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HOPF BIFURCATION ANALYSIS IN A MONOD-HALDANE PREDATOR-PREY MODEL WITH THREE DELAYS*

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Abstract

We analysis Hopf bifurcation in a Monod-Haldane predator-prey model with three delays in this paper. Fixing τ_1 and τ_2 and taking τ_3 as parameter, the direction and stability of Hopf bifurcation are studied by using center manifold theorem and normal form. At last some simulations are given to support our results.

Keywords Hopf bifurcation; Monod-Haldane predator-prey model; delays
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1 Introduction

The dynamics of predator-prey model with delay have attract many interest for researchers [1-8], For example [3, 8] discussed the effect of delay on the global stability for predator-prey system, [6] discussed Hopf bifurcation of a ratio-dependent predator-prey system with two delays, besides [4] studied Hopf bifurcation of delayed predator-prey model with stage-structure for prey.

We know that Holling functional response is usually used to represent the grasping for predator and the functional responses is usually monotonic. But in microbial dynamics or chemical kinetics, the functional response represents the uptake of substance by the microorganisms, and the nonmonotonic responses occur by experiment. For example the inhibitory effect on the growth rate occurs when the nutrient concentration reaches a sufficient level [9]. This case always exists when micro-organisms are used for waste decomposition or for water purification [11]. The response function $p(x) = \frac{mx}{a+bx+x^2}$ is proposed by Ardrew to present the inhibitory effect which is called Monod-Haldane function [10] at hight concentration. Besides Sokol and

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Howell [12] proposed a simple Monod-Haldane function of form $p(x) = \frac{mx}{a+x^2}$ to describe the uptake of phenol by a pure culture of Pseudomonas putida growing on phenol in a continuous culture. In [13] the ability of predator for prey is expressed by simplified Monod-Haldane function $p(x) = \frac{x}{a+x^2}$. So we think Monod-Haldane function in predator-prey model should be more realistic in some situation. Besides the diffusion between patch is introduced into predator-prey model should be more reasonable. As we know, some kinds of delays always exist during the works about predator-prey system, such as the hunting delay for predator, the delay caused by gestation or maturation for predator and so on. Recently the following predator-prey system [21] with three delays was studied:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big(r_1 - a_1x_1(t-\tau_1) - \dfrac{c_1y(t-\tau_3)}{1+kx_1^2(t)}\Big) + \delta(x_2(t) - x_1(t)), \\ \dot{x}_2(t) = x_2(t)(r_2 - a_2x_2(t)) + \delta(x_1(t) - x_2(t)), \\ \dot{y}(t) = y(t)\Big(d_1 + \dfrac{c_2x_1(t)}{1+kx_1^2(t)} - d_2y(t-\tau_2)\Big), \end{cases} \tag{1.1}$$

with the initial condition:

$$x_i(\theta) > 0, \quad y(\theta) > 0, \quad \theta \in [-\tau, 0], \quad i = 1, 2, \quad \tau = \max\{\tau_1, \tau_2, \tau_3\}, \tag{1.2}$$

where $x_1(t), x_2(t)$ denote the numbers of prey species in patch 1 and patch 2 respectively, $y(t)$ denotes the numbers of predator species in patch 1, c_1 is the capture rate. Monod-Haldane response function $\frac{x_1}{1+kx_1^2}$ expresses the capture ability of predator, c_2 denotes the conversion rate, $r_i\,(i = 1, 2)$ is the birth rate of prey species in patch i respectively, $a_i\,(i = 1, 2)$ and d_2 are the coefficients of intra-specific competition, d_1 is the birth rate for predator, the delays τ_1, τ_2 represent negative feedback of prey and predator in patch 1 respectively, τ_3 is the hunting delay. δ is the diffusion coefficient.

Assuming $\tau_3 = 0$, the author [2] studied the Hopf bifurcation of system (1.1) with two delays (τ_1 and τ_2) under four cases: (1) $\tau_1 \neq 0$, $\tau_2 = 0$, (2) $\tau_1 = 0$, $\tau_2 \neq 0$, (3) $\tau_1 = \tau_2 = \tau \neq 0$, (4) $\tau_1 \neq \tau_2$, $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$. But delay τ_3 always exists in reality, we should consider its importance for dynamics. Although there are many works about Hopf bifurcation with two delays [6,14-17]. But in my opinion, the Hopf bifurcations with three delays [19,20] are rarely. In [20], the author considered the following model HIV-1 system with three delays:

$$\begin{cases} \dot{x} = \lambda - dx(t) - \beta x(t)v(t), \\ \dot{y} = \beta \mathrm{e}^{-a\tau_1}x(t-\tau_1)v(t-\tau_1) - ay(t) - \alpha w(t)y(t), \\ \dot{z} = \alpha w(t)y(t) - bz(t), \\ \dot{v} = k\mathrm{e}^{-a_2\tau_2}y(t-\tau_2) - pv(t), \\ \dot{w} = c\mathrm{e}^{-a_3\tau_3}z(t-\tau_3) - qw(t). \end{cases} \tag{1.3}$$

The dynamics were discussed under three cases: (1) $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$, (2) $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, $\tau_3 = 0$, (3) $\tau_1 \in (0, \tau_{10})$, $\tau_2 \in (0, \tau_{20})$, $\tau_3 > 0$.

Stimulated by above work, we should discuss the Hopf bifurcation of system (1.1) with three delays in this paper. In the first part, based on work [2], we give dynamics of system (1.1) under two cases: (1) $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$, (2) $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, $\tau_3 = 0$, then we discuss Hopf bifurcation of system (1.1) with three delays ($\tau_1 \in (0, \tau_{10})$, $\tau_2 \in (0, \tau_{2*})$, $\tau_3 > 0$) and give the direction and stability of Hopf bifurcation by center manifold theorem and normal form [18], at last we should give simulation to support our results.

2 Preliminary Work

Before our discussion, we give the following result:

Theorem 2.1 *The solution of system* (1.1) *with initial* (1.2) *is positive.*

Proof By the fundamental theory of functional differential equation [22], system (1.1) with initial condition (1.2) has a unique solution $(x_1(t), x_2(t), y(t))$, $t \in (0, +\infty)$.

Now we prove $x_1(t) > 0$, $t > 0$. Otherwise, there exists a $t^* \in (0, +\infty)$, $x_1(t^*) = 0$. We define $t_1 = \inf\{t \colon x_1(t) = 0\}$, from the first equation of (1.1) we get $\dot{x_1}(t_1) = \delta x_2(t_1) < 0$, so $x_2(t_1) < 0$. Define $t_2 = \inf\{t : x_2(t) = 0\}$, we know $t_2 < t_1$. From the second equation of (1.1), we get $\dot{x_2}(t_2) = \delta x_1(t_2) < 0$, so $x_1(t_2) < 0$, which is a contradiction.

Similarly, we prove $x_2(t) > 0$, $t > 0$. Suppose on the contrary that there exists a $t^* \in (0, +\infty)$, $x_2(t^*) = 0$. We define $t_3 = \inf\{t : x_2(t) = 0\}$, from the second equation of (1.1) we get $\dot{x_2}(t_3) = \delta x_1(t_3) < 0$, so $x_1(t_3) < 0$. Define $t_4 = \inf\{t : x_1(t) = 0\}$, we know $t_4 < t_3$. From the first equation of (1.1), we get $\dot{x_1}(t_4) = \delta x_2(t_4) < 0$, so $x_2(t_4) < 0$, which is a contradiction.

From the third equation of (1.1), we get

$$y(t) = y(0) \exp\Big(\int_0^t \Big(d_1 + \frac{c_2 x_1(s)}{1 + k x_1^2(s)} - d_2 y(s - \tau_2)\Big) \mathrm{d}s \Big) > 0, \quad t > 0.$$

The solution is positive, which means that preys and predator always exist.

From [2], we know system (1.1) has at least a positive equilibrium $E^*(x_1^*, x_2^*, y^*)$ where $x_2^* = g_2(x_1^*)$, $y^* = \frac{1}{d_2}\big(d_1 + \frac{c_2 x_1^*}{1 + k(x_1^*)^2}\big)$ and

$$g_2(x_1) = \frac{x_1}{\delta}(a_1 x_1 + \delta - r_1) + \frac{c_1 x_1}{d_2 \delta (1 + k x_1^2)}\Big(d_1 + \frac{c_2 x_1}{1 + k x_1^2}\Big),$$

if the following hypothesis hold:

(H1) $r_2 - \delta > 0,\ c_2 < 2 d_1 \sqrt{k}$,

furthermore we have:

Lemma 2.1[2] *The solutions of system* (1.1) *with initial condition* (1.2) *are ultimately bounded, that is, there exist positive constants* M_1, M_2 *and* T *such that for* $t \geq T$:

$$x(t) \leq M_1, \quad x_2(t) \leq M_1, \quad y(t) \leq M_2.$$

By transformation $u_1(t) = x_1(t) - x_1^*$, $u_2(t) = x_2(t) - x_2^*$, $u_3(t) = y(t) - y^*$, system (1.1) becomes

$$\begin{cases} \dot{u}_1(t) = a_{11}u_1(t) + a_{12}u_2(t) + a_{13}u_3(t-\tau_3) + a_{14}u_1(t-\tau_1) + g_1u_1^2(t) \\ \qquad\quad +g_2u_1(t)u_1(t-\tau_1) + g_3u_1(t)u_3(t-\tau_3) + g_4u_1^3(t) + g_5u_1^2(t)u_3(t-\tau_3) + hot, \\ \dot{u}_2(t) = a_{21}u_1(t) + a_{22}u_2(t) + h_1u_2^2(t) + hot, \\ \dot{u}_3(t) = a_{31}u_1(t) + a_{32}u_3(t-\tau_2) + k_1u_1^2(t) + k_2u_1(t)u_3(t) + k_3u_3(t)u_3(t-\tau_2) \\ \qquad\quad +k_4u_1^3(t) + k_5u_1^2(t)u_3(t) + hot, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} &a_{11} = r_1 - a_1x_1^* - \frac{c_1y^*(1-kx_1^*)}{(1+k(x_1^*)^2)^2} - \delta, \quad a_{12} = \delta, \quad a_{13} = -\frac{c_1x_1^*}{1+k(x_1^*)^2}, \\ &a_{14} = -a_1x_1^*, \quad g_1 = -\frac{x_1^*y^*c_1k(k(x_1^*)^2-3)}{(1+k(x_1^*)^2)^3}, \quad g_2 = -a_1, \quad g_3 = \frac{c_1((kx_1^*)^2-1)}{(k(x_1^*)^2+1)^2}, \\ &g_4 = \frac{c_1ky^*(k^2(x_1^*)^4 - 6k(x_1^*)^2+1)}{(1+k(x_1^*)^2)^4}, \quad g_5 = -\frac{x_1^*c_1k(k(x_1^*)^2-3)}{(1+k(x_1^*)^2)^3}, \quad a_{21} = \delta, \\ &a_{22} = r_2 - 2a_2x_2^* - \delta, \quad h_1 = -a_2, \quad a_{31} = -\frac{y^*c_2(k(x_1^*)^2-1)}{(1+k(x_1^*)^2)^2}, \\ &a_{32} = -y^*d_2, \quad k_1 = \frac{kc_2x_1^*y^*(k(x_1^*)^2-3)}{(1+k(x_1^*)^2)^3}, \quad k_2 = \frac{c_2(1-(kx_1^*)^2)}{(1+k(x_1^*)^2)^2}, \\ &k_3 = -d_2, \quad k_4 = -\frac{y^*c_2k(k^2(x_1^*)^4 - 6k(x_1^*)^2+1)}{(1+k(x_1^*)^2)^4}, \quad k_5 = \frac{c_2x_1^*k(k(x_1^*)^2-3)}{(1+k(x_1^*)^2)^3}. \end{aligned}$$

For system (1.1), the characteristic equation of $E^*(x_1^*, x_2^*, y^*)$ is

$$\begin{aligned} &\lambda^3 + A\lambda^2 + B_1\lambda + (D\lambda^2 + E\lambda)\mathrm{e}^{-\lambda\tau_1} + (F\lambda^2 + G\lambda + H)\mathrm{e}^{-\lambda\tau_2} \\ &+(B_2\lambda + C)\mathrm{e}^{-\lambda\tau_3} + (I\lambda + L)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} = 0, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &A = -(a_{11} + a_{22}), \quad B_1 = a_{11}a_{22} - a_{12}a_{21}, \quad D = -a_{14}, \quad E = a_{14}a_{22}, \\ &F = -a_{32}, \quad G = a_{11}a_{32} + a_{22}a_{32}, \quad H = a_{32}(a_{12}a_{21} - a_{11}a_{22}), \\ &B_2 = -a_{13}a_{31}, \quad C = a_{13}a_{31}a_{22}, \quad I = a_{14}a_{32}, \quad L = -a_{14}a_{22}a_{32}. \end{aligned}$$

When $\tau_3 = 0$, denote $B = B_1 + B_2$, then equation (2.2) becomes

$$\begin{aligned}&\lambda^3+A\lambda^2+B\lambda+C+(D\lambda^2+E\lambda)\mathrm{e}^{-\lambda\tau_1}+(F\lambda^2+G\lambda+H)\mathrm{e}^{-\lambda\tau_2}\\&+(I\lambda+L)\mathrm{e}^{-\lambda(\tau_1+\tau_2)}=0,\end{aligned}\tag{2.3}$$

which is the the characteristic equation in [2]. The dynamics of system (1.1) is similar to [2], so when $\tau_1=\tau_2=\tau_3=0$, equation (2.3) becomes

$$\lambda^3+(A+D+F)\lambda^2+(B+E+G+I)\lambda+C+H+L=0.\tag{2.4}$$

All the roots of equation (2.4) have negative real parts if and only if

(H2) $A+D+F>0,\ (A+D+F)(B+E+G+I)>C+H+L,$

so the equilibrium point $E^*(x_1^*,x_2^*,y^*)$ of system (1.1) is locally asymptotically stable if (H2) holds.

Case (1) $\tau_1>0,\ \tau_2=\tau_3=0$, characteristic equation (2.3) becomes

$$\lambda^3+A_{12}\lambda^2+A_{11}\lambda+A_{10}+(B_{12}\lambda^2+B_{11}\lambda+B_{10})\mathrm{e}^{-\lambda\tau_1}=0,\tag{2.5}$$

where $A_{12}=A+F,\ A_{11}=B+G,\ A_{10}=C+H,\ B_{12}=D,\ B_{11}=E+I,\ B_{10}=L.$

Letting $\lambda=i\omega\,(\omega>0)$ be the root of equation (2.5), from the discussion in [2], we obtain

$$\omega^6+p_1\omega^4+q_1\omega^2+r_1=0.\tag{2.6}$$

By denoting $v=\omega^2$, equation (2.6) becomes

$$v^3+p_1v^2+q_1v+r_1=0.\tag{2.7}$$

Lemma 2.2[2] *For the third degree exponential polynomial equation* (2.5)*, we have the following conclusions*

(i) *When* $r_1\ge 0$ *and* $\Delta=p_1^2-3q_1\le 0$*, all roots of equation* (2.5) *have negative real parts for all* $\tau_1\ge 0$*, thus the equilibrium point* $E^*(x_1^*,x_2^*,y^*)$ *of system* (1.1) *is asymptotically stable for* $\tau_1\ge 0$*.*

(ii) *If either* $r_1<0$ *or* $r_1\ge 0$, $\Delta=p_1^2-3q_1>0$, $v_1^*=\frac{-p+\sqrt{\Delta}}{3}>0$ *and* $h'(v_1^*)\le 0$ *all hold, equation* (2.7) *has at least one positive root* v_k *and all roots of equation* (2.5) *have negative real parts for* $\tau_1\in[0,\tau_{10})$*, then system* (1.1) *at the positive equilibrium point* E^* *is local asymptotically stable for* $\tau_1\in[0,\tau_{10})$*.*

(iii) *If all the conditions stated in* (ii) *and* $h'(v_k)\neq 0$ *are satisfied, then system* (1.1) *exhibits the Hopf bifurcation at the positive equilibrium point* E^* *for* $\tau_1=\tau_k^{(j)}\,(j=0,1,\cdots)$*.*

The definitions of $r_1,p_1,q_1\Delta,h(v),\tau_{10},\tau_k^{(j)}$ could be founded in [2].

Case (2) $\tau_1\in(0,\tau_{10}),\ \tau_2>0,\ \tau_3=0$, we have:

Lemma 2.3[2] *For system* (1.1)*, suppose that* (H4),(H5) *in* [2] *hold, and* $\tau_1\in(0,\tau_{10})$*, then the positive equilibrium point* E^* *of system* (1.1) *is locally asymptotically stable for* $\tau_2\in[0,\tau_{2*})$ *and system* (1.1) *at the positive equilibrium point* E^*

undergoes a Hopf bifurcation when $\tau_2 = \tau_{2*}$. *That is, system* (1.1) *has a branch of periodic solutions bifurcating from the* E^* *near* $\tau_2 = \tau_{2*}$.

The definitions of (H4),(H5) and τ_{2*} could be founded in [2].

3 Hopf Bifurcation about Three Delays

When $\tau_1 > 0,\ \tau_2 > 0,\ \tau_3 > 0$, we take τ_3 as parameter, τ_1 and τ_2 in its stable interval, let $iw\,(w > 0)$ be a root of equation (2.2), we obtain

$$\begin{cases} C\cos(\omega\tau_3) + B_2\omega\sin(\omega\tau_3) = (L-\omega)(\sin(\omega\tau_1)\sin(\omega\tau_2) - \cos(\omega\tau_1)\cos(\omega\tau_2)) \\ \qquad +(F\omega^2 - H)\cos(\omega\tau_2) + D\omega^2\cos(\omega\tau_1) \\ \qquad -G\omega\sin(\omega\tau_2) - E\omega\sin(\omega\tau_1) + A\omega^2, \\ B_2\omega\cos(\omega\tau_3) - C\sin(\omega\tau_3) = (L-\omega)(\sin(\omega\tau_1)\cos(\omega\tau_2) + \cos(\omega\tau_1)\sin(\omega\tau_2)) \\ \qquad +(H - F\omega^2)\sin(\omega\tau_2) - D\omega^2\sin(\omega\tau_1) \\ \qquad -G\omega\cos(\omega\tau_2) - E\omega\cos(\omega\tau_1) + \omega^3 - B_1\omega. \end{cases}$$

Eliminating τ_3, we obtain

$$\omega^6 + A_1\omega^5 + A_2\omega^4 + A_3\omega^3 + A_4\omega^2 + A_5\omega + A_6 = 0, \tag{3.1}$$

where

$$\begin{aligned} A_1 &= -2F\sin(\omega\tau_2) - 2D\sin(\omega\tau_1), \\ A_2 &= 2FD(\cos(\omega\tau_1)\cos(\omega\tau_2) + \sin(\omega\tau_1)\sin(\omega\tau_2)) - 2(\cos(\omega\tau_1)\sin(\omega\tau_2) \\ &\quad + \cos(\omega\tau_2)\sin(\omega\tau_1)) + (2AF - 2G)\cos(\omega\tau_2) + (2AD - 2E)\cos(\omega\tau_1) \\ &\quad +F^2 + D^2 + A^2 - 2B_1, \\ A_3 &= 2A(\cos(\omega\tau_1)\cos(\omega\tau_2) - \sin(\omega\tau_1)\sin(\omega\tau_2)) + (2EF + 2L - 2GD)\cos(\omega\tau_1)\sin(\omega\tau_2) \\ &\quad +(2L + 2GD - 2EF)\cos(\omega\tau_2)\sin(\omega\tau_1) + 2F\cos(\omega\tau_1) + 2D\cos(\omega\tau_2) \\ &\quad +(2B_1D - 2AE)\sin(\omega\tau_1) + (2H + 2B_1F - 2AG)\sin(\omega\tau_2), \\ A_4 &= (2EG - 2HD - 2AL)\cos(\omega\tau_1)\cos(\omega\tau_2) + (2EG - 2HD + 2AL)\sin(\omega\tau_1)\sin(\omega\tau_2) \\ &\quad +2B_1\cos(\omega\tau_1)\sin(\omega\tau_2) + 2B_1\sin(\omega\tau_1)\cos(\omega\tau_2) \\ &\quad +(2B_1G - 2AH - 2LD)\cos(\omega\tau_2) + (2B_1E - 2FL)\cos(\omega\tau_1) + 2E\sin(\omega\tau_2) \\ &\quad +2G\sin(\omega\tau_1) + E^2 - 2FH + G^2 + B_1^2 - B_2^2 + 1, \\ A_5 &= (2EH - 2B_1L)\cos(\omega\tau_2)\sin(\omega\tau_1) - (2EH + 2B_1L)\cos(\omega\tau_1)\sin(\omega\tau_2) \\ &\quad -2H\cos(\omega\tau_1) - 2GL\sin(\omega\tau_1) - (2B_1H + 2EL)\sin(\omega\tau_2) - 2L, \\ A_6 &= 2HL\cos(\omega\tau_1) + L^2 + H^2 - C^2. \end{aligned}$$

Denote

$$H(\omega) = \omega^6 + A_1\omega^5 + A_2\omega^4 + A_3\omega^3 + A_4\omega^2 + A_5\omega + A_6, \tag{3.2}$$

we then give the following hypothesis:

(H6) Equation (3.1) has finite positive roots.

If (H6) holds, we denote $\omega_1, \omega_2, \cdots, \omega_k$ to be positive roots of equation (3.1).

For every fixed ω_i $(i = 1, 2, \cdots, k)$, then exists a sequence $\{\tau_{3i}^{(j)} | \, i = 1, 2, \cdots, k, j = 0, 1, 2, \cdots\}$, when $\tau_1 \in (0, \tau_{10})$, $\tau_2 \in (0, \tau_{2*})$,

$$\tau_{3i}^{(j)} = \frac{1}{\omega_i}\Big[\arccos\Big(\frac{\psi_1}{\psi_2}\Big) + 2j\pi\Big], \quad i = 1, 2, \cdots, k, \; j = 0, 1, \cdots, \tag{3.3}$$

where

$$\begin{aligned}\psi_1 = {} & (B_2 L\omega - B_2\omega^2)(\cos(\omega\tau_1)\sin(\omega\tau_2) + \sin(\omega\tau_1)\cos(\omega\tau_2)) \\ & + (CL - C\omega)(\sin(\omega\tau_1)\sin(\omega\tau_2) - \cos(\omega\tau_1)\cos(\omega\tau_2)) - (CE\omega + B_2 D\omega^3)\sin(\omega\tau_1) \\ & + (B_2 H\omega - B_2 F\omega^3 - CG\omega)\sin(\omega\tau_2) + (CD\omega^2 - B_2 E\omega^2)\cos(\omega\tau_1) \\ & + (CF\omega^2 - CH - B_2 G\omega^2)\cos(\omega\tau_2) + B_2\omega^4 + AC\omega^2 - B_1 B_2\omega^2, \\ \psi_2 = {} & B_2^2\omega^2 + C^2,\end{aligned}$$

then $\pm i\omega_i$ are a pair of purely imaginary roots of equation (3.1) when $\{\tau_3 = \tau_{3i}^{(j)}, i = 1, 2, \cdots, k, j = 0, 1, \cdots\}$. Let

$$\tau_{3*} = \min\Big\{\tau_{3i}^{(j)} = \frac{1}{\omega_i}\Big[\arccos\Big(\frac{\psi_1}{\psi_2}\Big) + 2j\pi\Big], \, i = 1, 2, \cdots, k, \; j = 0, 1, \cdots\Big\}, \tag{3.4}$$

then equation (2.2) has a pair of purely imaginary root $\pm i\omega_{3*}$ when $\tau_3 = \tau_{3*}$. We differentiate both sides of equation (2.2) with respect to τ_3 and derive

$$\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_3}\Big)^{-1} = \frac{f_{11}(\lambda) + f_{12}(\lambda)\mathrm{e}^{-\lambda\tau_1} + f_{13}(\lambda)\mathrm{e}^{-\lambda\tau_2} + f_{14}(\lambda)\mathrm{e}^{-\lambda\tau_3} + f_{15}(\lambda)\mathrm{e}^{-\lambda(\tau_1+\tau_2)}}{\lambda f_{21}(\lambda)\mathrm{e}^{-\lambda\tau_3}} - \frac{\tau_3}{\lambda},$$

where

$$\begin{aligned} & f_{11}(\lambda) = A_{11}\lambda^2 + A_{12}\lambda + A_{13}, \quad f_{12}(\lambda) = A_{21}\lambda^2 + A_{22}\lambda + A_{23}, \\ & f_{13}(\lambda) = A_{31}\lambda^2 + A_{32}\lambda + A_{33}, \quad f_{14}(\lambda) = B_2, \\ & f_{15}(\lambda) = A_{41}\lambda + A_{42}, \quad f_{21}(\lambda) = B_2\lambda + C \end{aligned}$$

with

$$\begin{aligned} & A_{11} = 3, \quad A_{12} = 2A, \quad A_{13} = B_1, \quad A_{21} = -D\tau_1, \quad A_{22} = -E\tau_1 + 2D, \\ & A_{23} = E, \quad A_{31} = -F\tau_2, \quad A_{32} = 2F - G\tau_2, \quad A_{33} = G - H\tau_2, \\ & A_{41} = -I(\tau_1 + \tau_2), \quad A_{42} = I - L(\tau_1 + \tau_2). \end{aligned}$$

We have

$$\mathrm{Re}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_3}\Big)^{-1}_{\lambda = i\omega_{3*}} = \mathrm{Re}\Big(\frac{R_1 + iR_2}{R_3 + iR_4}\Big) = \frac{R_1R_3 + R_2R_4}{R_3^2 + R_4^2}, \tag{3.5}$$

where

$$\begin{aligned}R_1=&A_{41}\omega_{3*}(\cos(\omega_{3*}\tau_1)\cos(\omega_{3*}\tau_2)-\sin(\omega_{3*}\tau_1)\sin(\omega_{3*}\tau_2))-A_{42}(\cos(\omega_{3*}\tau_1)\sin(\omega_{3*}\tau_2)\\&+\cos(\omega_{3*}\tau_2)\sin(\omega_{3*}\tau_1))-B_2\sin(\omega_{3*}\tau_{3*})+(A_{31}\omega_{3*}^2-A_{33})\sin(\omega_{3*}\tau_2)\\&+(A_{21}\omega_{3*}^2-A_{23})\sin(\omega_{3*}\tau_1)+A_{32}\omega_{3*}\cos(\omega_{3*}\tau_2)+A_{22}\omega_{3*}\cos(\omega_{3*}\tau_1)+A_{12}\omega_{3*},\\R_2=&-A_{42}(\cos(\omega_{3*}\tau_1)\cos(\omega_{3*}\tau_2)-\sin(\omega_{3*}\tau_1)\sin(\omega_{3*}\tau_2))\\&-A_{41}\omega_{3*}(\cos(\omega_{3*}\tau_1)\sin(\omega_{3*}\tau_2)+\cos(\omega_{3*}\tau_2)\sin(\omega_{3*}\tau_1))-B_2\cos(\omega_{3*}\tau_{3*})\\&+(A_{31}\omega_{3*}^2-A_{33})\cos(\omega_{3*}\tau_2)+(A_{21}\omega_{3*}^2-A_{23})\cos(\omega_{3*}\tau_1)\\&-A_{32}\omega_{3*}\sin(\omega_{3*}\tau_2)-A_{22}\omega_{3*}\sin(\omega_{3*}\tau_1)+A_{11}\omega_{3*}^2-A_{13},\\R_3=&B_2\omega_{3*}^2\sin(\omega_{3*}\tau_{3*})+C\omega_{3*}\cos(\omega_{3*}\tau_{3*}),\\R_4=&B_2\omega_{3*}^2\cos(\omega_{3*}\tau_{3*})-C\omega_{3*}\sin(\omega_{3*}\tau_{3*}).\end{aligned}$$

Suppose

(H7) $R_1R_3+R_2R_4\neq 0$

holds, then $\mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_3})^{-1}_{\lambda=i\omega_{3*}}\neq 0$.

We have the following results:

Theorem 3.1 *For system* (1.1)*, we assume that* (H1),(H2) *hold, and suppose condition* (i),(ii) *in Lemma* 2.2*,* (H4),(H5) *in* [2]*,* (H6),(H7) *are satisfied, and* $\tau_1\in(0,\tau_{10})$*,* $\tau_2\in(0,\tau_{2*})$*, then there exists a* $\delta>0$ *such that the following conclusions hold:*

(i) *The positive equilibrium* $E^*(x_1^*,x_2^*,y^*)$ *of system* (1.1) *is asymptotically stable when* $\tau_3\in(0,\tau_{3*})$ *and unstable when* $\tau_3\in[\tau_{3*},\tau_3+\delta)$*.*

(ii) *System* (1.1) *can undergo Hopf bifurcation at* $E^*(x_1^*,x_2^*,y^*)$ *when* $\tau_3=\tau_{3*}$*.*

4 Direction and Stability of Hopf Bifurcation

Now we should study the property of Hopf by center manifold theorem and normal form [18]. We assume $\tau_2<\tau_1<\tau_{3*}$, where $\tau_2\in(0,\tau_{2*})$, $\tau_1\in(0,\tau_{10})$, then system (1.1) undergoes Hopf bifurcation at E^* when $\tau_3=\tau_{3*}$.

Let $\tau_3=\tau_{3*}+\mu$, $t=s\tau_3$, $u_i(s\tau_3)=\hat{u}_i(s)$. For convenience, denoting $\hat{u}_i(s)$ as $u_i(t)$, system (2.1) could be written in $C([-1,0],R^3)$,

$$\dot{u}(t)=L_\mu u(t)+f(\mu,u_t), \tag{4.1}$$

$L_\mu: C\to R^3, f: R\times C\to R^3$ are given respectively

$$L_\mu(\phi)=(\tau_{3*}+\mu)(A_1\phi(0)+A_2\phi\Big(-\frac{\tau_2}{\tau_{3*}}\Big)+A_3\phi\Big(-\frac{\tau_1}{\tau_{3*}}\Big)+A_4\phi(-1)),$$

where

$$A_1=\begin{pmatrix}a_{11}&a_{12}&0\\a_{21}&a_{22}&0\\a_{31}&0&0\end{pmatrix},\ A_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&a_{32}\end{pmatrix},\ A_3=\begin{pmatrix}a_{14}&0&0\\0&0&0\\0&0&0\end{pmatrix},\ A_4=\begin{pmatrix}0&0&a_{13}\\0&0&0\\0&0&0\end{pmatrix},$$

and

$$f(\mu,u_t)=(\tau_{3*}+\mu)\begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix},$$

where

$$\begin{aligned}
f_1&=g_1\phi_1^2(0)+g_2\phi_1(0)\phi_1\Big(-\frac{\tau_1}{\tau_{3*}}\Big)+g_3\phi_1(0)\phi_3(-1)+g_4\phi_1^3(0)+g_5\phi_1^2(0)\phi_3(-1),\\
f_2&=h_1\phi_2^2(0),\\
f_3&=k_1\phi_1^2(0)+k_2\phi_1(0)\phi_3(0)+k_3\phi_3(0)\phi_3\Big(-\frac{\tau_2}{\tau_{3*}}\Big)+k_4\phi_1^3(0)+k_5\phi_1^2(0)\phi_3(0).
\end{aligned}$$

By Riesz representation theorem, there exists a $\eta(\theta,\mu)$ of bounded variation for $\theta\in[-1,0]$ such that

$$L_\mu\phi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta),\tag{4.2}$$

for $\phi\in C([-1,0],R^3)$. In fact, we choose

$$\eta(\theta,\mu)=\begin{cases}(\tau_{3*}+\mu)(A_1+A_2+A_3+A_4), & \theta=0,\\ (\tau_{3*}+\mu)(A_2+A_3+A_4), & \theta\in\big[-\frac{\tau_2}{\tau_{3*}},0\big),\\ (\tau_{3*}+\mu)(A_3+A_4), & \theta\in\big[-\frac{\tau_1}{\tau_{3*}},-\frac{\tau_2}{\tau_{3*}}\big),\\ (\tau_{3*}+\mu)A_4, & \theta\in\big(-1,-\frac{\tau_1}{\tau_{3*}}\big),\\ 0, & \theta=-1.\end{cases}$$

For $\phi\in C^1([-1,0],R^3)$, define

$$A(\mu)\phi=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta), & \theta=0,\end{cases}$$

and

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-1,0),\\ f(\mu,\phi), & \theta=0.\end{cases}$$

Then system (4.1) can be written as

$$\dot{u}_t=A(\mu)+R(\mu)u_t,$$

For $\psi\in C^1([0,1],(R^3)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta^T(t,0)\psi(-t), & s=0,\end{cases}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle=\overline{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}^{(}\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,$$

when $\eta(\theta)=\eta(\theta,0)$, then $A(0),A^{*}(0)$ are adjoint operators, From Section 3, we know $\pm i\omega_{3*}\tau_{3*}$ are eigenvalues of $A(0)$, thus they are also eigenvalues of $A^{*}(0)$. We compute the eigenvector of $A(0)$ and $A^{*}(0)$ corresponding to $i\omega_{3*}\tau_{3*}$ and $-i\omega_{3*}\tau_{3*}$.

Suppose that $q(\theta)=(1,q_2,q_3)\mathrm{e}^{i\omega_{3*}\tau_{3*}\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_{3*}\tau_{3*}$, by computing we obtain

$$q_2=\frac{a_{21}}{i\omega_3-a_{22}},\quad q_3=\frac{i\omega_3-a_{11}}{a_{13}}\mathrm{e}^{i\tau_{3*}\omega_{3*}}-\frac{a_{12}a_{21}}{a_{13}(i\omega_3-a_{22})}\mathrm{e}^{i\omega_{3*}\tau_{3*}}-\frac{a_{14}}{a_{13}}\mathrm{e}^{i\omega_{3*}(\tau_{3*}-\tau_1)}.$$

On the other hand, supposing $q^{*}(s)=D(1,q_2^{*},q_3^{*})\mathrm{e}^{i\omega_{3*}\tau_{3*}s}$, we obtain

$$q_2^{*}=-\frac{a_{12}}{i\omega_{3*}+a_{22}},\quad q_3^{*}=-\frac{a_{11}+i\omega_{3*}}{a_{31}}-\frac{a_{14}}{a_{31}}\mathrm{e}^{-i\omega_{3*}\tau_1}+\frac{a_{12}a_{21}}{a_{31}(i\omega_{3*}+a_{22})}.$$

From $\langle q^{*},q\rangle=1,\ \langle q^{*},\overline{q}\rangle=0$, we compute the value $\overline{D}$

$$\overline{D}=\frac{1}{1+q_2\overline{q}_2^{*}+q_3\overline{q}_3^{*}+\tau_2a_{32}q_3\overline{q}_3^{*}\mathrm{e}^{-i\omega_{3*}\tau_2}+\tau_1a_{14}\mathrm{e}^{-i\omega_{3*}\tau_1}+\tau_3a_{13}q_3\mathrm{e}^{-i\omega_{3*}\tau_{3*}}}.$$

We compute center manifold C_0 at $\mu=0$, let u_t be a solution of equation (4.1), and define

$$z(t)=\langle q^{*},u_t\rangle,\quad W(t,\theta)=u_t(\theta)-2\mathrm{Re}(z(t)q(\theta)),$$

on the center manifold C_0, then we have

$$W(t,\theta)=W(z,\overline{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots,$$

where $z,\overline{z}$ are the local coordinates for center manifold C_0 in the direction of q^{*} and $\overline{q}^{*}$. We note $W(t,\theta)$ is real if u_t is real. We consider only real solution, for the solution $u_t\in C_0$ of equation (4.1), since $\mu=0$

$$\begin{aligned}\dot{z}&=\langle q^{*},A(0)u_t\rangle+\langle q^{*},R(0)u_t\rangle\\&=i\omega_{3*}\tau_{3*}z+\langle\overline{q}^{*}(0),f(0,W(z,\overline{z},\theta)+2\mathrm{Re}(z,q(\theta)))\rangle\\&=i\omega_{3*}\tau_{3*}z+\overline{q}^{*}(0)f(0,W(z,\overline{z},\theta)+2\mathrm{Re}(z,q(\theta)))\\&=i\omega_{3*}\tau_{3*}z+\overline{q}^{*}(0)f_0(z,\overline{z}),\end{aligned}$$

we write equation

$$\dot{z}=i\omega_{3*}\tau_{3*}z(t)+g(z,\overline{z}),$$

where

$$g(z,\overline{z})=\overline{q}^{*}(0)f_0(z,\overline{z})=g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+g_{21}\frac{z^2\overline{z}}{2!}+\cdots, \tag{4.3}$$

then

$$g(z,\overline{z})=\overline{q}^{*}(0)f_0(z,\overline{z})=\overline{D}\tau_{3*}(1,\overline{q}_2^{*},\overline{q}_3^{*})(f_1,f_2,f_3)^{\top}. \tag{4.4}$$

From $u_t(\theta)=(u_{1t}(\theta),u_{2t}(\theta),u_{3t}(\theta))=W(t,\theta)+zq(\theta)+\overline{z}\overline{q}(\theta)$, $q(\theta)=(1,q_2,q_3)\mathrm{e}^{i\omega_{3*}\tau_{3*}\theta}$, we have

$$\begin{aligned}
u_{1t}(0)&=z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}\frac{\overline{z}^2}{2}+O(|\,z,\overline{z}\,|^3),\\
u_{2t}(0)&=q_2z+\overline{q}_2\overline{z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}\frac{\overline{z}^2}{2}+O(|\,z,\overline{z}\,|^3),\\
u_{3t}(0)&=q_3z+\overline{q}_3\overline{z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}\frac{\overline{z}^2}{2}+O(|\,z,\overline{z}\,|^3),\\
u_{1t}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)&=z\mathrm{e}^{-i\omega_{3*}\tau_2}+\overline{z}\mathrm{e}^{i\omega_{3*}\tau_2}+W_{20}^{(1)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\frac{z^2}{2}+W_{11}^{(1)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)z\overline{z}\\
&\quad+W_{02}^{(1)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3),\\
u_{2t}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)&=q_2z\mathrm{e}^{-i\omega_{3*}\tau_2}+\overline{q}_2\overline{z}\mathrm{e}^{i\omega_{3*}\tau_2}+W_{20}^{(2)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\frac{z^2}{2}+W_{11}^{(2)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)z\overline{z}\\
&\quad+W_{02}^{(2)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3),\\
u_{3t}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)&=q_3z\mathrm{e}^{-i\omega_{3*}\tau_2}+\overline{q}_3\overline{z}\mathrm{e}^{i\omega_{3*}\tau_2}+W_{20}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\frac{z^2}{2}+W_{11}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)z\overline{z}\\
&\quad+W_{02}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3),\\
u_{1t}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)&=z\mathrm{e}^{-i\omega_{3*}\tau_1}+\overline{z}\mathrm{e}^{i\omega_{3*}\tau_1}+W_{20}^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\frac{z^2}{2}+W_{11}^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)z\overline{z}\\
&\quad+W_{02}^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3),\\
u_{2t}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)&=q_2z\mathrm{e}^{-i\omega_{3*}\tau_1}+\overline{q}_2\overline{z}\mathrm{e}^{i\omega_{3*}\tau_1}+W_{20}^{(2)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\frac{z^2}{2}+W_{11}^{(2)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)z\overline{z}\\
&\quad+W_{02}^{(2)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3),\\
u_{3t}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)&=q_3z\mathrm{e}^{-i\omega_{3*}\tau_1}+\overline{q}_3\overline{z}\mathrm{e}^{i\omega_{3*}\tau_1}+W_{20}^{(3)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\frac{z^2}{2}+W_{11}^{(3)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)z\overline{z}\\
&\quad+W_{02}^{(3)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3),
\end{aligned}$$

$$\begin{aligned}
u_{1t}(-1) &= z\mathrm{e}^{-i\omega_{3*}\tau_{3*}} + \overline{z}\mathrm{e}^{i\omega_{3*}\tau_{3*}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z}\\
&\quad + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{2t}(-1) &= q_2 z\mathrm{e}^{-i\omega_{3*}\tau_{3*}} + \overline{q}_2\overline{z}\mathrm{e}^{i\omega_{3*}\tau_{3*}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z}\\
&\quad + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{3t}(-1) &= q_3 z\mathrm{e}^{-i\omega_{3*}\tau_{3*}} + \overline{q}_3\overline{z}\mathrm{e}^{i\omega_{3*}\tau_{3*}} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\overline{z}\\
&\quad + W_{02}^{(3)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3).
\end{aligned}$$

From (4.3),(4.4) we have

$$g(z,\overline{z}) = \overline{D}\tau_{3*}(1,\overline{q}_2^*,\overline{q}_3^*)\begin{pmatrix} K_{11}z^2 + K_{12}z\overline{z} + K_{13}\overline{z}^2 + K_{14}z^2\overline{z}\\ K_{21}z^2 + K_{22}z\overline{z} + K_{23}\overline{z}^2 + K_{24}z^2\overline{z}\\ K_{31}z^2 + K_{32}z\overline{z} + K_{33}\overline{z}^2 + K_{34}z^2\overline{z}\end{pmatrix},$$

where

$$\begin{aligned}
K_{11} &= g_1(q^{(1)}(0))^2 + g_2 q^{(1)}(0)q^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big) + g_3 q^{(1)}(0)q^{(3)}(-1),\\
K_{12} &= 2g_1 q^{(1)}(0)\overline{q}^{(1)}(0) + g_2\Big(q^{(1)}(0)\overline{q}^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big) + \overline{q}^{(1)}(0)q^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\Big)\\
&\quad + g_3\big(q^{(1)}(0)\overline{q}^{(3)}(-1) + \overline{q}^{(1)}(0)q^{(3)}(-1)\big),\\
K_{13} &= g_1(\overline{q}^{(1)}(0))^2 + g_2\overline{q}^{(1)}(0)\overline{q}^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big) + g_3\overline{q}^{(1)}(0)\overline{q}^{(3)}(-1),\\
K_{14} &= g_1(2q^{(1)}(0)W_{11}^{(1)}(0) + \overline{q}^{(1)}(0)W_{20}^{(1)}(0)) + g_2\Big(q^{(1)}(0)W_{11}^{(1)}\Big(-\frac{\tau_1}{\tau_{3*}}\Big)\\
&\quad + \frac{1}{2}\overline{q}^{(1)}\Big(\frac{\tau_1}{\tau_{3*}}\Big)W_{20}^{(1)}(0) + q^{(1)}\Big(\frac{\tau_1}{\tau_{3*}}\Big)W_{11}^{(1)}(0) + \frac{1}{2}\overline{q}^{(1)}(0)W_{20}^{(1)}\Big(\frac{\tau_1}{\tau_{3*}}\Big)\Big)\\
&\quad + g_3\Big(q^{(1)}(0)W_{11}^{(3)}(-1) + \frac{1}{2}\overline{q}^{(3)}(-1)W_{20}^{(1)}(0) + q^{(3)}(-1)W_{11}^{(1)}(0) + \frac{1}{2}\overline{q}^{(1)}(0)W_{20}^{(3)}(-1)\Big)\\
&\quad + g_4(3(q^{(1)}(0))^2\overline{q}^{(1)}(0)) + g_5((q^{(1)}(0))^2\overline{q}^{(3)}(-1) + 2q^{(1)}(0)\overline{q}^{(1)}(0)q^{(3)}(-1)).\\
K_{21} &= h_1(q^{(2)}(0))^2, K_{22} = 2h_1 q^{(2)}(0)\overline{q}^{(2)}(0), K_{23} = h_1(\overline{q}^{(2)}(0))^2,\\
K_{24} &= h_1(\overline{q}^{(2)}(0)W_{20}^{(2)}(0) + 2q^{(2)}(0)W_{11}^{(2)}(0)),\\
K_{31} &= k_1(q^{(1)}(0))^2 + k_2 q^{(1)}(0)q^{(3)}(0) + k_3 q^{(3)}(0)q^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big),\\
K_{32} &= 2k_1 q^{(1)}(0)\overline{q}^{(1)}(0) + k_2(q^{(1)}(0)\overline{q}^{(3)}(0) + \overline{q}^{(1)}(0)q^{(3)}(0))\\
&\quad + k_3\Big(q^{(3)}(0)\overline{q}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big) + \overline{q}^{(3)}(0)q^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\Big),
\end{aligned}$$

$$
\begin{aligned}
K_{33} &= k_1(\overline{q}^{(1)}(0))^2 + k_2\overline{q}^{(1)}(0)\overline{q}^{(3)}(0) + k_3\overline{q}^{(3)}(0)\overline{q}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big),\\
K_{34} &= k_1(\overline{q}^{(1)}(0)W_{20}^{(1)}(0) + 2q^{(1)}(0)W_{11}^{(1)}(0)) + k_2\Big(q^{(1)}(0)W_{11}^{(3)}(0) + q^{(3)}(0)W_{11}^{(1)}(0)\\
&\quad +\frac{1}{2}\overline{q}^{(1)}(0)W_{20}^{(3)}(0) + \frac{1}{2}\overline{q}^{(3)}(0)W_{20}^{(1)}(0)\Big) + k_3\Big(q^{(3)}(0)W_{11}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\\
&\quad +q^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)W_{11}^{(3)}(0) + \frac{1}{2}\overline{q}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)W_{20}^{(3)}(0) + \frac{1}{2}\overline{q}^{(3)}(0)W_{20}^{(3)}\Big(-\frac{\tau_2}{\tau_{3*}}\Big)\Big)\\
&\quad +3k_4((q^{(1)}(0))^2\overline{q}^{(1)}(0)) + k_5(2q^{(1)}(0)\overline{q}^{(1)}(0)q^{(3)}(0) + (q^{(1)}(0))^2\overline{q}^{(3)}(0)).
\end{aligned}
$$

Comparing the above coefficients with (4.3), we obtain

$$
\begin{aligned}
g_{20} &= 2\tau_{3*}\overline{D}(K_{11} + \overline{q}_2^*K_{21} + \overline{q}_3^*K_{31}),\\
g_{11} &= \tau_{3*}\overline{D}(K_{12} + \overline{q}_2^*K_{22} + \overline{q}_3^*K_{32}),\\
g_{02} &= 2\tau_{3*}\overline{D}(K_{13} + \overline{q}_2^*K_{23} + \overline{q}_3^*K_{33}),\\
g_{21} &= 2\tau_{3*}\overline{D}(K_{14} + \overline{q}_2^*K_{24} + \overline{q}_3^*K_{34}),
\end{aligned}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{02}}{\omega_{3*}\tau_{3*}}q(0)\mathrm{e}^{i\omega_{3*}\tau_{3*}\theta} + \frac{i\overline{g}_{02}}{2\omega_{3*}\tau_{3*}}\overline{q}(0)\mathrm{e}^{-i\omega_{3*}\tau_{3*}\theta} + E_1\mathrm{e}^{2i\omega_{3*}\tau_{3*}\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}}{\omega_{3*}\tau_{3*}}q(0)\mathrm{e}^{i\omega_{3*}\tau_{3*}\theta} + \frac{i\overline{g}_{11}}{\omega_{3*}\tau_{3*}}\overline{q}(0)\mathrm{e}^{-i\omega_{3*}\tau_{3*}\theta} + E_2,
\end{aligned}
$$

where E_1,E_2 could be determined by

$$
\begin{pmatrix}
2i\omega_{3*} - a_{11} - a_{14}\mathrm{e}^{-2i\omega_{3*}\tau_1} & -a_{12} & -a_{13}\mathrm{e}^{-2i\omega_{3*}\tau_{3*}}\\
-a_{21} & 2i\omega_{3*} - a_{22} & 0\\
-a_{31} & 0 & 2i\omega_{3*} - a_{32}\mathrm{e}^{-2i\omega_{3*}\tau_2}
\end{pmatrix} E_2 = 2\begin{pmatrix} K_{11}\\ K_{21}\\ K_{31}\end{pmatrix}
$$

and

$$
\begin{pmatrix}
a_{11} + a_{14} & a_{12} & a_{13}\\
a_{21} & a_{22} & 0\\
a_{31} & 0 & a_{32}
\end{pmatrix} E_2 = -\begin{pmatrix} K_{12}\\ K_{22}\\ K_{32}\end{pmatrix}.
$$

Then we can calculate the following value according to the above analysis:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\omega_{3*}\tau_{3*}}\Big(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(\tau_{3*}))}, \quad \beta_2 = 2\mathrm{Re}(C_1(0)),\\
T_2 &= -\frac{\mathrm{Im}C_1(0) + \mu_2\mathrm{Im}\lambda'(\tau_{3*})}{\omega_{3*}\tau_{3*}},
\end{aligned}
$$

so we have the following result [18]:

Theorem 4.1 (i) *The sign of μ_2 determines the direction of Hopf bifurcation, when $\mu_2 > 0\,(< 0)$, the bifurcation is supercritical (subcritical).*

(ii) *The sign of β_2 determines the stability of bifurcating periodic solution, when $\beta_2 > 0\,(< 0)$, the bifurcating periodic solutions are unstable (unstable).*

(iii) *The sign of T_2 determines the period of bifurcating periodic solution, when $T_2 > 0\,(< 0)$, the period of bifurcating periodic solutions increases (decreases).*

5 Numerical Simulation

Now we give simulation of system (1.1), all the parameters are same to [2], system (1.1) becomes:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big(0.8 - 0.2x_1(t-\tau_1) - \dfrac{0.4y(t-\tau_3)}{1+0.15x_1^2(t)}\Big) + 0.1(x_2 - x_1), \\ \dot{x}_2(t) = x_2(t)(0.75 - 0.15x_2(t)) + 0.1(x_1 - x_2), \\ \dot{y}(t) = y(t)\Big(0.6 + \dfrac{0.25x_1}{1+0.15x_1^2} - 0.3y(t-\tau_2)\Big). \end{cases} \tag{5.1}$$

By computation, system (5.1) has two positive equilibriums $E_1^*(1.02, 4.47, 2.282)$ and $E_2^*(3.6, 4.83, 3.01)$. In [2], the author gave simulation around $E_1^*(1.02, 4.47, 2.282)$ under four cases: (1) $\tau_1 > 0$, $\tau_2 = 0$, (2) $\tau_1 = 0$, $\tau_2 > 0$, (3) $\tau_1 = \tau_2 = \tau > 0$, (4) $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$.

Now we give simulation around $E_2^*(3.6, 4.83, 3.01)$ under three cases: (1) $\tau_1 > 0$, $\tau_2 = 0$, $\tau_3 = 0$, (2) $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, $\tau_3 = 0$, (3) $\tau_1 \in (0, \tau_{10})$, $\tau_2 \in (0, \tau_{2*})$, $\tau_3 > 0$.

In case (1), we get $\tau_{10} = 1.970\,98$. When $\tau_1 = 1.9 < 1.970\,98$, $E_2^*(3.6, 4.83, 3.01)$ is locally asymptotically stable, when $\tau_1 = 2.0 > 1.970\,98$, Hopf bifurcation occurs and periodic orbits bifurcate from E_2^*, which are illustrated by Figures 1 and 2.

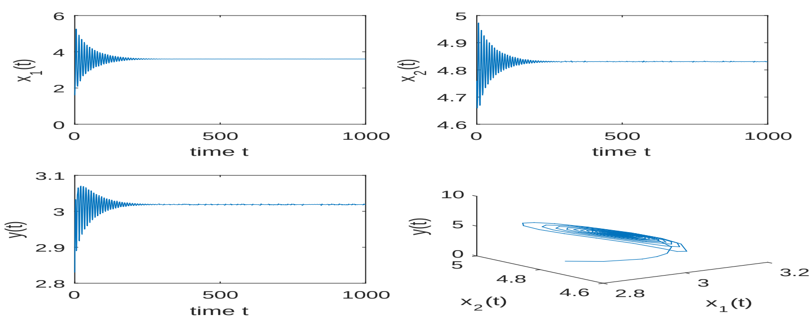

Figure 1: When $\tau_1 = 1.9 < \tau_{10} = 1.970\,98, \tau_2 = 0, \tau_3 = 0$, E_2^* is locally asymptotically stable.

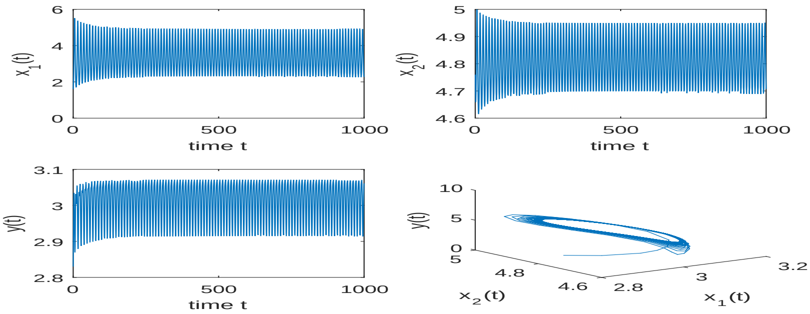

Figure 2: When $\tau_1 = 2.0 > \tau_{10} = 1.970\,98, \tau_2 = 0, \tau_3 = 0$, the periodic solutions bifurcates from E_2^*.

In case (2), fixing $\tau_1 = 1.93 \in (0, \tau_{10})$, we get $\tau_{2*} = 1.365\,92$. When $\tau_2 = 0.6 < \tau_{2*}$, E_2^* is locally asymptotically stable; when $\tau_2 = 1.5 > \tau_{2*}$, Hopf bifurcation occurs and periodic orbits bifurcate from E_2^*, which are illustrated by Figures 3 and 4.

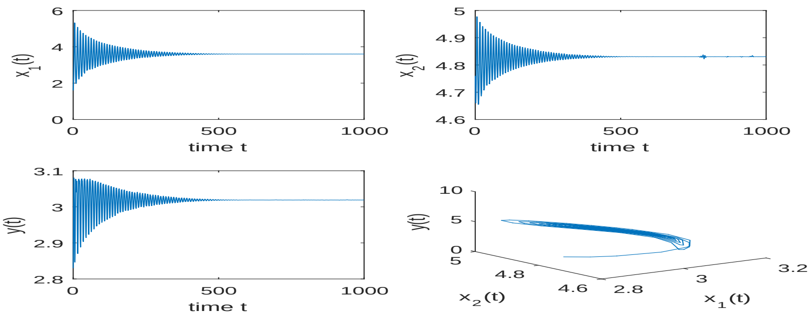

Figure 3: When $\tau_1 = 1.93 \in (0, \tau_{10}), \tau_2 = 0.6 < \tau_{2*} = 1.365\,92, \tau_3 = 0$, E_2^* is locally asymptotically stable.

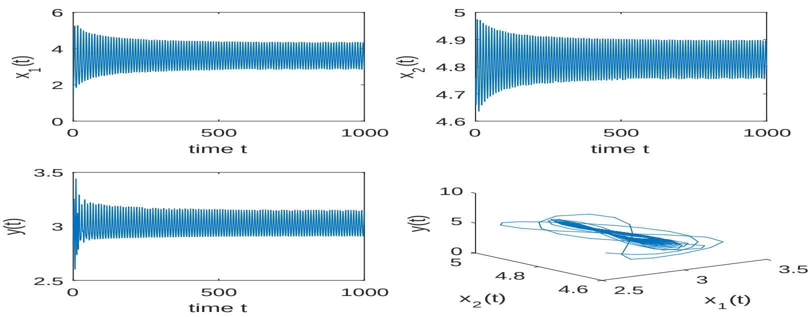

Figure 4: When $\tau_1 = 1.93 \in (0, \tau_{10}), \tau_2 = 1.5 > \tau_{2*} = 1.365\,92, \tau_3 = 0$, the periodic solutions bifurcates from E_2^*.

In case (3), fixing $\tau_1 = 1.94 \in (0, \tau_{10})$, $\tau_2 = 1 \in (0, \tau_{2*})$, we get $\tau_{3*} = 5.599\,73$. When $\tau_3 = 5.0 < \tau_{3*}$, E_2^* is asymptotically asymptotically stable; when $\tau_3 = 8.0 > \tau_{3*}$, Hopf bifurcation occurs and periodic orbits bifurcate from E_2^*, which are illustrated by Figures 5 and 6. In addition we get $\mu_2 = -4.442\,68{*}10^3$. $\beta_2 = 3.001\,2{*}10^3$, $T_2 = 5.031\,68 * 10^2$. From Theorem 4.1, the Hopf bifurcation is subcritical, the bifurcating periodic solutions are unstable, and the period of the bifurcating periodic solutions increases.

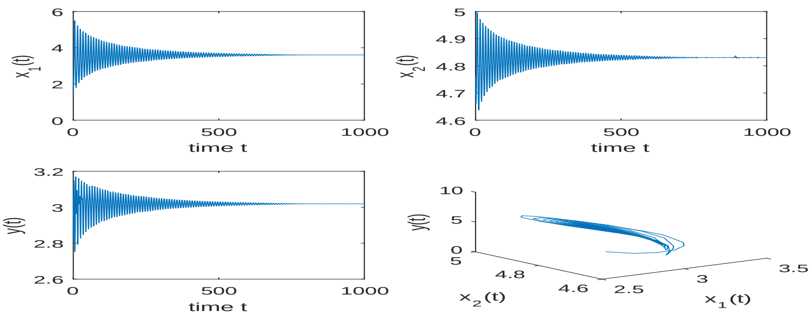

Figure 5: When $\tau_1 = 1.94 \in (0, \tau_{10}), \tau_2 = 1.0 \in (0, \tau_{2*}), \tau_3 = 5.0 < \tau_{3*}$, E_2^* is locally asymptotically stable.

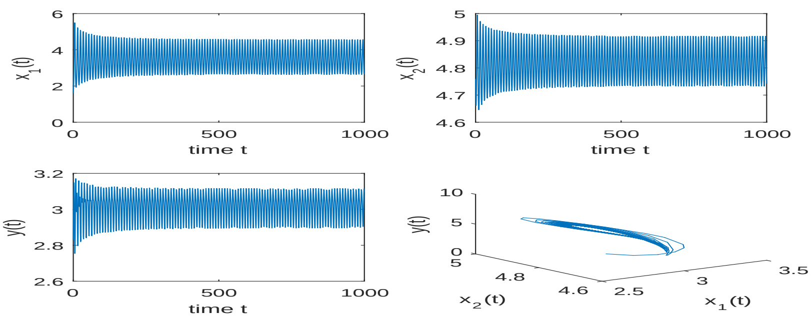

Figure 6: When $\tau_1 = 1.94 \in (0, \tau_{10}), \tau_2 = 1.0 \in (0, \tau_{2*}), \tau_3 = 8.0 > (0, \tau_{3*})$, the periodic solutions bifurcates from E_2^*.

6 Conclusion

In this paper, we analyse Hopf bifurcation in a Monold-Haldane predator-prey model with three delays and diffusion. First we give the description of this system and then based on the work [2] we give the dynamics of this system with three delays under two cases: (1) $\tau > 0$, $\tau_2 = 0$, $\tau_3 = 0$, (2) $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, $\tau_3 = 0$. Because when $\tau_3 = 0$, the dynamics of system (1.1) were studied in [2], dynamics of case (1) and case (2) are similar to those of case (II) and case (IV) respectively. Then we discuss Hopf bifurcation of system (1.1) when three delays coexists and give the direction and stability of Hopf bifurcation by center manifold theorem and normal form. At last, we conclude that there exist two positive equilibria $E_1^*(1.026, 4.24, 2.28)$, $E_2^*(3.6, 4.83, 3.01)$ and give the simulation around $E_2^*(3.6, 4.83,\ 3.01)$ under three cases: (1) $\tau_1 > 0,\ \tau_2 = 0,\ \tau_3 = 0$, (2) $\tau_1 \in (0, \tau_{10}),\ \tau_2 > 0,\ \tau_3 = 0$, (3) $\tau_1 \in (0, \tau_{10})$, $\tau_2 \in (0, \tau_{2*}),\ \tau_3 = 0$ to support our results. We know that the work about Hopf bifurcation with two delays are interesting. But the work about three delays should be more interesting and real contrary to [2].

Recently there exist many works about dynamics of predator-prey system. As we know, time delay always exists, so the dynamics of predator-prey system with delay should be more real. Besides diffusion exist under case that more than one kind of preys in different patches and the predator only prey on one prey, so we use system (1.1) to describe predator-prey system with diffusion. During these delays, one is the feedback of prey, the other one denote the feedback of predator and the last one denote the hunting delay for predator. Our result suggest that every delay could lead instability for system, and delay has great influence on the dynamics of predator-prey system. So how to control delays should be important for the stability of predator-prey system. When the delays are sufficiently small, equilibrium point is stable, but once one delay cross corresponding critical value, Hopf bifurcation occurs for system.

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