Ann. of Appl. Math.
36:1(2020), 91-110

A FAST AND HIGH ACCURACY NUMERICAL SIMULATION FOR A FRACTIONAL BLACK-SCHOLES MODEL ON TWO ASSETS[*†]

Hongmei Zhang[1‡], Fawang Liu[2,1], Shanzhen Chen[3], Ming Shen[1]

(1. *College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350108, Fujian, PR China;*

2. *School of Mathematical Sciences, Queensland University of Technology, Qld. 4001, Australia;*

3. *School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, Sichuan, PR China*)

Abstract

In this paper, a two dimensional (2D) fractional Black-Scholes (FBS) model on two assets following independent geometric Lévy processes is solved numerically. A high order convergent implicit difference scheme is constructed and detailed numerical analysis is established. The fractional derivative is a quasi-differential operator, whose nonlocal nature yields a dense lower Hessenberg block coefficient matrix. In order to speed up calculation and save storage space, a fast bi-conjugate gradient stabilized (FBi-CGSTAB) method is proposed to solve the resultant linear system. Finally, one example with a known exact solution is provided to assess the effectiveness and efficiency of the presented fast numerical technique. The pricing of a European Call-on-Min option is showed in the other example, in which the influence of fractional derivative order and volatility on the 2D FBS model is revealed by comparing with the classical 2D B-S model.

Keywords 2D fractional Black-Scholes model; Lévy process; fractional derivative; numerical simulation; fast bi-conjugrate gradient stabilized method

2000 Mathematics Subject Classification 26A33; 65M06; 65M12; 65T50

1 Introduction

During the last few decades the scientists have had a considerable interest in

*The work was supported by the Natural Science Foundation of Fujian Province 2017J01555, 2017J01502, 2017J01557 and 2019J01646; the National NSF of China 11201077; China Scholarship Fund; the Natural Science Foundation of Fujian Provincial Department of Education JAT160274.

†Manuscript received April 9, 2019; Revised October 13, 2019

‡Corresponding author. E-mail: zhm-fzu@163.com

modelling financial markets and pricing financial derivatives. The pioneering work was presented in the 1970s by Black, Scholes [4] and Merton [23], who set up the key principles of no arbitrage option pricing and derived a well known differential equation model, that is the B-S model. However, the model was proposed under many strict assumptions on the real financial market. More general models by relaxing some restrictions were constructed in ongoing researches [12, 13, 24].

It is frequently found that the option price presents the features of heavy tails and volatility skew or smile in real markets. Gaussian models fail to describe these phenomena while Lévy (or α-stable) distributions can do. Lévy processes allow extreme but realistic events, such as sudden jumps of market prices [7, 19, 20]. Therefore, more and more different Lévy processes have been introduced into the financial field to model price of financial derivatives. A modified Lévy-α-stable process was proposed by Koponen [17] and Boyarchenko and Levendorskiǐ [5] to model the dynamics of securities. This modification yields a damping effect in the tails of the Lévy stable distribution, which was known as the KoBoL process. Carr, Geman, Madan and Yor [6] raised a Lévy process (that is the CGMY process), which allowed for jump components displaying both finite and infinite activity and variation. A finite moment log stable (FMLS) process with the tail index $\alpha \in (0, 2]$, which can capture the highly skewed feature of the implied density for log returns, was applied to model S&P 500 option prices by Carr and Wu [7]. Schoutens [27, 28] summed up the application of Lévy process in finance. Of all the Lévy processes, the most interesting include the CGMY, KoBoL and FMLS processes.

Fractional derivatives are quasi-differential operators, which provide useful tools for a description of memory and hereditary properties and are closely related to Lévy processes. When the price log-returns are driven by a Lévy fractional stable distribution, after some suitable transformations, the price of an option on underlying assets can be modeled by a fractional partial differential equation (FPDE) [1, 8–10, 14, 34]. In this paper we focus on a 2D FBS model governing European Call-on-Min option. Assuming the two underlying assets S_1 and S_2 follow two independent geometric Lévy processes with maximal negative asymmetry (or skewness) [7] (that is the FMLS process), the option price V on these two assets is determined by a 2D FBS equation [11] as below

$$\frac{\partial V}{\partial t} + (r - v_\alpha)\frac{\partial V}{\partial x} + (r - v_\beta)\frac{\partial V}{\partial y} + v_\alpha \cdot {}_{-\infty}D_x^\alpha V + v_\beta \cdot {}_{-\infty}D_y^\beta V = rV, \tag{1}$$

where $x = \ln S_1$, $y = \ln S_2$, $v_\alpha = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}$, $v_\beta = -\frac{1}{2}\sigma^\beta \sec\frac{\beta\pi}{2}$, and the parameters r and $\sigma(\geq 0)$ are the risk-free rate and the volatility of the returns from the holding stock price, respectively. The fractional operators ${}_{-\infty}D_x^\alpha$ and ${}_{-\infty}D_y^\beta$ $(1 < \alpha, \beta \leq 2)$ are the left Riemann-Liouville fractional derivatives defined on infinite intervals [25]. Obviously, the FBS model (1) becomes the classical B-S model when $\alpha = \beta = 2$.

It is well known that it is difficult to derive an analysis solution for FPDE. So numerical simulation of the fractional pricing model has attracted considerable attentions [3, 8, 10, 15, 21, 32, 33]. These financial literatures are all concerned with one-dimensional situations on one asset. As far as the author knows, the exploration of high dimensional fractional pricing model on multi assets is still sparse. Chen [11] used a finite different scheme to solve model (1) and price the Call-on-Min and Basket options approximately. Karipova and Magdziarz [16] derived a subdiffusive B-S model for the fair price of basket options by the optimal martingale measure and used the approximate methods to compare the classical with subdiffusive prices.

The non-locality of the fractional derivative operator leads to the denseness of coefficient matrices, which makes the solution of the discrete system more difficult. Moreover, the financial markets are becoming more and more complex, with trading of numerous types of financial derivatives. The market requires updated information about the values of these derivatives every second of the day. These yield a huge demand for feasible, fast and high accuracy numerical simulations. In this paper, a new implicit numerical scheme with second-order accuracy is constructed to approximate the above 2D FBS model in finite intervals. Due to the non-local nature of the fractional derivative, the numerical discretization of the 2D FBS model results in a linear system with a dense lower Hessenberg block coefficient matrix, which needs high storage space and computational requirement. This is disadvantageous in the face of huge financial data and ever-changing financial market. In order to speed up calculation and save storage space, the FBi-CGSTAB method is presented to evaluate the linear system. It would be a contribution to a real world problem whose solution would otherwise have been hampered by the fact that the solution of the partial differential equation is required in a short period of time.

The rest of the paper is structured as follows: Section 2 derives a fully implicit difference scheme (FIDS) with second order accuracy in both temporal and spatial directions when the price of two underlying assets is limited to finite intervals. The detailed numerical analysis is established in Section 3. In Section 4, a fast algorithm combining the bi-conjugate gradient stabilized method with the fast fourier transform is utilised to solve the discrete system in order to reduce the storage memory and computational cost. A numerical experiment with an exact solution is proposed in Section 5 to value the effectiveness and efficiency of the proposed fast numerical technique. Compared with the classical B-S model, the influence of fractional derivative order and volatility on the 2D FBS model is revealed in another example. The last section is dedicated to conclusions.

2 The Fully Implicit Difference Discretization

In the section, a FIDS for the 2D FBS model is constructed. In computation, we truncate the infinite solution domain into finite domain $\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max})$. Moreover, in order to facilitate the verification of the accuracy of the numerical scheme, we consider a more general form

$$\frac{\partial V}{\partial t} + (r - v_\alpha)\frac{\partial V}{\partial x} + (r - v_\beta)\frac{\partial V}{\partial y} + v_\alpha \cdot {}_{x_{\min}} D_x^\alpha V + v_\beta \cdot {}_{y_{\min}} D_y^\beta V = rV + f,$$
$$(x, y) \in \Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max}),\ t \in [0, T) \tag{2}$$

with the following initial and boundary conditions:

$$\begin{aligned} &V(x, y, T) = \varphi(x, y), \quad (x, y) \in \Omega, \\ &V(x_{\min}, y, t) = V(x, y_{\min}, t) = 0, \\ &V(x_{\max}, y, t) = \psi(x_{\max}, y, t), \quad V(x, y_{\max}, t) = \phi(x, y_{\max}, t), \quad 0 < t < T. \end{aligned} \tag{3}$$

The fractional operators ${}_{x_{\min}} D_x^\alpha$ and ${}_{y_{\min}} D_y^\beta$ $(1 < \alpha, \beta \le 2)$ in equation (2) is the Riemann-Liouville fractional derivative [25] on a finite interval. The existence and uniqueness of the solution of this model were proved in [11]. In this paper, we focus on its numerical simulation.

Lemma 1[29] *Let $u \in L^1(\mathbb{R})$, ${}_{-\infty}D_x^{\gamma+2} u$ and its Fourier transform belongs to $L^1(\mathbb{R})$, then we have*

$$ {}_{-\infty}D_x^\gamma u(x) = \frac{\gamma - 2q}{2(p - q)} A_{h,p}^\gamma u(x) + \frac{2p - \gamma}{2(p - q)} A_{h,q}^\gamma u(x) + O(h^2), \tag{4}$$

uniformly for $x \in \mathbb{R}$, where p, q $(p \neq q)$ are integers and $A_{h,p}^\gamma u(x)$ is the shifted Grünwald-Letnikov difference operator [22] *given by*

$$A_{h,p}^\gamma u(x) = \frac{1}{h^\gamma} \sum_{k=0}^{\infty} g_k^{(\gamma)} u(x - (k - p)h)$$

with

$$\begin{cases} g_0^{(\gamma)} = 1, \quad g_1^{(\gamma)} = -\gamma < 0, \quad g_k^{(\gamma)} = \Big(1 - \dfrac{\gamma + 1}{k}\Big) g_{k-1}^{(\gamma)}, \quad k = 1, 2, \cdots; \\ 1 \ge g_2^{(\gamma)} \ge g_3^{(\gamma)} \ge \cdots \ge 0; \\ \displaystyle\sum_{k=0}^{\infty} g_k^{(\gamma)} = 0, \quad \sum_{k=0}^{m} g_k^{(\gamma)} < 0, \quad m \ge 1. \end{cases}$$

Remark 1 Considering a well defined function $u(x)$ on the bounded interval $[a, b]$. If $u(a) = 0$, the function $u(x)$ can be zero extended for $x < a$. Taking $(p, q) = (1, 0)$ in (4), the γ $(1 < \gamma \le 2)$ order left Riemann-Liouville fractional derivative of $u(x)$ at each point x_i can be approximated by the following operator with second order truncation error

$$
{}_aD_x^{\gamma}u(x_i) :={}_a\widetilde{D}_x^{\gamma}u(x_i) + O(h^2) = \frac{1}{h^{\gamma}}\sum_{k=0}^{i+1}\omega_k^{(\gamma)}u(x_{i-k+1}) + O(h^2), \tag{5}
$$

with

$$
\omega_0^{(\gamma)} = \frac{\gamma}{2}g_0^{(\gamma)}, \quad \omega_k^{(\gamma)} = \frac{\gamma}{2}g_k^{(\gamma)} + \frac{2-\gamma}{2}g_{k-1}^{(\gamma)}, \quad k \geq 1,
$$

which satisfies [29]:

$$
\begin{cases}
\omega_0^{(\gamma)} = \dfrac{\gamma}{2}, \ \ \omega_1^{(\gamma)} = \dfrac{2-\gamma-\gamma^2}{2} < 0, \ \ \omega_2^{(\gamma)} = \dfrac{\gamma(\gamma^2+\gamma-4)}{4}; \\
1 \geq \omega_0^{(\gamma)} \geq \omega_3^{(\gamma)} \geq \omega_4^{(\gamma)} \geq \cdots \geq 0; \\
\displaystyle\sum_{k=0}^{\infty}\omega_k^{(\gamma)} = 0, \ \ \sum_{k=0}^{m}\omega_k^{(\gamma)} < 0, \quad m \geq 2.
\end{cases}
$$

In the following we will construct a fully discrete scheme to approximate the equation (2). Let $t_n = (\mathcal{N} - n)\tau$ $(n = 0, 1, 2, \cdots, \mathcal{N})$ and $x_i = x_{\min} + ih_x$, $y_j = y_{\min} + jh_y$ $(i = 0, 1, 2, \cdots, M_x; j = 0, 1, 2, \cdots, M_y)$, where $\tau = T/\mathcal{N}$ is a temporal step size, $h_x = (x_{\max} - x_{\min})/M_x$ and $h_y = (y_{\max} - y_{\min})/M_y$ are spatial step sizes. We discretize the first-order spatial derivative by the central difference quotient and the R-L derivatives ${}_{x_{\min}}D_x^{\alpha}$ and ${}_{y_{\min}}D_y^{\beta}$ by ${}_{x_{\min}}\widetilde{D}_x^{\alpha}$ and ${}_{y_{\min}}\widetilde{D}_y^{\beta}$, respectively. As for the time derivative, the Crank-Nicolson scheme is employed. For simplicity, we define the following finite difference operators:

$$
\delta_x V_{i,j}^n = (r - v_{\alpha})\frac{V_{i+1,j}^n - V_{i-1,j}^n}{2h_x}, \quad \delta_y V_{i,j}^n = (r - v_{\beta})\frac{V_{i,j+1}^n - V_{i,j-1}^n}{2h_y},
$$

$$
\delta_x^{\alpha} V_{i,j}^n = \frac{v_{\alpha}}{h_x^{\alpha}}\sum_{k=0}^{i+1}\omega_k^{(\alpha)}V_{i-k+1,j}^n, \quad \delta_y^{\beta} V_{i,j}^n = \frac{v_{\beta}}{h_y^{\beta}}\sum_{k=0}^{j+1}\omega_k^{(\beta)}V_{i,j-k+1}^n,
$$

then equation (2) can be discretized as follows:

$$
\frac{V_{i,j}^{n+1} - V_{i,j}^n}{-\tau} + \frac{1}{2}(\delta_x + \delta_y + \delta_x^{\alpha} + \delta_y^{\beta} - r)V_{i,j}^{n+1} + \frac{1}{2}(\delta_x + \delta_y + \delta_x^{\alpha} + \delta_y^{\beta} - r)V_{i,j}^n = f_{i,j}^{n+\frac{1}{2}} + \varepsilon_{i,j}^n, \tag{6}
$$

where $V_{i,j}^n = V(x_i, y_j, t_n)$, $f_{i,j}^n = f(x_i, y_j, t_n)$, $f_{i,j}^{n+\frac{1}{2}} = \frac{1}{2}(f_{i,j}^{n+1} + f_{i,j}^n)$ and $\varepsilon_{i,j}^n = O(h_x^2 + h_y^2 + \tau^2)$ $(i = 1, 2, \cdots, M_x - 1; j = 1, 2, \cdots, M_y - 1; n = 0, 1, \cdots, \mathcal{N} - 1)$ is the truncation error.

Let $\widetilde{V}_{i,j}^n$ be the numerical solution of $V_{i,j}^n$. Multiplying $-\tau$ on both sides of equation (6) and omitting the truncation errors, we derive the following FIDS to approximate model (2):

$$\Big[1-\frac{\tau}{2}(\delta_x+\delta_y+\delta_x^\alpha+\delta_y^\beta-r)\Big]\widetilde{V}_{i,j}^{n+1}=\Big[1+\frac{\tau}{2}(\delta_x+\delta_y+\delta_x^\alpha+\delta_y^\beta-r)\Big]\widetilde{V}_{i,j}^{n}-\tau f_{i,j}^{n+\frac{1}{2}}, \tag{7}$$

whose initial and boundary conditions is

$$\begin{cases}\widetilde{V}_{i,j}^{\mathcal{N}}=\varphi_{i,j}, \quad i=1,2,\cdots,M_x-1;\ j=1,2,\cdots,M_y-1,\\ \widetilde{V}_{0,j}^{n}=\widetilde{V}_{i,0}^{n}=0, \quad n=1,2,\cdots,\mathcal{N},\\ \widetilde{V}_{M_x,j}^{n}=\psi_{M_x,j}^{n}, \quad \widetilde{V}_{i,M_y}^{n}=\phi_{i,M_y}^{n}, \quad n=1,2,\cdots,\mathcal{N}.\end{cases} \tag{8}$$

Let $\widetilde{V}^n=(\widetilde{V}_1^n,\widetilde{V}_2^n,\cdots,\widetilde{V}_{M_y-1}^n)^{\mathrm{T}}$ be a block vector and each block $\widetilde{V}_j^n=(\widetilde{V}_{1,j}^n,\widetilde{V}_{2,j}^n,\cdots,\widetilde{V}_{M_x-1,j}^n)$. Similarly, $F^n=(f_1^{n+\frac{1}{2}},f_2^{n+\frac{1}{2}},\cdots,f_{M_y-1}^{n+\frac{1}{2}})^{\mathrm{T}}$ with $f_j^{n+\frac{1}{2}}=(f_{1,j}^{n+\frac{1}{2}}, f_{2,j}^{n+\frac{1}{2}},\cdots,f_{M_x-1,j}^{n+\frac{1}{2}})$ $(j=1,2,\cdots,M_y-1)$. Denote $\zeta_\alpha=\frac{v_\alpha\tau}{2h_x^\alpha}$, $\zeta_\beta=\frac{v_\beta\tau}{2h_y^\beta}$, $\xi_\alpha=\frac{\tau(r-v_\alpha)}{4h_x}$, $\xi_\beta=\frac{\tau(r-v_\beta)}{4h_y}$ and $\eta=\frac{\tau r}{2}$, then the formula (7) can be expressed in the following matrix form:

$$(I+M)\widetilde{V}^{n+1}=(I-M)\widetilde{V}^n+\mathcal{C}^n-\tau F^n, \tag{9}$$

where I is a unit matrix of order $(M_x-1)\times(M_y-1)$, M is a block matrix which has $(M_y-1)\times(M_y-1)$ blocks and the size of each block matrix is $(M_x-1)\times(M_x-1)$, which is

$$M=\begin{pmatrix} A+B_0 & B_1 & 0 & \cdots & \cdots & \cdots & 0\\ B_2 & A+B_0 & B_1 & \ddots & & & 0\\ B_3 & B_2 & A+B_0 & B_1 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ B_{M_y-3} & & \ddots & B_2 & A+B_0 & B_1 & 0\\ B_{M_y-2} & & & B_3 & B_2 & A+B_0 & B_1\\ B_{M_y-1} & \cdots & \cdots & \cdots & B_3 & B_2 & A+B_0\end{pmatrix}, \tag{10}$$

where the matrix $A=(a_{i,j})_{(M_x-1)\times(M_x-1)}$ has entries

$$a_{ij}=\begin{cases}\eta-\zeta_\alpha\omega_1^{(\alpha)}, & i=j,\ j=1,\cdots,M_x-1,\\ \xi_\alpha-\zeta_\alpha\omega_2^{(\alpha)}, & i=j+1,\ j=1,\cdots,M_x-2,\\ -\xi_\alpha-\zeta_\alpha\omega_0^{(\alpha)}, & i=j-1,\ j=2,\cdots,M_x-1,\\ -\zeta_\alpha\omega_{i-j+1}^{(\alpha)}, & i-j\geq 2,\ j=1,\cdots,M_x-3,\\ 0, & \text{otherwise},\end{cases} \tag{11}$$

$$B_j=\begin{cases}-\zeta_\beta\omega_1^{(\beta)}I_{M_x-1}, & j=0,\\ (-\xi_\beta-\zeta_\beta\omega_0^{(\beta)})I_{M_x-1}, & j=1,\\ (\xi_\beta-\zeta_\beta\omega_2^{(\beta)})I_{M_x-1}, & j=2,\\ -\zeta_\beta\omega_j^{(\beta)}I_{M_x-1}, & j=3,4,\cdots,M_y-1,\end{cases} \tag{12}$$

and

$$\mathcal{C}^n = (\mathcal{C}_1^{n+\frac{1}{2}}, \mathcal{C}_2^{n+\frac{1}{2}}, \cdots, \mathcal{C}_{M_y-1}^{n+\frac{1}{2}})^{\mathrm{T}},$$

with each block

$$\begin{aligned}
\mathcal{C}_j^{n+\frac{1}{2}} &= \big(0, \cdots, 0, (\xi_\alpha + \zeta_\alpha\omega_0^{(\alpha)})(\widetilde{V}_{M_x,j}^{n+1} + \widetilde{V}_{M_x,j}^{n})\big), \quad j = 1, 2, \cdots, M_y - 2, \\
\mathcal{C}_{M_y-1}^{n+\frac{1}{2}} &= \big((\xi_\beta + \zeta_\beta\omega_0^{(\beta)})(\widetilde{V}_{1,M_y}^{n+1} + \widetilde{V}_{1,M_y}^{n}),\ \cdots,\ (\xi_\beta + \zeta_\beta\omega_0^{(\beta)})(\widetilde{V}_{M_x-2,M_y}^{n+1} + \widetilde{V}_{M_x-2,M_y}^{n}), \\
&\quad (\xi_\alpha + \zeta_\alpha\omega_0^{(\alpha)})(\widetilde{V}_{M_x,M_y-1}^{n+1} + \widetilde{V}_{M_x,M_y-1}^{n}) + (\xi_\beta + \zeta_\beta\omega_0^{(\beta)})(\widetilde{V}_{M_x-1,M_y}^{n+1} + \widetilde{V}_{M_x-1,M_y}^{n})\big).
\end{aligned}$$

Of course each $(M_x - 1) \times (M_x - 1)$ bloch $M_{i,j}$ of the $(M_y - 1) \times (M_y - 1)$ block matrix M can also be expressed by

$$\begin{aligned}
&(M_{j,j})_{p,p} = \eta - \zeta_\alpha\omega_1^{(\alpha)} - \zeta_\beta\omega_1^{(\beta)}, \\
&(M_{j,j})_{p,p-1} = \xi_\alpha - \zeta_\alpha\omega_2^{(\alpha)}, \\
&(M_{j,j})_{p,p+1} = -\xi_\alpha - \zeta_\alpha\omega_0^{(\alpha)}, \\
&(M_{j,j})_{p,q} = -\zeta_\alpha\omega_{p-q+1}^{(\alpha)}, \quad p - q \geq 2,\ q = 1, 2, \cdots, M_x - 3, \\
&(M_{j,j})_{p,q} = 0,\ q - p \geq 2, \quad q = 3, \cdots, M_x - 1, \\
&(M_{j,j-1})_{p,q} = \delta_{p,q}(\xi_\beta - \zeta_\beta\omega_2^{(\beta)}), \\
&(M_{j,j+1})_{p,q} = \delta_{p,q}(-\xi_\beta - \zeta_\beta\omega_0^{(\beta)}), \\
&(M_{i,j})_{p,q} = \delta_{p,q}(-\zeta_\beta\omega_{i-j+1}^{(\beta)}), \quad i - j \geq 2,\ j = 1, \cdots, M_y - 3, \\
&(M_{i,j})_{p,q} = 0,\ j - i \geq 2, \quad j = 3, \cdots, M_y - 1,
\end{aligned} \tag{13}$$

where $\delta_{p,q}$ is the Kronecker delta.

3 Stability and Convergence of the FIDS

Using the Taylor expansion, we have

$$\begin{aligned}
&\frac{\tau^2}{4}\Big(\delta_x + \delta_x^\alpha - \frac{r}{2}\Big)\Big(\delta_y + \delta_y^\beta - \frac{r}{2}\Big)(V_{i,j}^{n+1} - V_{i,j}^n) \\
= &\frac{\tau^3}{4}\Big[\Big((r - \nu_\alpha)\frac{\partial}{\partial x} + \nu_\alpha \cdot_{x_{\min}} D_x^\alpha - \frac{r}{2}\Big)\Big((r - \nu_\beta)\frac{\partial}{\partial y} + \nu_\beta \cdot_{y_{\min}} D_y^\beta - \frac{r}{2}\Big)V_t\Big]_{i,j}^{n+\frac{1}{2}} \\
&+O(\tau^5 + \tau^3(h_x^2 + h_y^2)),
\end{aligned}$$

then the discrete scheme (6) can be transformed into

$$\begin{aligned}
&\Big[1 - \frac{\tau}{2}\Big(\delta_x + \delta_x^\alpha - \frac{r}{2}\Big)\Big]\Big[1 - \frac{\tau}{2}\Big(\delta_y + \delta_y^\beta - \frac{r}{2}\Big)\Big]V_{i,j}^{n+1} \\
= &\Big[1 + \frac{\tau}{2}\Big(\delta_x + \delta_x^\alpha - \frac{r}{2}\Big)\Big]\Big[1 + \frac{\tau}{2}\Big(\delta_y + \delta_y^\beta - \frac{r}{2}\Big)\Big]V_{i,j}^n - \tau f_{i,j}^{n+\frac{1}{2}} + \tau\varepsilon_{i,j}^n.
\end{aligned} \tag{14}$$

Similarly, the FIDS (7) can be transformed into

$$
\begin{aligned}
&\left[1-\frac{\tau}{2}\Big(\delta_x+\delta_x^\alpha-\frac{r}{2}\Big)\right]\left[1-\frac{\tau}{2}\Big(\delta_y+\delta_y^\beta-\frac{r}{2}\Big)\right]\widetilde{V}_{i,j}^{n+1}\\
=&\left[1+\frac{\tau}{2}\Big(\delta_x+\delta_x^\alpha-\frac{r}{2}\Big)\right]\left[1+\frac{\tau}{2}\Big(\delta_y+\delta_y^\beta-\frac{r}{2}\Big)\right]\widetilde{V}_{i,j}^{n}-\tau f_{i,j}^{n+\frac{1}{2}}.
\end{aligned}\tag{15}
$$

Let

$$
\begin{aligned}
G_x^\alpha &= \xi_\alpha I_{M_y-1}\otimes D_{M_x-1}-\zeta_\alpha I_{M_y-1}\otimes W_{M_x-1}^{(\alpha)},\\
G_y^\beta &= \xi_\beta D_{M_y-1}\otimes I_{M_x-1}-\zeta_\beta W_{M_y-1}^{(\beta)}\otimes I_{M_x-1},
\end{aligned}\tag{16}
$$

where $\otimes$ is the Kronecker product, I_N is the identity matrix of order N, $W_N^{(\gamma)}$ ($\gamma=\alpha$ or β) and D_N are Toeplitz matrices of order N with the forms

$$
W_N^{(\gamma)}=\begin{pmatrix}\omega_1^{(\gamma)} & \omega_0^{(\gamma)} & \cdots & 0\\ \omega_2^{(\gamma)} & \omega_1^{(\gamma)} & \ddots & \vdots\\ \vdots & \ddots & \ddots & \omega_0^{(\gamma)}\\ \omega_N^{(\gamma)} & \omega_{N-1}^{(\gamma)} & \cdots & \omega_1^{(\gamma)}\end{pmatrix},\quad D_N=\begin{pmatrix}0 & -1 & \cdots & 0\\ 1 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & -1\\ 0 & \cdots & 1 & 0\end{pmatrix}.\tag{17}
$$

Then the matrix form of the discrete scheme (15) is

$$
\begin{aligned}
&\left[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\right]\left[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\right]\widetilde{V}^{n+1}\\
=&\left[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\right]\left[I-\Big(G_y^\beta+\frac{\eta}{2}I\Big)\right]\widetilde{V}^{n}+\mathcal{C}^n-\tau F^n.
\end{aligned}\tag{18}
$$

Lemma 2[18] *Let* $A\in\mathbb{R}^{m\times n},\ B\in\mathbb{R}^{r\times s},\ C\in\mathbb{R}^{n\times p},\ D\in\mathbb{R}^{s\times t}$, *then*

$$(A\otimes B)(C\otimes D)=AC\otimes BD\ (\in R^{mr\times pt}).$$

Moreover, if $A,B\in\mathbb{R}^{n\times n}$, I *is a unit matrix of order* n, *then matrixes* $I\otimes A$ *and* $B\otimes I$ *commute.*

Lemma 3[18] *For all matrices* A *and* B, $(A\otimes B)^{\mathrm{T}}=A^{\mathrm{T}}\otimes B^{\mathrm{T}}$.

Lemma 4[29] *When* $1<\gamma\le 2$, *the eigenvalue* λ *of the matrix* $W_N^{(\gamma)}$ *defined in (13) satisfies* $\mathrm{Re}(\lambda)<0$. *Moreover, matrix* $W_N^{(\gamma)}$ *is negative definite, and the real parts of the eigenvalues of the matrix* $c_1W_N^{(\gamma)}+c_2(W_N^{(\gamma)})^{\mathrm{T}}$ *are less than* 0, *where* $c_1,c_2\ge 0,\ c_1^2+c_2^2\ne 0$.

Lemma 5[18] *Let* $A\in\mathbb{R}^{n\times n}$ *have eigenvalues* $\{\lambda_i\}_{i=1}^n$, *and* $B\in\mathbb{R}^{m\times m}$ *have eigenvalues* $\{\mu_j\}_{j=1}^m$. *Then the* mn *eigenvalues of* $A\otimes B$, *which represents the kronecker product of matrix* A *and* B, *are*

$$\lambda_1\mu_1,\cdots,\lambda_1\mu_m,\lambda_2\mu_1,\cdots,\lambda_2\mu_m,\cdots,\lambda_n\mu_1,\cdots,\lambda_n\mu_m.$$

Lemma 6[26] *For* $A\in\mathbb{C}^{n\times n}$, *let* $H=\frac{A+A^*}{2}$ *be the hermitian part of* A *and* A^* *be the conjugate transpose of* A, *then for any eigenvalue* λ *of* A, *it has*

$$\lambda_{\min}(H)\le\mathrm{Re}(\lambda)\le\lambda_{\max}(H),$$

where $\mathrm{Re}(\lambda)$ *represents the real part of* λ, $\lambda_{\min}(H)$ *and* $\lambda_{\max}(H)$ *are respectively the minimum and maximum of the eigenvalues of* H.

Theorem 1 *The FIDS* (15) *is unconditionally stable.*

Proof According to Lemma 2, it has

$$\begin{aligned}G_x^{\alpha}G_y^{\beta}=G_y^{\beta}G_x^{\alpha}=\xi_\alpha\xi_\beta D_{M_y-1}\otimes D_{M_x-1}-\xi_\alpha\zeta_\beta W_{M_y-1}^{(\beta)}\otimes D_{M_x-1}\\ -\zeta_\alpha\xi_\beta D_{M_y-1}\otimes W_{M_x-1}^{(\alpha)}+\zeta_\alpha\zeta_\beta W_{M_y-1}^{(\beta)}\otimes W_{M_x-1}^{(\alpha)},\end{aligned}$$

then

$$\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)=\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big),$$

which yields that any two of the matrices

$$\Big(I-\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big),\ \ \Big(I-\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big),\ \ \Big(I+\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big)^{-1}\ \text{and}\ \Big(I+\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big)^{-1}$$

can be commuted.

Here we suppose $\widehat{V}^n$ is the approximate solution of the implicit difference scheme (15) and denote $\epsilon^n=\widetilde{V}^n-\widehat{V}^n,\ n=0,1,\cdots,\mathcal{N}$, then ϵ^n satisfies:

$$\Big[I+\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big]\Big[I+\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big]\epsilon^{n+1}=\Big[I-\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big]\Big[I-\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big]\epsilon^{n}.$$

Therefore,

$$\begin{aligned}\epsilon^n&=\Big[I+\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I+\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big]\Big[I-\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big]\epsilon^{n-1}\\ &=\cdots\cdots\\ &=\Big\{\Big[I+\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_y^{\beta}+\frac{\eta}{2}I\Big)\Big]\Big\}^n\cdot\Big\{\Big[I+\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_x^{\alpha}+\frac{\eta}{2}I\Big)\Big]\Big\}^n\epsilon^0.\end{aligned}$$

Calculate the symmetric form of G_x^{α} by Lemma 3 as:

$$\begin{aligned}\frac{G_x^{\alpha}+(G_x^{\alpha})^{\mathrm{T}}}{2}&=\frac{1}{2}[\xi_\alpha I_{M_y-1}\otimes(D_{M_x-1}+D_{M_x-1}^{\mathrm{T}})-\zeta_\alpha I_{M_y-1}\otimes(W_{M_x-1}^{(\alpha)}+(W_{M_x-1}^{(\alpha)})^{\mathrm{T}})]\\ &=-\frac{1}{2}\zeta_\alpha I_{M_y-1}\otimes(W_{M_x-1}^{(\alpha)}+(W_{M_x-1}^{(\alpha)})^{\mathrm{T}}).\end{aligned}$$

Similarly,

$$\frac{G_y^{\beta}+(G_y^{\beta})^{\mathrm{T}}}{2}=-\frac{1}{2}\zeta_\beta(W_{M_y-1}^{(\beta)}+(W_{M_y-1}^{(\beta)})^{\mathrm{T}})\otimes I_{M_x-1}.$$

According to Lemma 4, the real parts of the eigenvalues of $[W_{M_x-1}^{(\alpha)}+(W_{M_x-1}^{(\alpha)})^{\mathrm{T}}]/2$ and $[W_{M_y-1}^{(\beta)}+(W_{M_y-1}^{(\beta)})^{\mathrm{T}}]/2$ are all negative for $1<\alpha,\beta\le 2$. Furthermore, from the consequences of Lemma 5 it has the real part of the eigenvalues of $[G_x^{\alpha}+(G_x^{\alpha})^{\mathrm{T}}]/2$

and $[G_y^\beta + (G_y^\beta)^{\mathrm{T}}]/2$ are all positive for $\zeta_\alpha, \zeta_\beta > 0$. Let λ_α and λ_β be the eigenvalues of matrices G_x^α and G_y^β, respectively, then the real parts of λ_α and λ_β are both great than zero by Lemma 6. Since

$$\Big(1 + \frac{\eta}{2} + \lambda_\beta\Big)^{-1}\Big(1 - \frac{\eta}{2} - \lambda_\beta\Big) \quad \text{and} \quad \Big(1 + \frac{\eta}{2} + \lambda_\alpha\Big)^{-1}\Big(1 - \frac{\eta}{2} - \lambda_\alpha\Big)$$

are the eigenvalues of matrices

$$\Big[I + \Big(G_y^\beta + \frac{\eta}{2}I\Big)\Big]^{-1}\Big[I - \Big(G_y^\beta + \frac{\eta}{2}I\Big)\Big] \quad \text{and} \quad \Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]^{-1}\Big[I - \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big],$$

respectively, the spectral radius of these two matrices are both less than 1 for $\eta > 0$, which yields that

$$\Big\{\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]\Big\}^n \quad \text{and} \quad \Big\{\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]\Big\}^n$$

converge to zero matrix when n approximates to infinity. Therefore the FIDS (15) is unconditionally stable.

Proposition 1 *G_x^α and G_y^β defined in* (16) *satisfy*

$$\Big\|\Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]^{-1}\Big[I + \Big(G_y^\beta + \frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2 \leq 1,$$
$$\Big\|\Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]^{-1}\Big[I - \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]\Big\|_2 \leq 1,$$
$$\Big\|\Big[I + \Big(G_y^\beta + \frac{\eta}{2}I\Big)\Big]^{-1}\Big[I - \Big(G_y^\beta + \frac{\eta}{2}I\Big)\Big]\Big\|_2 \leq 1,$$

where $\|\cdot\|_2$ *is the* 2*-norm.*

Proof From the proof of Theorem 1, we know the real parts of the eigenvalues of $G_x^\alpha + (G_x^\alpha)^{\mathrm{T}}$ and $G_y^\beta + (G_y^\beta)^{\mathrm{T}}$ are all greater than 0. Furthermore, $G_x^\alpha + (G_x^\alpha)^{\mathrm{T}}$ and $G_y^\beta + (G_y^\beta)^{\mathrm{T}}$ are both real symmetric matrices, which yields $G_x^\alpha + (G_x^\alpha)^{\mathrm{T}}$ and $G_y^\beta + (G_y^\beta)^{\mathrm{T}}$ are positive definite.

For any real vector $v = (v_1, v_2, \cdots, v_{M_y-1})^{\mathrm{T}}$ with $v_j = (v_{1,j}, v_{2,j}, \cdots, v_{M_x-1,j})$ $(j = 1, 2, \cdots, M_y - 1)$ and $\eta > 0$, we have

$$\begin{aligned} & v^{\mathrm{T}}\Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]^{\mathrm{T}}\Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]v \\ = & \Big(1 + \frac{\eta}{2}\Big)^2 v^{\mathrm{T}}v + \Big(1 + \frac{\eta}{2}\Big)v^{\mathrm{T}}(G_x^\alpha + (G_x^\alpha)^{\mathrm{T}})v + \|G_x^\alpha v\|_2^2 \geq v^{\mathrm{T}}v. \end{aligned}$$

Substituting v and v^{T} by $(I + (G_x^\alpha + \frac{\eta}{2}I))^{-1}v$ and $v^{\mathrm{T}}((I + (G_x^\alpha + \frac{\eta}{2}I))^{\mathrm{T}})^{-1}$ in the above formula, respectively, it obtains

$$v^{\mathrm{T}}\Big\{\Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]^{\mathrm{T}}\Big\}^{-1}\Big[I + \Big(G_x^\alpha + \frac{\eta}{2}I\Big)\Big]^{-1}v \leq v^{\mathrm{T}}v,$$

which leads to

$$\Big\|\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2=\sup_{v\neq 0}\frac{\sqrt{v^{\mathrm{T}}((I+(G_x^\alpha+\frac{\eta}{2}I))^{\mathrm{T}})^{-1}(I+(G_x^\alpha+\frac{\eta}{2}I))^{-1}v}}{\sqrt{v^{\mathrm{T}}v}}\leq 1.$$

Similarly, it can be obtained that $\|(I+(G_y^\beta+\frac{\eta}{2}I))^{-1}\|_2\leq 1$. Therefore,

$$\begin{aligned}&\Big\|\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2\\ \leq&\Big\|\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2\cdot\Big\|\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2\leq 1.\end{aligned}$$

Moreover, according to the positive definiteness of matrices $G_x^\alpha+(G_x^\alpha)^{\mathrm{T}}$ and $G_y^\beta+(G_y^\beta)^{\mathrm{T}}$, for any real vector v, it gets

$$\begin{aligned}&v^{\mathrm{T}}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{\mathrm{T}}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]v\\ =&\Big(1-\frac{\eta}{2}\Big)^2v^{\mathrm{T}}v-\Big(1-\frac{\eta}{2}\Big)v^{\mathrm{T}}(G_x^\alpha+(G_x^\alpha)^{\mathrm{T}})v+\|G_x^\alpha v\|_2^2\\ \leq&\Big(1+\frac{\eta}{2}\Big)^2v^{\mathrm{T}}v+\Big(1+\frac{\eta}{2}\Big)v^{\mathrm{T}}(G_x^\alpha+(G_x^\alpha)^{\mathrm{T}})v+\|G_x^\alpha v\|_2^2\\ =&v^{\mathrm{T}}\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{\mathrm{T}}\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]v.\end{aligned}$$

Then

$$v^{\mathrm{T}}\Big\{\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{\mathrm{T}}\Big\}^{-1}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{\mathrm{T}}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}v\leq v^{\mathrm{T}}v,$$

which means

$$\Big\|\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2\leq 1.$$

Consequently,

$$\Big\|\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]\Big\|_2\leq 1$$

holds because of the commutativity of the matrices $(I+(G_x^\alpha+\frac{\eta}{2}I))^{-1}$ and $I-(G_x^\alpha+\frac{\eta}{2}I)$.

Similarly,

$$\Big\|\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]\Big\|_2\leq 1$$

is valid. The proof is completed.

Theorem 2 *Let $V_{i,j}^n$ be the exact solution of model* (2) *and $\widetilde{V}_{i,j}^n$ be the solution of the discrete equation* (15). *Then for $1<\alpha,\beta\leq 2$, it obtains*

$$\|V^n-\widetilde{V}^n\|_2\leq C(h_x^2+h_y^2+\tau^2),\quad n=1,2,\cdots,\mathcal{N},$$

where C is a positive constant.

Proof Let $e_{i,j}^n = V_{i,j}^n - \widetilde{V}_{i,j}^n$. Subtracting (14) from (15) leads to

$$\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]e^{n+1} = \Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]\Big[I-\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]e^n+\tau\varepsilon^n,$$

where

$$e^n=(e_{1,1}^n,e_{2,1}^n,\cdots,e_{M_x-1,1}^n,e_{1,2}^n,e_{2,2}^n,\cdots,e_{M_x-1,2}^n,\cdots,e_{1,M_y-1}^n,e_{2,M_y-1}^n,\cdots,e_{M_x-1,M_y-1}^n)^{\mathrm{T}},$$
$$\varepsilon^n=(\varepsilon_{1,1}^n,\varepsilon_{2,1}^n,\cdots,\varepsilon_{M_x-1,1}^n,\varepsilon_{1,2}^n,\varepsilon_{2,2}^n,\cdots,\varepsilon_{M_x-1,2}^n,\cdots,\varepsilon_{1,M_y-1}^n,\varepsilon_{2,M_y-1}^n,\cdots,\varepsilon_{M_x-1,M_y-1}^n)^{\mathrm{T}}.$$

Then

$$\begin{aligned} e^n &= Qe^{n-1}+\tau\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\varepsilon^{n-1} \\ &= \cdots\ \cdots \\ &= Q^n e^0+\tau\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\sum_{k=0}^{n-1}Q^k\varepsilon^{n-k-1}, \end{aligned}$$

where

$$Q=\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I-\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big].$$

Since $e^0=0$ and $\|Q^k\|_2\le\|Q\|_2^k$, according to Proposition 1 it gets

$$\begin{aligned} \|e^n\|_2 &\le \tau\Big\|\Big[I+\Big(G_y^\beta+\frac{\eta}{2}I\Big)\Big]^{-1}\Big[I+\Big(G_x^\alpha+\frac{\eta}{2}I\Big)\Big]^{-1}\Big\|_2\sum_{k=0}^{n-1}\|Q^k\|_2\|\varepsilon^{n-k-1}\|_2 \\ &\le \tau\sum_{k=0}^{n-1}\|\varepsilon^{n-k-1}\|_2\le C(h_x^2+h_y^2+\tau^2). \end{aligned}$$

The proof is completed.

4 Fast Bi-conjugate Gradient Stabilized (FBi-CGSTAB) Method for Solving the FIDS

The implicit difference discretization of equation (2) results in a linear system with a dense lower Hessenberg block coefficient matrix. Solving the linear system (9) directly, such as by Gaussian elimination method, requires a computational cost of $O(M_x^3M_y^3)$ per time step and storage memory of $O(M_x^2M_y^2)$, which represents a high computational expense. Moreover, the ever-changing market information and enormous financial data lead to the complexity of the market. These urgently require fast computing. In the section, we present an efficient iterative method to solve (9) by combing the bi-conjugate gradient stabilized (Bi-CGSTAB) method [30] and fast Fourier transform (FFT), which significantly reduces the computational cost to $O(M_xM_y\log(M_xM_y))$ per time iteration and the storage space to $O(M_xM_y)$.

By a simple calculation, M in formula (9) can be decomposed as

$$M = \eta I_{M_x-1} \otimes I_{M_y-1} + G_x^\alpha + G_y^\beta. \tag{19}$$

According to the formulae (16) and (17), it only needs to store $\omega^{(\alpha)} = (\omega_0^{(\alpha)}, \omega_1^{(\alpha)}, \cdots, \omega_{M_x}^{(\alpha)})^{\mathrm{T}}$ and $\omega^{(\beta)} = (\omega_0^{(\beta)}, \omega_1^{(\beta)}, \cdots, \omega_{M_y}^{(\beta)})^{\mathrm{T}}$ instead of the full matrix M. Moreover, the elements of the vector $C^{n+\frac{1}{2}}$ and $F^{n+\frac{1}{2}}$ also need to be stored. Then the total memory requirement has been significantly reduced $M_x + M_y + 2 + (M_x - 1)(M_y - 1) = O(M_x M_y)$.

In the following, we consider the computational cost.

Since using the conjugate gradient squared method to solve the nonsymmetric linear system (9) may arise the irregular convergence patterns, in order to avoid the problem, the Bi-CGSTAB method is applied to (9) ([2, 30]).

Algorithm 1: The Bi-CGSTAB Method for (9)

Step1: In each time level t^k, let $\mathbf{V}^{(0)} = \mathbf{V}^{k-1}$, $\mathbf{b} = [I - M]\mathbf{V}^{k-1} + \mathcal{C}^{k-1} - \tau F^{k-1}$
Step2: Compute $\mathbf{r}^{(0)} = \mathbf{b} - (I + M)\mathbf{V}^{(0)}$;
  Choose an arbitrary vector $\mathbf{R}$ such that $(\mathbf{R}, \mathbf{r}^{(0)}) \neq 0$, e.g., $\mathbf{R} = \mathbf{r}^{(0)}$
Step3: $\rho_0 = a_0 = w_0 = 1, \mathbf{v}^{(0)} = \mathbf{p}^{(0)} = \mathbf{0}$
Step4: **for** $i = 1, 2, \cdots$
  $\rho_i = (\mathbf{R}, \mathbf{r}^{(i-1)})$
  if $\rho_i = 0$, **invalid**
  else continue
   $\beta_i = (\rho_i/\rho_{i-1})(a_{i-1}/w_{i-1})$
   $\mathbf{p}^{(i)} = \mathbf{r}^{(i-1)} + \beta_i(\mathbf{p}^{(i-1)} - w_{i-1}\mathbf{v}^{(i-1)})$
   $\mathbf{v}^{(i)} = (I + M)\mathbf{p}^{(i)}$
   $a_i = \rho_i/(\mathbf{R}, \mathbf{v}^{(i)})$
   $\mathbf{s} = \mathbf{r}^{(i-1)} - a_i\mathbf{v}^{(i)}$
  Check the norm of $\mathbf{s}$, if ($\|\mathbf{s}\| < tol$), then $\mathbf{V}^{(i)} = \mathbf{V}^{(i-1)} + a_i\mathbf{p}^{(i)}$, stop
  $\mathbf{t} = (I + M)\mathbf{s}$
  $w_i = (\mathbf{t}, \mathbf{s})/(\mathbf{t}, \mathbf{t})$
  $\mathbf{V}^{(i)} = \mathbf{V}^{(i-1)} + a_i\mathbf{p}^{(i)} + w_i\mathbf{s}$
  $\mathbf{r}^{(i)} = \mathbf{s} - w_i\mathbf{t}$
  Check accuracy; If $\mathbf{V}^{(i)}$ is accurate enough then quit, else continue
 end for
Step5: Output $\mathbf{V}^{(k)} = \mathbf{V}^{(i)}$, error=$\|\mathbf{s}\|/\|\mathbf{b}\|$.

In Algorithm 1, in addition to the operation of matrix-vector multiplication, the computational costs of other operations are only $O(M_x M_y)$, so we have to compute the matrix-vector multiplication in an efficient way (that is FFT) to reduce the computational complexity. Since the matrices D_{M_x-1}, D_{M_y-1}, $W_{M_x-1}^{(\alpha)}$ and $W_{M_y-1}^{(\beta)}$ are all Toeplitz matrices, and noting the decomposition in (16) and (19), we only need to consider the computational cost of $(\mathfrak{L}_{M_y-1}\otimes I_{M_x-1})\mathbf{v}$ for the Toeplitz matrix $\mathfrak{L}_{M_y-1}$ ($\mathfrak{L}(i,j)=l_{j-i}$) and $\mathbf{v}\in\mathbb{R}^{(M_x-1)(M_y-1)}$.

First we embed $\mathfrak{L}_{M_y-1}$ into a $2(M_x-1)(M_y-1)\times 2(M_x-1)(M_y-1)$ block-circulant-circulant-block matrix $\mathbf{C}_{2(M_x-1)(M_y-1)}$ of the form

$$\mathbf{C}_{2(M_x-1)(M_y-1)}=\begin{pmatrix}\mathfrak{L}_{M_y-1}\otimes I_{M_x-1} & \mathfrak{D}_{M_y-1}\otimes I_{M_x-1}\\ \mathfrak{D}_{M_y-1}\otimes I_{M_x-1} & \mathfrak{L}_{M_y-1}\otimes I_{M_x-1}\end{pmatrix},$$

where

$$\mathfrak{D}_{M_y-1}=\begin{pmatrix}0 & l_{2-M_y} & \cdots & l_{-2} & l_{-1}\\ l_{M_y-2} & 0 & l_{2-M_y} & \ddots & l_{-2}\\ \vdots & l_{M_y-2} & 0 & \ddots & \vdots\\ l_2 & \ddots & \ddots & \ddots & l_{2-M_y}\\ l_1 & l_2 & \cdots & l_{M_y-2} & 0\end{pmatrix}.$$

Then the matrix-vector multiplication can be calculated as follows:

$$\begin{aligned}\mathbf{C}_{2(M_x-1)(M_y-1)}\begin{pmatrix}\mathbf{v}\\ \mathbf{0}\end{pmatrix}&=\begin{pmatrix}\mathfrak{L}_{M_y-1}\otimes I_{M_x-1} & \mathfrak{D}_{M_y-1}\otimes I_{M_x-1}\\ \mathfrak{D}_{M_y-1}\otimes I_{M_x-1} & \mathfrak{L}_{M_y-1}\otimes I_{M_x-1}\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{0}\end{pmatrix}\\ &=\begin{pmatrix}(\mathfrak{L}_{M_y-1}\otimes I_{M_x-1})\mathbf{v}\\ (\mathfrak{D}_{M_y-1}\otimes I_{M_x-1})\mathbf{v}\end{pmatrix}.\end{aligned}$$

It is well known that the circulant matrix $\mathbf{C}_{2(M_x-1)(M_y-1)}$ can be decomposed as

$$\mathbf{C}_{2(M_x-1)(M_y-1)}=(F_{2(M_y-1)}\otimes F_{2(M_x-1)})^{-1}\mathrm{diag}(\mathbf{c})(F_{2(M_y-1)}\otimes F_{2(M_x-1)}),$$

where $\mathbf{c}$ is the corresponding two dimensional FFT of the first column vector of $\mathbf{C}_{2(M_x-1)(M_y-1)}$, and $F_{2(M-1)}$ is the $2(M-1)\times 2(M-1)$ discrete Fourier transform matrix given by

$$F_{2(M-1)}(j,k)=\frac{1}{\sqrt{2(M-1)}}\exp\Big(\frac{\pi\mathbf{i}jk}{M-1}\Big),\quad \mathbf{i}\equiv\sqrt{-1},$$

for $1\le j,k\le 2(M-1)-2$. For any vector $\mathbf{v}$, the matrix-vector multiplication $(\mathfrak{L}_{M_y-1}\otimes I_{M_x-1})\mathbf{v}$ can be carried out in $O((M_x-1)(M_y-1)\log(M_x-1)(M_y-1))$ operations via the FFT (see [31]). Therefore, by FBi-CGSTAB method, the computational cost of solve linear system (9) has been significantly reduced to $O(M_xM_y\log(M_xM_y))$ per time iteration.

5 Numerical Experiments

Example 1 Consider the following fractional diffusion-convection-reaction equation with a source term

$$\begin{cases} \dfrac{\partial V}{\partial t} + (r - v_\alpha)\dfrac{\partial V}{\partial x} + (r - v_\beta)\dfrac{\partial V}{\partial y} + v_\alpha \cdot {}_0D_x^\alpha V + v_\beta \cdot {}_0 D_y^\beta V = rV + f, \\ \qquad\qquad\qquad\qquad\qquad (x, y) \in \Omega = (0, 1) \times (0, 1),\ t \in [0, T), \\ V(x, y, T) = x^3 y^4, \quad (x, y) \in \Omega, \\ V(0, y, t) = V(x, 0, t) = 0, \\ V(1, y, t) = y^4 \mathrm{e}^{T-t}, \quad V(x, 1, t) = x^3 \mathrm{e}^{T-t}, \quad 0 < t \le T, \end{cases} \tag{20}$$

where the source term

$$f(x, t) = \mathrm{e}^{T-t}\Big[- (1 + r)x^3y^4 + 3(r - v_\alpha)x^2y^4 + 4(r - v_\beta)x^3y^3 + v_\alpha \frac{\Gamma(4)}{\Gamma(4 - \alpha)} x^{3-\alpha}y^4 + v_\beta \frac{\Gamma(5)}{\Gamma(5 - \beta)} x^3 y^{4-\beta} \Big].$$

The exact solution of this equation is $V(x, t) = x^3y^4\mathrm{e}^{T-t}$.

In Example 1, we always take the values of parameters as $\alpha = 1.7,\ \beta = 1.8,\ r = 0.05,\ \sigma = 0.25$ and $T = 1.0$.

Tables 1 and 2 list the errors in the maximum-norm and its corresponding convergence order of equation (20). From these tables, it can be seen that the numerical solution obtained by the FIDS (7) is very close to the exact solution. Furthermore, the convergence order of FIDS is in good agreement with the conclusion of Theorem 2. Here the convergence orders of the FIDS are computed by $\log_{\frac{h_1}{h_2}} \frac{error_1}{error_2}$ for the spatial step size $h = h_x = h_y = \frac{1}{M}$ and $\log_{\frac{\Delta\tau_1}{\Delta\tau_2}} \frac{error_1}{error_2}$ for the temporal step size $\Delta\tau = \frac{1}{N}$, respectively. The notation $error_i$ corresponds to the error when $h = h_i$ or $\Delta\tau = \Delta\tau_i$.

Table 1: Numerical errors and orders of convergence for Example 1 when $N = 1000,\ M_x = M_y := M$.

M	Max-error	order
2^3	3.4836×10^{-4}	
2^4	9.3998×10^{-5}	1.89
2^5	2.4365×10^{-5}	1.95
2^6	6.2067×10^{-6}	1.97
2^7	1.5781×10^{-6}	1.98

Table 2: Numerical errors and orders of convergence for Example 1 when $N = M_x = M_y$.

N	Max-error	order
2^4	4.1772×10^{-4}	
2^5	1.1199×10^{-4}	1.90
2^6	2.8894×10^{-5}	1.95
2^7	7.3267×10^{-6}	1.98
2^8	1.8445×10^{-6}	1.99

The consumed CPU times to run the Gaussian left division method, the Bi-CGSTAB method and the FBi-CGSTAB method for fixed temporal step $\tau = 1/300$ are respectively displayed in Table 3. Especially, for the Gaussian left division method and the Bi-CGSTAB method, the computer does not work properly because there is no sufficient storage space when $M_x = M_y = 2^8$. From which it can be seen that the FBi-CGSTAB method has significantly reduced the computational requirement and the storage space. We carried out all numerical computations by using MATLAB on Lenovo L430 laptop with configuration: Intel(R) Core(TM) i5-7500, 3.40GHz and 8.0G RAM.

Table 3: Comparison of the consumed CPU time of FBi-CGSTAB versus Gauss left division and Bi-CGSTAB under the same accuracy in Example 1.

	$M_x = M_y$	CPU time
Gaussian left division	2^5	20.4063s
	2^6	779.7031s $\approx$ 13min
	2^7	3.6428e+04s $\approx$ 10h7min8s
	2^8	***
Bi-CGSTAB	2^5	1.5469s
	2^6	37.6563s
	2^7	0.1034e+04s$\approx$ 17min14s
	2^8	***
FBi-CGSTAB	2^5	0.5313s
	2^6	5.6250s
	2^7	32.0625s
	2^8	287.7696s$\approx$ 4min48s

Example 2 We now consider the following FBS model on two assets for a European Call-on-Min option

$$\begin{cases} \dfrac{\partial V}{\partial t} + (r - v_\alpha)\dfrac{\partial V}{\partial x} + (r - v_\beta)\dfrac{\partial V}{\partial y} + v_\alpha \cdot \ {}_0D_x^\alpha V + v_\beta \cdot {}_0\, D_y^\beta V = rV, \\ \qquad\qquad\qquad (x, y) \in \Omega = (\ln 0.1, \ln 100) \times (\ln 0.1, \ln 100),\ t \in [0, T), \\ V(x, y, T) = \max\{\min(\mathrm{e}^x, \mathrm{e}^y) - K, 0\}, \quad (x, y) \in \Omega, \\ V(x, y, t) = \max\{\min(\mathrm{e}^x, \mathrm{e}^y) - K \cdot \mathrm{e}^{-r(T-t)}, 0\}, \quad (x, y) \in \partial\Omega,\ t \in (0, T], \end{cases} \tag{21}$$

where K is the strike price and T is the expiry time.

We use the FIDS (7) to solve this model approximately. Here taking the parameter $K = 50,\ T = 1,\ r = 0.05$. The numerical results of the European Call-on-Min option are plotted in Figure 1 against the stock prices $S_1 = \mathrm{e}^x$ and $S_2 = \mathrm{e}^y$ with $\alpha = \beta = 1.5$ and $\sigma = 0.25$, which shows a fat tail characteristics.

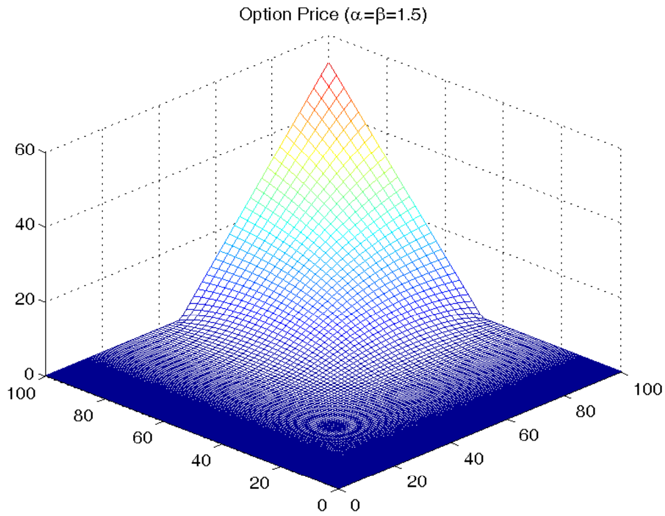

Figure 1: Computed Prices of a European Call-on-Min Option

To illustrate the influence of α and β on the option price, we calculate the price of European Call-on-Min option numerically for different α and β, and plot the difference curves between the numerical solutions of the FBS model (21) and the classical B-S model in Figure 2 at $t = 0$.

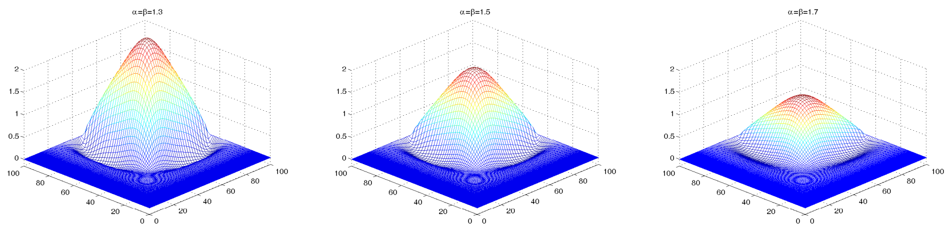

Figure 2: $V_{FBS} - V_{BS}$ for different α and β when $\sigma = 0.25$

From Figure 2, one can see that the call option price obtained by FBS model increases as α and β decrease when the stock prices S_1 and S_2 are greater than a critical value (but close to the strike price), which likely implies that the option price exhibits a jump (or convective) nature when α and β are close to 1, the jump nature of the FMLS process delivers much higher prices. While α and β are close to 2, the corresponding option price mainly presents a diffusive nature. Furthermore, in order to observe the effect of volatility rate σ on the option price, the difference curves for different σ are presented in Figure 3 when $\alpha = \beta = 1.5$. Figure 3 shows that the difference value increases as the volatility rate σ increases when S_1 and S_2 are greater than a critical value, which suggests that the FBS model is more sensitive to volatility change compared with the classical B-S model. It would be indicated that the FBS model can better capture the dynamic process of option price changes.

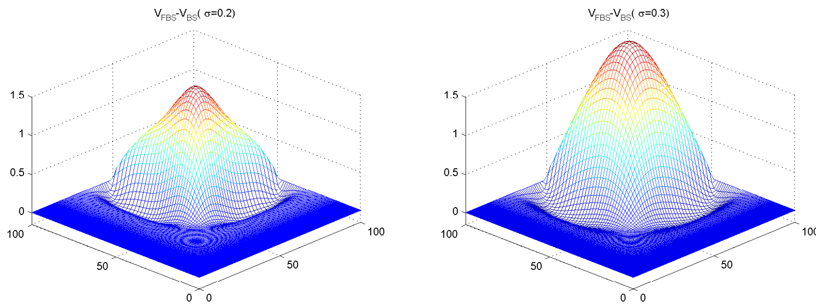

Figure 3: $V_{FBS} - V_{BS}$ for different σ

6 Conclusion

The complexity of the financial markets and non-locality of the fractional derivative operator yield a huge demand for feasible, fast and high accuracy numerical simulations to FBS models. In the paper, a FIDS is constructed for a 2D FBS model, which is unconditional stable and second-order convergent. Then, in order to solve the resultant linear system quickly and effectively, a FBi-CGSTAB is presented which significantly reduces the computational cost to $O(M_xM_y\log(M_xM_y))$ per iteration and the storage space to $O(M_xM_y)$. One numerical example with exact solution is chosen to confirm our theoretical analysis. The comparison of CPU time spent on running the Gaussian left division method, the Bi-CGSTAB mehtod and the FBi-CGSTAB method highlights the remarkable effectiveness of the FBi-CGSTAB in rapid computation. It is worth mentioning that this fast and high accuracy numerical technique can be applied to similar models.

A European Call-on-Min option is priced by the FBS model and the proposed numerical technique, which can also be used to price other more complex options, such as barrier options. For different α, β and σ, the difference curves of option prices between the FBS model and the classical B-S model are plotted. From these graphical features, we may think that the FBS model based on the Lévy process not only presents the feature of fat tail, but also can capture some extreme but realistic events, such as sudden jumps of prices, and thus more correctly simulate the dynamics of option prices in markets with jumps than the classical B-S model. We believe that these findings provide useful information for further applications of the fractional calculus in financial market.

References

[1] J.P. Aguilar, C. Coste and J. Korbel, Series representation of the pricing formula for the European option driven by space-time fractional diffusion, *Fractal Fract.*, arXiv:1712.04990v22018.

[2] R. Barrett, M. Berry, T.F. Chen, J. Demmel, J.M. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine and H.V. der Vorst, Templates for the Solution of Linear Systems: Building Blocks for Iteratve Methods, SIAM, Philadephia, 1994.

[3] S.K. Bhowmik, Fast and efficient numerical methods for an extended Black-Scholes model, *Comput. Math. Appl.*, **67**(2014),636-654.

[4] F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, **81**(1973),637-659.

[5] S. Boyarchenko and S. Levendorskiǐ, Non-Gaussian Merton-Black-Scholes Theory, vol.9, World Scientific, Singapore, 2002.

[6] P. Carr, H. Geman, D.B. Madan and M. Yor, Stochastic volatility for Lévy processes, *Math. Finance*, **13**(2003),345-382.

[7] P. Carr and L. Wu, The finite moment log stable process and option pricing, *J. Finance*, **58**:2(2003),597-626.

[8] Á. Cartea and D. del-Castillo-Negrete, Fractional diffusion models of option prices in markets with jumps, *Phys. A*, **374**:2(2007),749-763.

[9] W.T. Chen, X. Xu and S.P. Zhu, Analytically pricing European-style options under the modified Black-Scholes equation with a spatial-fractional derivative, *Quart. Appl. Math.*, **72**:3(2014),597-611.

[10] W. Chen and S. Wang, A finite difference method for pricing European and American options under a geometric Lévy process, *J. Ind. Manag. Optim.*, **11**(2015),241-264.

[11] W. Chen and S. Wang, A 2nd-order FDM for a 2D fractional Black-Scholes equation, *Numerical Analysis and Its Applications*, NAA 2016, *Lecture Notes in Comput. Sci.*, **10187**(2017),46-57.

[12] M.H.A. Davis, V.G. Panas and T. Zariphopoulou, Eouropean option pricing with transaction costs, *SIAM J. Control Optim.*, **31**:2(1993),470-493.

[13] S. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Finan. Stud.*, **6**(1993),327-343.

[14] G. Jumarie, Stock exchange fractional dynamics defined as fractional exponential growth driven by (usual) Gaussian white noise. Application to fractional Black-Scholes equations, *Insurance Math. Econom.*, **42**:1(2008),271-287.

[15] G. Jumarie, Derivation and solutions of some fractional Black Scholes equations in coarse-grained space and time. Application to Mertons optimal portfolio, *Comput. Math. Appl.*, **59**:3(2010),1142-1164.

[16] G. Karipova and M. Magdziarz, Pricing of basket options in subdiffusive fractional BlackCScholes model, *Chaos Solitons Fractals*, **102**(2017),245-253.

[17] I. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, *Phys. Rev. E*, **52**(1995),1197-1199.

[18] A.J. Laub, Matrix Analysis for Scientists and Engineers, SIAM, 2005.

[19] B. Mandelbrot, The pareto-levy law and the distribution of income, *Internat. Econom. Rev.*, **1**:2(1960),79-106.

[20] B. Mandelbrot, The variation of certain speculative prices, *J. Bus. Univ. Chicago*, **36**(1963),39-419.

[21] O. Marom and E. Momoniat, A comparison of numerical solution of fractional diffusion models in finance, *Nonlinear Anal. Real World Appl.*, **10**(2009),3435-3442.

[22] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, *J. Comput. Appl. Math.*, **172**(2004),65-77.

[23] R.C. Merton, Theory of Rational Option Pricing, *Bell J. Econom. Manag. Science* (*The RAND Corporation*), **4**:1(1973),141-183.

[24] R.C. Merton, Option pricing when underlying stock returns are discontinuous, *J. Finan. Econom.*, **3**:1-2(1976),125-144.

[25] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.

[26] A. Quarteroni, R. Sacco and F. Saleri, Numerical Mathematics, 2nd ed., Springer, 2007.

[27] W. Schoutens, Levy Process in Finance, Wiley Ser. Probab. Stat., 2003.

[28] W. Schoutens, Exotic options under levy models: An overview, *J. Comput. Appl. Math.*, **189**:1(2006),526-538.

[29] W.Y. Tian, H. Zhou and W.H. Deng, A class of second order difference approximation for solving space fractional diffusion equations, arXiv:1201.5949v3 [math.NA] 7 Mar., 2012.

[30] H.A. Van der Vorst, Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems, *SIAM J. Sci. and Stat. Comput.*, **13**:2(1992),631-644.

[31] H. Wang, K. Wang and T. Sircar, A direct $O(N\log^2 N)$ finite difference method for fractional diffusion equations, *J. Comput. Phys.*, **229**(2010),8095-8104.

[32] H. Zhang, F. Liu, I. Turner and S. Chen, The numerical simulation of the tempered fractional Black-Scholes equation for European double barrier option, *Appl. Math. Model.*, **40**(2016),5819-5834.

[33] H. Zhang, F. Liu, I. Turner, S. Chen and Q. Yang, Numerical simulation of a finite moment Log stable model for a European call option, *Numer. Algorithms*, **75**:3(2017),569-585.

[34] H. Zhang, F. Liu, S. Chen, V. Anh and J. Chen, Fast numerical simulation of a new time-space fractional option pricing model governing European call option, *App. Math. Comput.*, **339**(2018),186-198.

(*edited by Liangwei Huang*)