectral parameters are all related to pole energy:

$$\exp(-\beta\varepsilon_1) = r\sin(u+\eta), \quad \exp(-\beta\varepsilon_3) = r\sin u, \quad \exp(-\beta\varepsilon_5) = r\sin\eta.$$

We obtain

$$R(u,\eta) = r\begin{pmatrix} \sin(u+\eta) & 0 & 0 & 0 \\ 0 & \sin u & \sin\eta & 0 \\ 0 & \sin\eta & \sin u & 0 \\ 0 & 0 & 0 & \sin(u+\eta) \end{pmatrix}$$

or

$$ABA = BAB, \quad A, B \in GL(X),$$

$A, B : X \to X$ are the bilinear operators, $X$ is a linear space.

## 6 Chern-Simons-Higgs Equation

Consider a nonlinear Schrödinger equation

$$i\frac{\partial\varphi}{\partial t} = \frac{1}{2m}\Delta\psi - g|\psi|^2\psi. \tag{6.1}$$

Let $\rho = |\psi(t,x)|^2$, then we have

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}|\psi(t,x)|^2 = \psi\frac{\partial\overline{\psi}}{\partial t} + \overline{\psi}\frac{\partial}{\partial t}\psi = -\partial_k\Big\{\frac{i}{2m}(\psi\partial_k\overline{\psi} - \overline{\psi}\partial_k\psi)\Big\}.$$

Thus, if we use the notation $J = (J^\mu) = (\rho, J^k)$, where

$$J = (J^k)_{k=1,2},$$
$$J^k = -\frac{i}{2m}(\psi\partial^k\overline{\psi} - \overline{\psi}\partial^k\psi) = \frac{i}{2m}(\psi\partial_k\overline{\psi} - \overline{\psi}\partial_k\psi), \quad k = 1, 2.$$

So we have

$$\partial_\mu J^\mu = 0, \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \nabla \cdot J = 0. \tag{6.2}$$

It is clear that equation (6.1) is the Euler-Lagrange equation of the action density [5]

$$\mathcal{L} = i\overline{\psi}\partial_0\psi - \frac{1}{2m}|\partial_j\psi|^2 + \frac{g}{2}|\psi|^4. \tag{6.3}$$

We introduce a real-valued gauge filed $A = (A_\mu)$ ($A_\mu \in \mathbb{R}$, $\mu = 0, 1, 2$) and covariant derivatives $D_\mu = \partial_\mu - iA_\mu$, $\mu = 0, 1, 2$. Hence we arrive at the gauged Schrödinger equation

$$iD_0\psi = -\frac{1}{2m}D_j^2\psi - g|\psi|^2\psi. \tag{6.4}$$

We introduce Maxwell equation

$$\partial_\mu F^{\mu\nu} = -J^\mu, \tag{6.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu, \nu = 0, 1, 2$. The strength tensor gives rise to the induced electric and magnetic fields, $E = (E_1, E_2, 0)$ and $B = (0, 0, B)$, by the standard prescription

$$E_j = F_{0j}, \quad j = 1, 2, \quad B = F_{12}.$$

Therefore, (6.5) becomes

$$B = \frac{1}{\kappa}\rho, \tag{6.6}$$
$$E_j = -\frac{1}{\kappa}\varepsilon_{jk}J^k, \quad k = 1, 2. \tag{6.7}$$

Equation (6.5) or system (6.6),(6.7) is the simplest Chern-Simons-Higgs equation.

Equations (6.4),(6.5) are gauged Schrödinger-Chern-Simons equations. It is clear that this system is the Euler-Lagrange equations of the action density

$$\mathcal{L} = -\frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha + i\overline{\psi}D_0\psi - \frac{1}{2m}|D_j\psi|^2 + \frac{g}{2}|\psi|, \tag{6.8}$$

where $\varepsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha = \frac{1}{2}\varepsilon^{\mu\nu\alpha}A_\mu F_{\nu\alpha}$ is the Chern-Simons term.

In the following we shall look for the explicit static solutions which are independent of $x_0 = t$ of the nonrelativistic Chern-Simons theory (6.8), hence equations (6.4)-(6.6) become

$$\begin{cases} A_0\psi = -\dfrac{1}{2m}D_j^2\psi - g|\psi|^2\psi, \\ F_{12} = \dfrac{1}{\kappa}|\psi|^2, \\ \partial_j A_0 = -\dfrac{i}{2mk}\varepsilon_{jk}(\psi\overline{D_k\psi} - \overline{\psi}D_k\psi). \end{cases} \tag{6.9}$$

By some calculations, (6.9) can be reduced into

$$\begin{cases} D_\pm\psi = 0, \\ F_{12} = \dfrac{1}{\kappa}|\psi|^2, \\ A_0 = \mp\dfrac{1}{2m\kappa}|\psi|^2. \end{cases} \tag{6.10}$$

When $g = \pm\frac{1}{mk}$, equations (6.10) can be viewed as a first integral of (6.9). Also

$$F = \mp\frac{1}{2}\Delta\ln|\psi|^2.$$

It can be shown as before that $(6.10)_1$ implies that the zeros of $\psi$ are discrete and finite. Let the zeros be $p_1, p_2, \cdots, p_N \in \mathbb{R}^2$. By the subtitution $u = \ln|\psi|^2$, we transform $(6.10)_1$, $(6.10)_2$ into the equivalent Liouville equation

$$\Delta u = \pm\frac{2}{\kappa}\mathrm{e}^u + 4\pi\sum_{j=1}^N \delta_{p_j}. \tag{6.11}$$

If let $\kappa = \mp|\kappa|$, (6.11) becomes

$$\Delta u = -\frac{2}{|\kappa|}\mathrm{e}^u + 4\pi\sum_{j=1}^N \delta_{p_j}. \tag{6.12}$$

Using the complex variable $z = x_1 + ix_2$ and Liouville method, we can rewrite $u$ as

$$u(z) = \ln\left(\frac{4|\kappa||F'(z)|^2}{(1+|F(z)|^2)^2}\right), \tag{6.13}$$

where $F(z)$ is any holomorphic function of $z$ so that $p_1, p_2, \cdots, p_N$ are the zeros of $F'(z)$. In particular, we may choose $F(z)$ to be a polynomial in $z$ of degree $N+1$. The solution pair $(\phi, A)$ of the self-dual equations are given by the scheme

$$\begin{cases} \psi(z) = \exp\Big(\dfrac{1}{2}u(z) + i\theta(z)\Big), \quad \theta(z) = \displaystyle\sum_{j=1}^N \arg(z - k_j), \\ A_1(z) = -\mathrm{Re}\{i\partial_\pm \ln\psi(z)\}, \\ A_2(z) = -\mathrm{Im}\{i\partial_\pm \ln\psi(z)\}, \end{cases} \tag{6.14}$$

$$|\psi(z)|^2 = \mathrm{e}^{u(x)} = O(|x|^{-2N-4}), \quad x = (x_1, x_2), \quad |x| \to \infty.$$

Consider Abelian Chern-Simons-Higgs system [6]

$$\mathcal{L} = \frac{\kappa}{4}\mathrm{e}^{\mu\nu\rho}A_\mu F_{\nu\rho} + |D_\mu\phi|^2 - V(|\phi|). \tag{6.15}$$

The theory possesses two vacua: $\phi = 0$ and $|\phi| = v$, one is symmetric and the other is asymmetric.

The energy of the system is

$$\begin{aligned} E &= \int \mathrm{d}^2x[|D_0\phi|^2 + |D_1\phi|^2 + V(|\phi|)] \\ &= \int \mathrm{d}^2x\left[|\partial_0\phi|^2 + \frac{\kappa^2 B^2}{4\mathrm{e}^2|\phi|^2} + |D_1\phi|^2 + V(|\phi|)\right]. \end{aligned} \tag{6.16}$$

The term $\frac{\kappa^2 B^2}{4\mathrm{e}^2|\phi|^2}$ is fundamental, it forces the magnetic field to stay away from the $\phi = 0$, that is, the magnetic field goes to zero faster than $|\phi|^2$.

The equations of motion are

$$D_\mu D^\mu\phi = -\frac{\delta V}{\delta\phi^*}, \tag{6.17}$$

$$\frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = -i\mathrm{e}(\phi^* D^\mu\phi - \phi D^\mu\phi^*). \tag{6.18}$$

Performing the Bogomol'nyi factorization, one obtains

$$E = \int \mathrm{d}^2x\left[\left|D_0\phi \pm \frac{iE^2}{k}(|\phi|^2 - v^2)\phi\right|^2 + |D_\pm\phi|^2 \pm \mathrm{e}v^2 B\right], \tag{6.19}$$

where $D_\pm\phi = (D_1 \pm iD_2)\phi$, then

$$E \geqslant \mathrm{e}v^2|\Phi_B|, \tag{6.20}$$

where $\Phi_B = \int \mathrm{d}^2xB = 2\pi n/\mathrm{e}$.

Consider the self-dual Maxwell-Chern-Simons Lagrange is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}k\varepsilon_{\mu\nu\rho}A_\mu F_{\nu\rho} + |D_\mu\phi|^2 + \frac{1}{2}(\partial_\mu N)^2 - U_{BPS}(|\phi|, N) \tag{6.21}$$

with the BPS potential

$$U_{BPS}(|\phi|, N) = \frac{1}{2}[\mathrm{e}|\phi|^2 + kN - \mathrm{e}v^2]^2 + \mathrm{e}^2N^2|\phi|^2. \tag{6.22}$$

The Bogomol'nyi completion of the energy is

$$\begin{aligned} E &= \int \mathrm{d}^2x\left\{\frac{1}{2}F_{i0}^2 + \frac{1}{2}F_{12}^2 + |D_0\phi|^2 + |D_i\phi|^2 + \frac{1}{2}(\partial_0 N)^2 + \frac{1}{2}(\partial_i N)^2 + U_{BPS}\right\} \\ &= \int \mathrm{d}^2x\left\{\frac{1}{2}(F_{i0}\pm\partial_i N)^2 + \frac{1}{2}(F_{12}\pm(\mathrm{e}|\phi|^2 + kN - \mathrm{e}v^2))^2 + |D_0\phi \mp i\mathrm{e}\phi N|^2 + |D_\pm|^2 + \frac{1}{2}(\partial_0 N)^2\right\} \\ &\geq \mathrm{e}v^2\Phi_B, \end{aligned} \tag{6.23}$$

and the seld-duality equations are

$$D_{\pm}\phi = 0, \tag{6.24}$$

$$F_{12} \pm (\mathrm{e}|\phi|^2 - \mathrm{e}v^2 + kN) = 0. \tag{6.25}$$

Consider the domain wall energy of system (6.21)

$$\begin{aligned} T &= \int \mathrm{d}x \Big[ |\partial_x \phi|^2 + \frac{1}{2}|\partial_x N|^2 + U_{BPS}(|\phi|v, N) \Big] \\ &= \int \mathrm{d}x \Big[ |\partial_x \phi \pm \mathrm{e}N\phi|^2 + \frac{1}{2}(\partial_x N \pm (\mathrm{e}|\phi|^2 + kN - \mathrm{e}v^2))^2 \\ &\quad \mp \Big( \mathrm{e}N|\phi|^2 + \frac{1}{2}kN^2 - \mathrm{e}v^2 N \Big) \Big]. \end{aligned} \tag{6.26}$$

When $A_x = 0$, the energy reads

$$\begin{aligned} T &= \int \mathrm{d}x \Big[ |\partial_x \phi|^2 + \frac{1}{2}|\partial_x N|^2 + U_{BPS}(|\phi|, N) \Big] \\ &\quad + \frac{1}{2}(\partial_x A_0)^2 + \frac{1}{2}(\partial_x A_y)^2 + \mathrm{e}^2 A_0^2 |\phi|^2 + \mathrm{e}^2 A_y^2 |\phi|^2 \\ &= \int \mathrm{d}x \Big[ |\partial_x \phi \pm \mathrm{e}N\phi|^2 + \frac{1}{2}(\partial_x N \pm (\mathrm{e}|\phi|^2 + kN - \mathrm{e}v^2))^2 \\ &\quad \mp \Big( \mathrm{e}N|\phi|^2 + \frac{1}{2}kN^2 - \mathrm{e}v^2 N \Big) + \frac{1}{2}(\partial_x A_0 \pm \partial'_x A_y)^2 \\ &\quad + ((A_0 \pm' A_y)\mathrm{e}|\phi|^2)^2 \mp' \partial_x \Big( A_y \partial_x A_0 - \frac{1}{2}kA_y^2 \Big) \Big], \end{aligned} \tag{6.27}$$

where the Gauss law has been used to write the completion and the second set of $\pm s$ is independent of the first set and is marked with $'$.

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