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DIFFUSIVE LIMITS OF THE BOLTZMANN EQUATION IN BOUNDED DOMAIN*†

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Abstract

The goal of this paper is to study the important diffusive expansion via an alternative mathematical approach other than that in [21].

Keywords steady and unsteady Boltzmann equation; hydrodynamics limits; boundary effects

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1 Introduction

1.1 Hilbert Expansion with No Boundary Layer Approximations

The hydrodynamic limit of the Boltzmann equation has been the subject of many studies since the pioneering work by Hilbert, who introduced his famous expansion in the Knudsen number ε in [37, 38], realizing the first example of the program he proposed in the sixth of his famous questions [39]. Mathematical results on the closeness of the Hilbert expansion of the Bolzmann equation to the solutions of the compressible Euler equations for small Knudsen number ε, were obtained by

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Caflisch [14], and Lachowicz [45], while Nishida [47], Asano and Ukai [4] proved this by different methods.

On a longer time scale ε^{-1}, where diffusion effects become significant, the problem can be faced only in the low Mach numbers regime (Mach number of order ε or smaller) due to the lack of scaling invariance of the compressible Navier-Stokes equations. Hence the Boltzmann solution has been proved to be close to the incompressible Navier-Stokes-Fourier system. Mathematical results were given, among the others, in [11, 18, 31, 33, 34] for smooth solutions. For weak solutions (renormalized solutions), partial results were given, among the others, in [7–10], and the full result for the convergence of the renormalized solutions has been obtained by Golse and Saint-Raymond [27].

Much less is known about the steady solutions. It is worth to notice that, even for fixed Knudsen numbers, the analog of DiPerna-Lions' renormalized solutions [19] is not available for the steady case, due to lack of L^1 and entropy estimates. In [29, 30], steady solutions were constructed in convex domains near Maxwellians, and their positivity was left open. The only other results are for special, essentially one dimensional geometry (see [3] for results at fixed Knudsen numbers and [1, 2, 22, 23] for results at small Knudsen numbers in certain special 2D geometry). In a recent paper [20], via a new $L^2 - L^\infty$ framework, we have constructed the steady solution to the Boltzmann equation close to Maxwellians, in 3D general domains, for a gas in contact with a boundary with a prescribed temperature profile modeled by the diffuse reflection boundary condition. The question about positivity of these steady solutions was resolved as a consequence of their dynamical stability. As pointed in [25], despite the importance of steady Navier-Stokes-Fourier equations in applications, it has been an outstanding open problem to derive them from the steady Boltzmann theory.

The goal of our paper is to employ the $L^2 - L^\infty$ framework developed in [20] to study the hydrodynamical limit of the solutions to the steady Boltzmann equation, in the low Mach numbers regime, in a general domain with boundary where a temperature profile is specified. We refer to [15, 16, 41–44] for the recent development of $L^2 - L^\infty$ framework in various directions.

Let Ω be a bounded open region of $\mathbb{R}^d$ for either $d = 2$ or $d = 3$. We consider the Boltzmann equation for the distribution density $F(t, x, v)$ with $t \in \mathbb{R}_+ := [0, \infty)$, $x \in \Omega$, $v \in \mathbb{R}^3$. In the diffusive regime, the time evolution of the gas, subject to the action of a field $\vec{G}$, is described by the following *rescaled* Boltzmann equation:

$$\partial_t F + \varepsilon^{-1} v \cdot \nabla_x F + \vec{G} \cdot \nabla_v F = \varepsilon^{-2} Q(F, F), \tag{1.1.1}$$

where the Boltzmann collision operator is defined as

$$
\begin{aligned}
Q(F,H)(v) &:= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)[F(v')H(u')-F(v)H(u)]\mathrm{d}\omega\mathrm{d}u \\
&:= Q_+(F,H)(v) - Q_-(F,H)(v),
\end{aligned}
$$

with $v' = v - [(v-u)\cdot\omega]\omega,\ u' = v + [(v-u)\cdot\omega]\omega$. Here, B is chosen as *the hard spheres cross section* throughout this paper,

$$
B(V,\omega) = |V\cdot\omega|. \tag{1.1.2}
$$

The interaction of gas with the boundary $\partial\Omega$ is given by the diffuse reflection boundary condition, defined as follows: Let

$$
M_{\rho,u,T} := \frac{\rho}{(2\pi T)^{\frac{3}{2}}}\exp\Big[-\frac{|v-u|^2}{2T}\Big]
$$

be the local Maxwellian with density ρ, mean velocity u, and temperature T. For a prescribed function T^w on $\partial\Omega$, we define

$$
M^w = \sqrt{\frac{2\pi}{T^w}}M_{1,0,T^w}. \tag{1.1.3}
$$

We impose *the diffuse reflection boundary condition* as

$$
F = \mathcal{P}^w_\gamma(F), \quad \text{on} \ \ \gamma_-, \tag{1.1.4}
$$

where

$$
\mathcal{P}^w_\gamma F(x,v) := M^w(x,v)\int_{n(x)\cdot u>0} F(x,u)\{n(x)\cdot u\}\mathrm{d}u. \tag{1.1.5}
$$

Here, we denote by $n(x)$ the outward normal to $\partial\Omega$ at $x\in\partial\Omega$ and decompose the phase boundary $\gamma := \partial\Omega\times\mathbb{R}^3$ as

$$
\begin{aligned}
\gamma_\pm &:= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 \colon n(x)\cdot v \gtrless 0\}, \\
\gamma_0 &:= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 \colon n(x)\cdot v = 0\}.
\end{aligned} \tag{1.1.6}
$$

We remind that the boundary condition (1.1.4), (1.1.5) ensures the zero net mass flow at the boundary:

$$
\int_{\mathbb{R}^3} F(x,v)\{n(x)\cdot v\}\mathrm{d}v = 0, \quad \text{for any } x\in\partial\Omega.
$$

The rescaled Boltzmann equation (1.1.1) is studied under the assumption of low Mach numbers, meaning that the average velocity is small compared to the sound speed. This can be achieved by looking for solutions

$$
F - \mu = \mathfrak{M}\sqrt{\mu}f, \tag{1.1.7}
$$

with *the global Maxwellian*

$$
\mu(v) = M_{1,0,1} = \frac{1}{(2\pi)^{3/2}}\exp\Big[-\frac{|v|^2}{2}\Big]. \tag{1.1.8}
$$

Here, the number $\mathfrak{M}$ is proportional to the Mach number. The case of $\mathfrak{M} = \varepsilon$ corresponds to the incompressible Navier-Stokes-Fourier limit (INSF) that will be discussed in this paper. The case of $\mathfrak{M} \ll \varepsilon$ corresponds to the incompressible Stokes-Fourier limit and the results of this paper also cover this case which will not be discussed explicitly.

The condition (1.1.7), once assumed initially, needs to be checked at later times. By multiplying (1.1.1) by v and integrating on velocities, we see that the change of mean velocity is proportional to $\vec{G}$. Thus, we need to assume $\vec{G} = \mathfrak{M}\Phi$ with a bounded Φ. Moreover, to make (1.1.7) compatible with the boundary conditions, we need to assume that $T^w = 1 + \mathfrak{M}\vartheta^w$. In particular, for the INSF case, we have

$$\vec{G} = \varepsilon\Phi, \quad T^w = 1 + \varepsilon\vartheta^w. \tag{1.1.9}$$

The presence of the boundary represents a major issue in pursuing such a program. The usual approach is based on the representation of the solution by means of an Hilbert-like expansion in the bulk, suitably corrected at the boundary to satisfy the boundary conditions [1, 2, 22, 23]:

$$F = \mu + \varepsilon\sqrt{\mu}[f_1 + \varepsilon f_2 + \cdots + \varepsilon^k f_{k+1} + \varepsilon f_1^B + \varepsilon^2 f_2^B + \cdots + \varepsilon^{k+1} f_{k+1}^B + \varepsilon^k R]. \tag{1.1.10}$$

Here, the functions f_k are corrections in the bulk, while f_k^B are boundary layer corrections which solve Milne-like problems, and $R = R^\varepsilon$ denotes the remainder. It is important to choose sufficiently large $k \geq 1$ so that the nonlinear collision term can be controlled. The corrections at the boundary are computed by means of a boundary layer expansion which, in a general domain, presents some issues hard to deal with. The usual strategy is to solve the k-th term of the boundary layer expansion by looking at it in terms of the rescaled distance from the boundary (see e.g. [48]). Using of such a variable, the problem looks like a half-space linear problem (Milne problem) [5] with a correction due to the geometry which can be interpreted as an external field of the order of the Knudsen number. The field, due to the k-th term of the boundary layer expansion, is usually included as source term in the equation for the $(k+1)$-th term [48], but the lack of regularity makes this hard to control.

This strategy has been used in [12] in the much simpler case of the neutron transport equations, but recently in [51] it has been proved that the result in [12] breaks down exactly because of the lack of regularity (see the recent work in the Boltzmann case [50]). Therefore, the geometric field, even if of small size, has to be included in the equation for the k-term of the expansion, as in [2, 24] for the case of the gravity and [51] for the geometrical field in the neutron transport equation in a disk:

$$F = \mu + \varepsilon\sqrt{\mu}[f_1 + \varepsilon f_2 + \cdots + \varepsilon^k f_{k+1} + \varepsilon f_{1,\varepsilon}^B + \varepsilon^2 f_{2,\varepsilon}^B + \cdots + \varepsilon^{k+1} f_{k+1,\varepsilon}^B + \varepsilon^k R],$$

where $f^B_{k,\varepsilon}$ depends on ε. Unfortunately, this strategy fails even for a general 2D domain because the analysis of the derivatives' singularities presents severe difficulties (see [35, 36] for the analysis at $\varepsilon \approx 1$).

In this paper, we *avoid* the boundary layer expansion to cope with a general geometry. This is possible because, in the incompressible regime, the first term of the bulk expansion in (1.1.10) which violates the boundary condition is of order ε^2, while the main hydrodynamic contribution, is of order ε. We will discuss more about this in Section 1.3.

1.2 Expansion with Remainder

The Hilbert expansion suggests that the solution can be written as

$$F \ = \ \mu + \varepsilon\sqrt{\mu}\big[f_1 + \varepsilon f_2 + \varepsilon^{\alpha} R\big], \tag{1.2.1}$$

where μ is the standard Maxwellian in (1.1.8).

To determine f_1, f_2 and R_s, we define the linearized collision operator as

$$Lf = -\frac{1}{\sqrt{\mu}}[Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}, f\mu)], \tag{1.2.2}$$

and the nonlinear collision operator as

$$\Gamma(f, g) = \frac{1}{2\sqrt{\mu}}[Q(\sqrt{\mu}f, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \sqrt{\mu}f)]. \tag{1.2.3}$$

The null space of L, Null$L$ is a five-dimensional subspace of $L^2(\mathbb{R}^3)$ spanned by

$$\Big\{\sqrt{\mu},\ v\sqrt{\mu},\ \frac{|v|^2 - 3}{2}\sqrt{\mu}\Big\}.$$

We denote the orthogonal projection of f onto Null$L$ as

$$\mathbf{P}f = a\sqrt{\mu} + v \cdot b\sqrt{\mu} + c\frac{|v|^2 - 3}{2}\sqrt{\mu}, \tag{1.2.4}$$

and $(\mathbf{I} - \mathbf{P})$ the projection on the orthogonal complement of Null$L$. The inverse operator L^{-1} is defined as follows: $L^{-1}g$ is the unique solution of $L(L^{-1}g) = g$, and $\mathbf{P}(L^{-1}g) = 0$.

The first correction f_1 is given by

$$f_1 := \Big[\rho + u \cdot v + \frac{|v|^2 - 3}{2}\vartheta\Big]\sqrt{\mu}, \tag{1.2.5}$$

where (ρ, u, ϑ) represents the density, velocity, and temperature fluctuations. The density and the temperature fluctuations satisfy the Boussinesq relation

$$\nabla_x(\rho + \vartheta) = 0, \tag{1.2.6}$$

and the velocity and the temperature fluctuations satisfies the INSF system

$$\begin{aligned}
&\partial_t u + u \cdot \nabla_x u + \nabla_x p = \mathfrak{v}\Delta u + \Phi, \quad \nabla_x \cdot u = 0 && \text{in } \Omega,\\
&\partial_t \vartheta + u \cdot \nabla_x \vartheta = \kappa \Delta \vartheta && \text{in } \Omega,\\
&u(x,0) = u_0(x), \quad \vartheta(x,0) = \vartheta_0(x) && \text{in } \Omega,\\
&u(x) = 0, \quad \vartheta(x) = \vartheta^w(x) && \text{on } \partial\Omega,
\end{aligned} \tag{1.2.7}$$

where $\mathfrak{v}$ is the viscosity and κ is the heat conductivity.

The second correction f_2 is given by

$$\begin{aligned}
f_2 := \frac{1}{2}\sum_{i,j=1}^{3} \mathcal{A}_{ij}[\partial_{x_i} u_j + \partial_{x_j} u_i] + \sum_{i=1}^{3} \mathcal{B}_i \partial_{x_i}\vartheta\\
- L^{-1}[\Gamma(f_1, f_1)] + \Big[\rho_2 + u_2 \cdot v + \frac{|v|^2 - 3}{2}\vartheta_2\Big]\sqrt{\mu},
\end{aligned} \tag{1.2.8}$$

where $\mathcal{A}_{ij}$ and $\mathcal{B}_i$ are given by

$$\mathcal{A}_{ij} = L^{-1}\bigg(\sqrt{\mu}\Big(v_i v_j - \frac{|v|^2}{3}\delta_{i,j}\Big)\bigg), \quad \mathcal{B}_i = L^{-1}\big(\sqrt{\mu}v_i(|v|^2 - 5)\big).$$

Note that the only restriction on ρ_2, u_2 and ϑ_2 turns out to be that $\nabla_x[\rho\vartheta + \rho_2 + \vartheta_2] = \nabla_x p$. For simplicity we choose

$$\rho \equiv -\vartheta + \fint \vartheta, \quad \text{and} \quad u_2 \equiv 0 \equiv \rho_2, \quad \vartheta_2 \equiv p - \fint p - \rho\vartheta, \tag{1.2.9}$$

where

$$\fint \vartheta := \frac{1}{|\Omega|}\int_\Omega \vartheta(x)\mathrm{d}x \quad \text{and} \quad \fint p := \frac{1}{|\Omega|}\int_\Omega p(x)\mathrm{d}x.$$

These choices imply

$$\iint_{\Omega\times\mathbb{R}^3} f_1\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0 = \int_{\mathbb{R}^3} f_2\sqrt{\mu}\mathrm{d}v. \tag{1.2.10}$$

By choosing the reference Maxwellian, we can assume

$$\iint_{\Omega\times\mathbb{R}^3} R\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0. \tag{1.2.11}$$

The equation for R is obtained by plugging (1.2.1) into (1.1.1):

$$\partial_t R + \varepsilon^{-1} v \cdot \nabla_x R + \varepsilon\frac{1}{\sqrt{\mu}}\Phi \cdot \nabla_v\big[\sqrt{\mu}R\big] + \varepsilon^{-2}LR = \varepsilon^{-1}L_1 R + \varepsilon^{-\frac{1}{2}}\Gamma(R,R) + \varepsilon^{-\frac{1}{2}}A, \tag{1.2.12}$$

with

$$L_1 R := 2\Gamma(f_1 + \varepsilon f_2, R), \tag{1.2.13}$$

and

$$\begin{aligned}
A = &- [\partial_t f_1 + v \cdot \nabla_x f_2 - \Phi \cdot v\sqrt{\mu}] - 2\Gamma(f_1, f_2)\\
&- \varepsilon\Big\{\partial_t f_2 + \Phi \cdot \Big[\frac{1}{\sqrt{\mu}}\nabla_v\sqrt{\mu}(f_1 + \varepsilon f_2)\Big] - \Gamma(f_2, f_2)\Big\}.
\end{aligned}$$

It is important to observe the fact that $\mathbf{P}[\partial_t f_1 + v\cdot\nabla_x f_2 - \Phi\cdot v\sqrt{\mu}] = 0$ since (u,ϑ,p) solves (1.2.7). As a consequence, A is given by

$$A = -(\mathbf{I}-\mathbf{P})[v\cdot\nabla_x f_2] - 2\Gamma(f_1,f_2) - \varepsilon\Big\{\partial_t f_2 + \Phi\cdot\frac{1}{\sqrt{\mu}}\nabla_v\big[\sqrt{\mu}(f_1+\varepsilon f_2)\big] - \Gamma(f_2,f_2)\Big\}, \tag{1.2.14}$$

which implies the crucial fact that $\mathbf{P}A = O(\varepsilon)$. This is the only but essential point of our expansion where the specific hydrodynamic equations play a role. We also remark that, by (1.2.10),

$$\int_{\mathbb{R}^3} A\sqrt{\mu}\mathrm{d}v = 0. \tag{1.2.15}$$

It is well-known that (see [17])

$$Lf = \nu f - Kf,$$

where the collision frequency is defined as

$$\nu(v) = \frac{1}{\sqrt{\mu}}Q_-(\sqrt{\mu}f,\mu) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v-u)\cdot\omega|\sqrt{\mu}(u)\mathrm{d}\omega\mathrm{d}u.$$

For the hard sphere cross section (1.1.2), there are positive numbers C_0 and C_1 such that, for $\langle v\rangle := \sqrt{1+|v|^2}$,

$$C_0\langle v\rangle \le \nu(v) \le C_1\langle v\rangle. \tag{1.2.16}$$

Moreover the compact operator K is defined as

$$Kf = \frac{1}{\sqrt{\mu}}[Q_+(\mu,\sqrt{\mu}f) + Q_+(\sqrt{\mu}f,\mu) - Q_-(\mu,\sqrt{\mu}f)] = \int_{\mathbb{R}^3}[\mathbf{k}_1(v,u) - \mathbf{k}_2(v,u)]f(u)\mathrm{d}u.$$

The operator L is symmetric on $L^2(\mathbb{R}^3)$: $(f,Lg)_2 = (g,Lf)_2$ where $(\cdot\,,\cdot)_2$ is the L^2 inner product.

The following spectral inequality holds for L:

$$(f,Lf)_2 \gtrsim \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})f\|^2_{L^2(\mathbb{R}^3)}. \tag{1.2.17}$$

1.3 Boundary Conditions

We assume that (ρ,u,ϑ) satisfies the boundary conditions of (1.2.7) with (1.2.9). As a consequence, for $x\in\partial\Omega$,

$$M_{1+\varepsilon\rho,\varepsilon u,1+\varepsilon\vartheta}\big|_{\gamma_-} = \mathcal{P}^w_\gamma(M_{1+\varepsilon\rho,\varepsilon u,1+\varepsilon\vartheta}).$$

Moreover, by expanding $M_{1+\varepsilon\rho,\varepsilon u,1+\varepsilon\vartheta}$ in ε, we get

$$M_{1+\varepsilon\rho,\varepsilon u,1+\varepsilon\vartheta} = \mu + \varepsilon f_1\sqrt{\mu} + \varepsilon^2\varphi_\varepsilon, \tag{1.3.1}$$

where $|\varphi_\varepsilon| \le O(\|\rho\|_{L^\infty(\partial\Omega)}\|\theta^w\|_{L^\infty(\partial\Omega)})\langle v\rangle^4\mu(v)$.

Therefore, on γ_-

$$\mu+\varepsilon f_1\sqrt{\mu}+\varepsilon^2\varphi_\varepsilon\sqrt{\mu}=\mathcal{P}^w_\gamma(\mu+\varepsilon f_1\sqrt{\mu}+\varepsilon^2\varphi_\varepsilon\sqrt{\mu}). \tag{1.3.2}$$

On the other hand, from (1.1.4) and (1.2.1), on γ_-,

$$\mu+\varepsilon f_1\sqrt{\mu}+\varepsilon^2 f_2\sqrt{\mu}+\varepsilon^{\frac{3}{2}}R\sqrt{\mu}=\mathcal{P}^w_\gamma(\mu+\varepsilon f_1\sqrt{\mu}+\varepsilon^2 f_2\sqrt{\mu}+\varepsilon^{\frac{3}{2}}R\sqrt{\mu}).$$

Subtracting above two equations, we obtain the boundary condition for R:

$$R|_{\gamma_-}=\sqrt{\mu}^{-1}\mathcal{P}^w_\gamma(\sqrt{\mu}R)+\varepsilon^{1-\alpha}r,$$

with

$$r=\frac{1}{\sqrt{\mu}}\mathcal{P}^w_\gamma\big(\sqrt{\mu}[f_2-\varphi_\varepsilon]\big)-[f_2-\varphi_\varepsilon]. \tag{1.3.3}$$

We expand M^w in (1.1.3) with $T^w=1+\varepsilon\vartheta^w$ in ε to obtain

$$M^w(x,v)=\sqrt{2\pi}\mu(v)+\varepsilon\vartheta^w\sqrt{2\pi}\Big(\frac{|v|^2}{2}-2\Big)\mu(v)+\varepsilon^2 O(|\vartheta^w|^2)\langle v\rangle^4\mu(v). \tag{1.3.4}$$

Therefore we can write

$$\sqrt{\mu}^{-1}\mathcal{P}^w_\gamma(\sqrt{\mu}R)=P_\gamma R+\varepsilon\mathcal{Q}R,$$

with

$$P_\gamma R(x,v):=\sqrt{2\pi}\sqrt{\mu(v)}\int_{n(x)\cdot u>0}R(u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{1.3.5}$$

$$\mathcal{Q}R:=\varepsilon^{-1}\Big[\frac{1}{\sqrt{\mu}}\mathcal{P}^w_\gamma(\sqrt{\mu}R)-P_\gamma R\Big]. \tag{1.3.6}$$

Note that the boundary operator $\mathcal{Q}$ is bounded uniformly in ε because of (1.3.4).

Hence the boundary condition for R becomes

$$R=P_\gamma R+\varepsilon\mathcal{Q}R+\varepsilon^{\frac{1}{2}}r, \quad \text{on} \quad \gamma_- \tag{1.3.7}$$

with $\mathcal{Q}$ in (1.3.6) and r in (1.3.3).

From

$$\int_{n\cdot v<0}\mu\{n\cdot v\}\mathrm{d}v=-1=\int_{n\cdot v<0}M^w\{n\cdot v\}\mathrm{d}v$$

and (1.3.3) and (1.3.6), it follows that

$$\int_{n(x)\cdot v<0}\mathcal{Q}R\sqrt{\mu}\{n(x)\cdot v\}\mathrm{d}v=0=\int_{n(x)\cdot v<0}r\sqrt{\mu}\{n(x)\cdot v\}\mathrm{d}v, \quad \text{for any } x\in\partial\Omega. \tag{1.3.8}$$

Notations We use $\|\cdot\|_p$ and $\|\cdot\|_{L^p}$ for both of the $L^p(\bar{\Omega}\times\mathbb{R}^3)$ norm and the $L^p(\bar{\Omega})$ norm, and $(\,\cdot\,,\,\cdot\,)$ for the $L^2(\bar{\Omega}\times\mathbb{R}^3)$ inner product, where $\bar{\Omega}:=\Omega\cup\partial\Omega$. We subscript this to denote the variables, thus $\|\cdot\|_{L^p_y}$ means $L^p(\{y\in Y\})$. We denote $\|\cdot\|_\nu\equiv\|\nu^{1/2}\cdot\|_2$ and $\|f\|_{H^k}=\|f\|_2+\sum_{i=1}^k\|\nabla^i_x f\|_2$. We also denote $\|\cdot\|_{L^pL^q}:=\|\cdot\|_{L^p(L^q)}$

$:= \big\| \| \cdot \|_{L^q} \big\|_{L^p}$. For the phase boundary integration, we define $\mathrm{d}\gamma = |n(x) \cdot v| \mathrm{d}S(x) \mathrm{d}v$ where $\mathrm{d}S(x)$ is the surface measure and define $|f|_p^p = \int_\gamma |f(x,v)|^p \mathrm{d}\gamma$ and the corresponding space as $L^p(\partial\Omega \times \mathbb{R}^3; \mathrm{d}\gamma) = L^p(\partial\Omega \times \mathbb{R}^3)$. Further $|f|_{p,\pm} = |f \mathbf{1}_{\gamma_\pm}|_p$. We also use $|f|_p^p = \int_{\partial\Omega} |f(x)|^p \mathrm{d}S(x)$. Denote $f_\pm = f_{\gamma_\pm}$. $X \lesssim Y$ is equivalent to $X \leq CY$, where C is a constant not depending on X and Y. We subscript this to denote dependence on parameters, thus $X \lesssim_\alpha Y$ means $X \leq C_\alpha Y$.

1.4 Main Results

We first focus on the steady case. The following (p_s, u_s, ϑ_s) is a solution to the steady INSF with Dirichlet boundary conditions and subject to the external field Φ:

$$\begin{aligned} &u_s \cdot \nabla_x u_s + \nabla_x p_s = \mathfrak{v} \Delta u_s + \Phi, \quad \nabla_x \cdot u_s = 0 && \text{in } \Omega, \\ &u_s \cdot \nabla_x \vartheta_s = \kappa \Delta \vartheta_s && \text{in } \Omega, \\ &u_s(x) = 0, \quad \vartheta_s(x) = \vartheta^w(x) && \text{on } \partial\Omega. \end{aligned} \tag{1.4.1}$$

Note that, if Φ is a potential field, $u_s \equiv 0$ is a solution to the above system. Therefore, in order to have a stationary solution with non vanishing velocity field, we may assume that Φ is not a potential field, such that $\nabla_x \cdot \Phi = 0$. (See [25])

The steady solution to the Boltzmann equation is obtained with the same procedure discussed before for the unsteady case:

$$F_s = \mu + \varepsilon\sqrt{\mu}[f_{s,1} + \varepsilon f_{s,2} + \varepsilon^{1/2} R_s], \tag{1.4.2}$$

where $f_{s,1}$ and $f_{s,2}$ are given by (1.2.5) and (1.2.8) with ρ, u, ϑ, and p replaced by ρ_s, ϑ_s, u_s and p_s. The remainder has to satisfy the following equation

$$v \cdot \nabla_x R_s + \varepsilon^2 \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v \big[\sqrt{\mu} R_s\big] + \varepsilon^{-1} L R_s = L_1 R_s + \varepsilon^{1/2} \Gamma(R_s, R_s) + \varepsilon^{1/2} A_s, \tag{1.4.3}$$

with the boundary condition (1.3.7). Here A_s is given by (1.2.14) with f_1, f_2 replaced by $f_{s,1}, f_{s,2}$, and satisfies the mean zero condition (1.2.15).

Theorem 1.1 *Assume Ω is an open bounded subset of $\mathbb{R}^3$ with C^3 boundary $\partial\Omega$. We also assume the hard sphere cross section* (1.1.2).

If $\Phi = \Phi(x) \in H^2(\Omega) \cap C^1(\Omega)$, $\vartheta^w \in H^{7/2}(\Omega)$ and

$$\|\vartheta^w\|_{H^{1+}(\partial\Omega)} + \|\Phi\|_{L^{\frac{3}{2}+}(\Omega)} \ll 1, \tag{1.4.4}$$

then, for $0 < \varepsilon \ll 1$, there is a unique positive solution $F_s \geq 0$, given by (1.4.2) *with R_s satisfying* (1.4.3) *and the boundary condition* (1.3.7)*. Here, $f_{1,s}$ and $f_{2,s}$ are given by* (1.2.5) *and* (1.2.8) *where (u_s, ϑ_s, p_s) solves* (1.4.1)*.*

Moreover,

$$\begin{aligned}&\|R_s\|_2+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_\nu\ll 1,\quad \varepsilon\|wR_s\|_\infty\ll 1,\\&\|R_s\|_{L^3_x}\lesssim 1,\\&\|f_{s,1}\|_{L^6_xL^2_v\cap L^\infty_{x,v}}+\|f_{s,2}\|_{L^6_xL^2_v\cap L^\infty_{x,v}}\lesssim 1,\end{aligned}\tag{1.4.5}$$

where $w(v)=\mathrm{e}^{\beta|v|^2}$ *with* $0<\beta\ll 1$.

We remark that in the expansion (1.4.2), the remainder $\sqrt{\varepsilon}R_s$ is of higher order in L^p for $2\le p<6$. On the other hand, $\sqrt{\varepsilon}R_s$ is of order $\varepsilon^{-1/2}$ in L^∞, so the expansion $F_s=\mu+\varepsilon\sqrt{\mu}[f_{1,s}+\varepsilon^{\frac12}R_s]$ is *not* proved to be valid in L^∞. It is important to note that the key difficulty in this paper is to control the 'strong' nonlinear terms $\sqrt{\varepsilon}\Gamma(R_s,R_s)$, in the absence of boundary layer approximations. The hard spheres cross section is needed to control the term $\varepsilon v\cdot\Phi f$ coming from the external field.

We use the quantitative L^2-L^∞ approach developed in [20], in the presence of ε. We start with the energy estimates to get

$$\frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})R_s\|_\nu\lesssim\sqrt{\varepsilon}\|\Gamma(R_s,R_s)\|_2+1.$$

The missing $\mathbf{P}R_s$ can be estimated by the coercivity estimates in [20], with carefully chosen proper test functions in the weak formulation, such that (Proposition 2.2):

$$\|\mathbf{P}R_s\|_2\lesssim\frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})R_s\|_\nu+\sqrt{\varepsilon}\|\Gamma(R_s,R_s)\|_2+1.$$

By using a double iteration of the Duhamel formula along the characteristics [20], we may bootstrap such L^2 estimates to L^∞ estimate as

$$\|R_s\|_\infty\lesssim\frac{1}{\varepsilon^{d/2}}\|\mathbf{P}R_s\|_2+\frac{1}{\varepsilon^{d/2}}\|(\mathbf{I}-\mathbf{P})R_s\|_2+1\lesssim\frac{1}{\varepsilon^{d/2}},$$

where the dimension is $d=3$.

We split

$$|\Gamma(R_s,R_s)|\lesssim|\Gamma(\mathbf{P}R_s,R_s)|+|\Gamma((\mathbf{I}-\mathbf{P})R_s,R_s)|.$$

Since we expect $\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_\nu\lesssim 1$, the second part of the nonlinear term is estimated as

$$\sqrt{\varepsilon}\|\Gamma((\mathbf{I}-\mathbf{P})R_s,R_s)\|_2\lesssim\sqrt{\varepsilon}\|(\mathbf{I}-\mathbf{P})R_s\|_2\|R_s\|_\infty\lesssim 1.$$

Unfortunately, in 3D, $\|R_s\|_\infty\lesssim\frac{1}{\varepsilon^{3/2}}$ leads to

$$\sqrt{\varepsilon}\|\Gamma(\mathbf{P}R_s,\mathbf{P}R_s)\|_2\lesssim\|\mathbf{P}R_s\|_2\|\mathbf{P}R_s\|_\infty\lesssim\sqrt{\varepsilon}\frac{1}{\varepsilon^{3/2}}=\frac{1}{\varepsilon},$$

which is way out of control to close the estimates.

The key observation is that in our L^∞ estimate, higher integrability of $\mathbf{P}R_s$ helps to reduce ε singularity in the estimate of $\|R_s\|_\infty$. Indeed, if we have

$$\|\mathbf{P}R_s\|_{L^3} \lesssim 1,$$

then, in $d = 3$, we are able to improve the L^∞ estimate as (Proposition 3.3):

$$\|R_s\|_\infty \lesssim \frac{1}{\varepsilon}\|\mathbf{P}R_s\|_3 + \frac{1}{\varepsilon^{3/2}}\|(\mathbf{I}-\mathbf{P})R_s\|_2 + 1 \lesssim \frac{1}{\varepsilon}.$$

Now with such an improvement, we have $\|\mathbf{P}R_s\|_6 \lesssim \|\mathbf{P}R_s\|_3^{1/2}\|\mathbf{P}R_s\|_\infty^{1/2} \lesssim \varepsilon^{-1/2}$, and the nonlinearity is *exactly* controllable:

$$\sqrt{\varepsilon}\|\Gamma(\mathbf{P}R_s, \mathbf{P}R_s)\|_2 \lesssim \sqrt{\varepsilon}\|\mathbf{P}R_s\|_3\|\mathbf{P}R_s\|_6 \lesssim \sqrt{\varepsilon} \times \frac{1}{\sqrt{\varepsilon}} = 1.$$

In the absence of the external field and the boundary, $\Phi \equiv 0$ and $\Omega = \mathbb{R}^3$, such gain of integrability, $\|\mathbf{P}R_s\|_{L^3} \lesssim 1$, is well-known from the Averaging Lemma [26] and the sharp Sobolev embedding $H^{1/2} \subset L^3$ (See also the case for a convex bounded domain with $\Phi \equiv 0$ in [26]). We need to extend this estimate properly to case of the bounded domain Ω with the presence of the external field $\Phi \neq 0$. We first consider an extension of R_s to the whole space, denoted by $\bar{R}_s$, such that $\bar{R}_s \in L^2$ and

$$v \cdot \nabla_x \bar{R}_s + \varepsilon^2 \Phi \cdot \nabla_v \bar{R}_s \in L^2.$$

This would require that $\bar{R}_s$ is continuous along all exterior trajectories, matching with given incoming and outgoing data of R_s on the boundary. For a general domain Ω with $\varepsilon^2\Phi \neq 0$, the exterior trajectories can be complicated and they can connect the outgoing set γ_+ and incoming set γ_-, arbitrarily near the grazing set γ_0. It is not clear that an extension $\bar{R}_s$ would satisfy both $\bar{R}_s \in L^2$ and $v \cdot \nabla_x \bar{R}_s + \varepsilon^2\Phi \cdot \nabla_v \bar{R}_s \in L^2$, due to a possible discontinuity of R_s [41].

We circumvent this difficulty via an extension lemma, Lemma 2.4, which asserts that, for the function cutoff from the grazing set γ_0,

$$R_{s,\delta} \sim \mathbf{1}_{\{|v|<\frac{1}{\delta}\}}\mathbf{1}_{\{|n(x)\cdot v|>\delta \text{ or } \mathrm{dist}(x,\partial\Omega)>\delta\}} R_s, \quad \text{for } \delta \ll 1, \tag{1.4.6}$$

such an extension $\bar{R}_{s,\delta}$ is indeed possible. Here, $\mathrm{dist}(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x - y|$. Luckily, $\mathbf{P}R_{s,\delta} \sim \mathbf{P}R_s$ thanks to the estimate $\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_2 \sim 1$. In the presence of the external field $\Phi \neq 0$, a direct application of averaging lemma leads to $\mathbf{P}\bar{R}_{s,\delta} \in H^{\frac{1}{4}} \not\subset L^3$. To show $\mathbf{P}\bar{R}_{s,\delta} \in L^3$ in the whole space (Proposition 2.1), instead, we utilize the Duhamel formula along the trajectories, and employ the approach in [40], and take advantage of small field $\varepsilon^2\Phi$. This is different from the classical proof based on Fourier transform in [26].

We also remark that in the presence of an external field, even the construction of the solution to the linear problem is delicate. In fact, an extension similar to (1.4.6) must be used again to gain compactness from the averaging lemma. Moreover, as in [20], our construction cannot yield the positivity of F_s directly, which is left for the unsteady case.

Next we investigate the stability properties of the stationary solution. To discuss this, we study the unsteady problem. The solution to (1.1.1) is written as

$$F(t) = F_s + \varepsilon\sqrt{\mu}[\tilde{f}_1 + \varepsilon\tilde{f}_2 + \varepsilon^{1/2}\tilde{R}(t)]. \tag{1.4.7}$$

Here, $\tilde{f}_1$ is given by

$$\tilde{f}_1 = \Big[\tilde{\rho} + \tilde{u}\cdot v + \frac{|v|^2-3}{2}\tilde{\vartheta}\Big]\sqrt{\mu}$$

where $(u_s+\tilde{u}, \vartheta_s+\tilde{\vartheta}, p_s+\tilde{p})$ solves (1.2.7), and $f_{2,s}+\tilde{f}_2$ satisfies (1.2.8). Therefore $(\tilde{u}, \tilde{\vartheta}, \tilde{p})$ satisfies

$$\begin{aligned}
&\partial_t\tilde{u} + \tilde{u}\cdot\nabla_x\tilde{u} + \tilde{u}\cdot\nabla_x u_s + u_s\cdot\nabla_x\tilde{u} + \nabla_x\tilde{p} = \mathfrak{v}\Delta\tilde{u}, \quad \nabla_x\cdot\tilde{u} = 0 && \text{in } \Omega,\\
&\partial_t\tilde{\vartheta} + \tilde{u}\cdot\nabla_x\tilde{\vartheta} + \tilde{u}\cdot\nabla_x\vartheta_s + u_s\cdot\nabla_x\tilde{\vartheta} = \kappa\Delta\tilde{\vartheta} && \text{in } \Omega,\\
&\tilde{u} = 0, \quad \tilde{\vartheta} = 0 && \text{on } \partial\Omega.
\end{aligned} \tag{1.4.8}$$

From the choice of (1.2.9), we have $\tilde{\rho}(t,x) = -\tilde{\vartheta}(t,x) + ⨍\,\tilde{\vartheta}(t)$ and $\tilde{\vartheta}_2 = \tilde{p} - ⨍\,\tilde{p} + 2\vartheta_s\tilde{\vartheta} + (\tilde{\vartheta})^2 - \tilde{\vartheta}⨍\,\vartheta_s - \vartheta_s⨍\,\tilde{\vartheta} - \tilde{\vartheta}⨍\,\tilde{\vartheta}$.

Then the equation of $\tilde{R}$ is given by

$$\begin{aligned}
&\partial_t\tilde{R} + \varepsilon^{-1}v\cdot\nabla_x\tilde{R} + \varepsilon\Phi\cdot\nabla_v\tilde{R} + \varepsilon^{-2}L\tilde{R}\\
={}&\varepsilon^{-1}L_1\tilde{R} + \varepsilon^{-1}L_{\varepsilon^{1/2}R_s}\tilde{R} + \varepsilon^{-1}L_{R_s}(\tilde{f}_1+\varepsilon\tilde{f}_2)\\
&+\varepsilon^{-1/2}\Gamma(\tilde{R},\tilde{R}) + \varepsilon\frac{\Phi\cdot v}{2}\tilde{R} + \varepsilon^{-1/2}\tilde{A},
\end{aligned} \tag{1.4.9}$$

where $\tilde{A} = A - A_s$. Here we have used the notation $L_\phi\psi := -[\Gamma(\phi,\psi)+\Gamma(\psi,\phi)]$. Note that, due to symmetry, for all $\psi_1, \psi_2 \in L^2$,

$$(L_\phi\psi_1, \psi_2) = (L_\phi\psi_1, (\mathbf{I}-\mathbf{P})\psi_2). \tag{1.4.10}$$

The boundary condition of $\tilde{R}$ is given by

$$\tilde{R}|_{\gamma_-} = P_\gamma\tilde{R} + \varepsilon\mathcal{Q}\tilde{R} + \varepsilon^{1/2}\tilde{r}, \tag{1.4.11}$$

where

$$\tilde{r} := \varepsilon^{-1}[\mu^{-\frac{1}{2}}P_\gamma^w(\tilde{f}_1\sqrt{\mu}) - \tilde{f}_1] + [\mu^{-\frac{1}{2}}P_\gamma^w(\tilde{f}_2\sqrt{\mu}) - \tilde{f}_2].$$

Note that, since M^w only depends on ϑ^w, by taking the difference of (1.3.2), written for the unsteady and steady solutions respectively, we obtain

$$\mu^{-\frac{1}{2}}P_\gamma^w(\tilde{f}_1\sqrt{\mu}) - \tilde{f}_1 = \varepsilon\{\mu^{-\frac{1}{2}}P_\gamma^w(\tilde{\varphi}_\varepsilon\sqrt{\mu}) - \tilde{\varphi}_\varepsilon\},$$

with $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon - \varphi_{\varepsilon,s}$. Thus, apparently $\varepsilon^{-1}[\mu^{-\frac{1}{2}}P_\gamma^w(\tilde{f}_1\sqrt{\mu}) - \tilde{f}_1]$ term in $\tilde{r}$ is actually $O(1)$.

We define the energy and the dissipation as

$$
\begin{aligned}
\mathcal{E}_\lambda(t) &:= \sup_{0\le s\le t} \|\mathrm{e}^{\lambda s}\tilde{R}(s)\|_2^2 + \sup_{0\le s\le t} \|\mathrm{e}^{\lambda s}\partial_t\tilde{R}(s)\|_2^2,\\
\mathcal{D}_\lambda(t) &:= \frac{1}{\varepsilon^2}\int_0^t \|\mathrm{e}^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde{R}\|_\nu^2 + \frac{1}{\varepsilon^2}\int_0^t \|\mathrm{e}^{\lambda s}(\mathbf{I}-\mathbf{P})\partial_t\tilde{R}\|_\nu^2\\
&\quad + \int_0^t \|\mathrm{e}^{\lambda s}\mathbf{P}\tilde{R}\|_2^2 + \int_0^t \|\mathrm{e}^{\lambda s}\mathbf{P}\partial_t\tilde{R}\|_2^2 + \int_0^t |\mathrm{e}^{\lambda s}\tilde{R}|_{2,\gamma}^2 + \int_0^t |\mathrm{e}^{\lambda s}\partial_t\tilde{R}|_{2,\gamma}^2.
\end{aligned}
$$

Theorem 1.2 *We assume the same hypotheses of Theorem* 1.1. *Suppose* $F_0 = F_s + \varepsilon\sqrt{\mu}[\tilde{f}_1(0) + \varepsilon\tilde{f}_2(0) + \varepsilon^{1/2}\tilde{R}(0)] \ge 0$, *and* $\tilde{u}(0), \tilde{\vartheta}(0) \in H^8(\Omega)$ *and*

$$
\|\tilde{u}(0)\|_{H^2(\Omega)} + \|\tilde{\vartheta}(0)\|_{H^2(\Omega)} \ll 1, \tag{1.4.12}
$$

and

$$
\mathcal{E}(0) + \varepsilon^{3/2}\|w\partial_t\tilde{R}_0\|_\infty + \left\| \int_{\mathbb{R}^3} |\tilde{R}_0(x,v)|\langle v\rangle^2\sqrt{\mu}\mathrm{d}v \right\|_{L^3(\Omega)} \ll 1, \quad \varepsilon\|w\tilde{R}_0\|_\infty \lesssim 1, \tag{1.4.13}
$$

where $w(v) = \mathrm{e}^{\beta|v|^2}$ *with* $0 < \beta \ll 1$.

Then there exists a unique global solution $F \ge 0$ *given by* (1.4.7) *with* $\tilde{R}$ *solving* (1.4.9) *and the boundary condition* (1.4.11). *Here,* $\tilde{f}_1$ *and* $\tilde{f}_2$ *are given by* (1.2.5) *and* (1.2.8) *where* $(\tilde{u}, \tilde{\vartheta}, \tilde{p})$ *solves* (1.4.8).

Moreover, for some $0 < \lambda \ll 1$,

$$
\begin{aligned}
&\mathcal{E}_\lambda(\infty) + \mathcal{D}_\lambda(\infty) + \sup_{0\le t\le\infty} \varepsilon^{3/2}\|w\partial_t\tilde{R}(t)\|_\infty \ll 1, \qquad \sup_{0\le t\le\infty} \varepsilon\|w\tilde{R}(t)\|_\infty \lesssim 1,\\
&\|w\mathrm{e}^{\lambda t}\tilde{f}_1\|_{L^6_x L^\infty_{t,v}\cap L^\infty_{t,x,v}} + \|w\mathrm{e}^{\lambda t}\tilde{f}_2\|_{L^6_x L^\infty_{t,v}\cap L^\infty_{t,x,v}} \lesssim 1.
\end{aligned} \tag{1.4.14}
$$

Here, we recall that the notation $\|\cdot\|_{L^pL^q}$ means $\|\cdot\|_{L^p(L^q)} := \big\|\|\cdot\|_{L^q}\big\|_{L^p}$.

We remark that such an asymptotical stability implies positivity of steady solution F_s (Section 3.7). Moreover, since $R = R_s + \tilde{R}$, we conclude that the expansion (1.4.7) is valid in $L^\infty_t L^2_x$ and $L^3_x L^2_t$. We use similar ideas as in the steady case, but the analysis is more intricate.

We start with the energy estimates, as the steady case, to get

$$
\frac{1}{\varepsilon^2}\int_0^t \|(\mathbf{I}-\mathbf{P})\tilde{R}\|_\nu^2 \lesssim \varepsilon\int_0^t \|\Gamma(\tilde{R},\tilde{R})\|_2^2 + 1.
$$

The missing $\mathbf{P}\tilde{R}$ can be estimated by the coercivity estimates in [20], with carefully chosen proper test functions in the weak formulation together with the local conservation laws (Proposition 3.2):

$$
\int_0^t \|\mathbf{P}\tilde{R}\|_2^2 \lesssim \frac{1}{\varepsilon^2}\int_0^t \|(\mathbf{I}-\mathbf{P})\tilde{R}\|_\nu^2 + \varepsilon\int_0^t \|\Gamma(\tilde{R},\tilde{R})\|_2^2 + 1.
$$

Furthermore, as in the steady case, via a similar extension argument, Lemma 3.3 and Proposition 3.1, we establish a gain of integrability as

$$\|\mathbf{P}\tilde{R}\|_{L^3_{x,v}L^2_t} \lesssim 1 + \sqrt{\varepsilon}\|\Gamma(\tilde{R},\tilde{R})\|_{L^2_{t,x,v}} + \sqrt{\varepsilon}\|\Gamma(R_s,\tilde{R})\|_{L^2_{t,x,v}}. \tag{1.4.15}$$

Hence, the nonlinearity can be bounded by interpolations:

$$\begin{aligned}\sqrt{\varepsilon}\|\Gamma(\mathbf{P}\tilde{R},\mathbf{P}\tilde{R})\|_{L^2_{t,x,v}} &\lesssim \sqrt{\varepsilon}\|\mathbf{P}\tilde{R}\|_{L^6_{x,v}L^\infty_t} \cdot \|\mathbf{P}\tilde{R}\|_{L^3_{x,v}L^2_t} \\ &\lesssim \sqrt{\varepsilon}\|\mathbf{P}\tilde{R}\|^{1/2}_{L^3_{x,v}L^\infty_t}\|\mathbf{P}\tilde{R}\|^{1/2}_{L^\infty_{x,v}L^\infty_t} \cdot \|\mathbf{P}\tilde{R}\|_{L^3_{x,v}L^2_t}.\end{aligned}$$

On the other hand, $\sqrt{\varepsilon}\|\Gamma(R_s,\tilde{R})\|_{L^2_{t,x,v}}$ in (1.4.15) needs an extra care since we cannot take L^2_t-norm to the steady solution R_s. It turns out, by a closer look, that we only need to consider

$$\sqrt{\varepsilon}\|\Gamma((\mathbf{I}-\mathbf{P})R_s,\mathbf{P}\tilde{R})\|_{L^2_{t,x,v}} \lesssim \sqrt{\varepsilon}\|(\mathbf{I}-\mathbf{P})R_s\|_{L^6_{x,v}}\|\mathbf{P}\tilde{R}\|_{L^3_x L^2_t}.$$

Thanks to the good bound of $\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_2 \lesssim 1$ and further by the interpolation $L^6 \subset L^2 \cap L^\infty$, we bound the above by

$$\sqrt{\varepsilon} \times \varepsilon^{-1/3}\big[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_2\big]^{1/3}\big[\varepsilon\|R_s\|_\infty\big]^{2/3}.$$

Similarly to the steady case, by using a double iteration of the Duhamel formula along the characteristics [20], we may bootstrap such L^2 estimates to an improved L^∞ estimate as

$$\|\tilde{R}\|_{L^\infty} \lesssim \frac{1}{\varepsilon}\|\mathbf{P}\tilde{R}\|_{L^\infty_t L^3_{x,v}} + \frac{1}{\varepsilon^{3/2}}\|(\mathbf{I}-\mathbf{P})\tilde{R}\|_{L^\infty_t L^2_{x,v}} + 1,$$

where the dimension is $d=3$. Clearly, a new difficulty is to estimate $\|\mathbf{P}\tilde{R}\|^{1/2}_{L^3_{x,v}L^\infty_t}$ which is not controllable from (1.4.15).

The key new idea is to repeat energy estimates $\|\tilde{R}_t\|_{L^\infty_t L^2_{x,v}}$ and $\|\mathbf{P}\tilde{R}_t\|_{L^3_{x,v}L^2_t}$ estimates for the time derivative $\tilde{R}_t$:

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_0^t \|(\mathbf{I}-\mathbf{P})\tilde{R}_t\|^2_\nu + \int_0^t \|\mathbf{P}\tilde{R}_t\|_2^2 \lesssim \varepsilon\int_0^t \|\Gamma(\tilde{R},\tilde{R}_t)\|_2^2 + 1,\\ &\|\mathbf{P}\tilde{R}_t\|_{L^3_{x,v}L^2_t} \lesssim \varepsilon\int_0^t \|\Gamma(\tilde{R},\tilde{R}_t)\|_2^2 + \int_0^t \|\Gamma(\tilde{R}_t,\tilde{R})\|_2^2 + 1.\end{aligned}$$

We then estimate L^∞_t via H^1_t (Lemma 3.6) as

$$\|\mathbf{P}\tilde{R}\|_{L^3_{x,v}L^\infty_t} + \|\mathbf{P}\tilde{R}\|_{L^\infty_t L^3_{x,v}} \lesssim \|\mathbf{P}\tilde{R}\|_{L^3_{x,v}L^2_t} + \|\mathbf{P}\tilde{R}_t\|_{L^3_{x,v}L^2_t} + 1.$$

Fortunately, in order to close the estimates, we don't need to improve $\|\tilde{R}_t\|_{L^\infty}$, but only need to control the new nonlinear term

$$\begin{aligned}\sqrt{\varepsilon}\|\Gamma(\mathbf{P}\tilde{R},\mathbf{P}\tilde{R}_t)\|_{L^2_{t,x,v}} &\lesssim \sqrt{\varepsilon}\|\mathbf{P}\tilde{R}\|_{L^6_{x,v}L^\infty_t}\cdot\|\mathbf{P}\tilde{R}_t\|_{L^3_{x,v}L^2_t}\\ &\lesssim \|\mathbf{P}\tilde{R}\|^{1/2}_{L^3_{x,v}L^\infty_t}\{\sqrt{\varepsilon}\|\mathbf{P}\tilde{R}\|^{1/2}_{L^\infty_{x,v}L^\infty_t}\}\|\mathbf{P}\tilde{R}_t\|_{L^3_{x,v}L^2_t},\end{aligned}$$

which can be exactly closed.

We remark that our method works also for a general 2D domain. Now the gain of integrability is expected as $H^{1/2}\subset L^{\frac{4}{2-1}}=L^4$, and it is not *critical* and analysis is much less delicate than the 3D case.

2 Steady Problems

2.1 Domain, Trace, and Green Identity

Assume $\partial\Omega$ is C^3. Then for any $x_0\in\partial\Omega$, there exist $0<r_0,r_1\ll 1$ and C^3 function $\eta:\{x_\parallel=(x_{\parallel,1},x_{\parallel,2})\in\mathbb{R}^2:|x_\parallel|<r_1\}\to\partial\Omega\cap B(x_0,r_0)$ such that if $x\in\partial\Omega\cap B(x_0,r_0)$ then there exists a unique $x_\parallel\in\mathbb{R}^2$ with $|x_\parallel|<r_1$ satisfying $x=\eta(x_\parallel)$. Here, we have used the notation $B(x_0,r_0):=\{x\in\mathbb{R}^3:|x-x_0|<r_0\}$. Without loss of generality we assume that $|\partial_{x_{\parallel,i}}\eta(x_\parallel)|\neq 0$ for $i=1,2$.

Assume $\mathrm{dist}(x,\partial\Omega)\ll 1$ and $x_0\in\partial\Omega$ such that $\mathrm{dist}(x,x_0)=\mathrm{dist}(x,\partial\Omega)$. Then there exists an η which is a parametrization of $\partial\Omega$ around x_0. Clearly

$$\nabla_{x_\parallel}|\eta(x_\parallel)-x|^2=(\partial_{x_{\parallel,1}}|\eta(x_\parallel)-x|^2,\partial_{x_{\parallel,2}}|\eta(x_\parallel)-x|^2)=0,\quad\text{for some}\ \ x_\parallel. \tag{2.1.1}$$

On the other hand, if $|\eta(x_\parallel)-x|\ll 1$,

$$\partial^2_{x_{\parallel,i}}|\eta(x_\parallel)-x|^2=\partial_{x_{\parallel,i}}\big[2\partial_i\eta(x_\parallel)\cdot(\eta(x_\parallel)-x)\big]=O(|\eta(x_\parallel)-x|)+2|\partial_i\eta(x_\parallel)|^2\neq 0.$$

Then, by the implicit function theorem, there exists a unique $x_\parallel(x)\in C^2$ satisfying (2.1.1). Moreover,

$$\begin{pmatrix}\partial_{x_i}x_{\parallel,1}\\ \partial_{x_i}x_{\parallel,2}\end{pmatrix}=\begin{pmatrix}|\partial_1\eta|^2+\partial_1^2\eta\cdot(\eta-x) & \partial_1\eta\cdot\partial_2\eta+\partial_1\partial_2\eta\cdot(\eta-x)\\ \partial_1\eta\cdot\partial_2\eta+\partial_1\partial_2\eta\cdot(\eta-x) & |\partial_2\eta|+\partial_2^2\eta\cdot(\eta-x)\end{pmatrix}^{-1}\begin{pmatrix}-\partial_1\eta_i\\ -\partial_2\eta_i\end{pmatrix},$$

where $\eta=\eta(x_\parallel)$. Then we define $x_\perp\in C^2$ for $\mathrm{dist}(x,\partial\Omega)\ll 1$,

$$x_\perp(x):=[x-\eta(x_\parallel(x))]\cdot n(x_\parallel(x)). \tag{2.1.2}$$

Note that $\mathrm{dist}(x,\partial\Omega)=|x_\perp(x)|$ if $\mathrm{dist}(x,\partial\Omega)\ll 1$.

By the compactness of $\partial\Omega$, we conclude that if $\mathrm{dist}(x,\partial\Omega)<4r$ for some $0<r\ll_\Omega 1$ then there exists $(x_\parallel(x),x_\perp(x))\in C^2$ such that $x=\eta(x_\parallel(x))+x_\perp(x)n(x_\parallel(x))$.

Finally we define the C^2 function $\xi:\mathbb{R}^3\to\mathbb{R}$ as

$$\xi(x):=x_\perp(x)\chi\Big(\frac{|\mathrm{dist}(x,\Omega)|^2}{4r^2}\Big)+r\Big[1-\chi\Big(\frac{|\mathrm{dist}(x,\Omega)|^2}{r^2}\Big)\Big], \tag{2.1.3}$$

where

$$\chi \in C_c^\infty(\mathbb{R}) \text{ such that } 0 \le \chi \le 1,\ \chi'(x) \ge -4 \times \mathbf{1}_{\frac{1}{2} \le |x| \le 1} \text{ and } \chi(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2}, \\ 0 & \text{if } |x| \ge 1. \end{cases} \tag{2.1.4}$$

Then $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$. If $|\xi(x)| \ll 1$ then $\xi(x) = x_\perp(x)$.

Moreover $n(x) \equiv \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ at the boundary $x \in \partial\Omega$. From now we define

$$n(x) := \frac{\nabla\xi(x)}{|\nabla\xi(x)|} \quad \text{for} \quad x \in \mathbb{R}^3. \tag{2.1.5}$$

We use this new coordinate (2.1.2) to extend Φ on the whole space, and denote this extension by $\bar{\Phi}$, with $\|\bar{\Phi}\|_\infty \le \|\Phi\|_\infty$: For $0 < \delta \ll 1$,

$$\bar{\Phi}(x) := \Phi(x)\mathbf{1}_{x\in\bar{\Omega}} + \Phi(\eta(x_\parallel(x)))\chi\Big(\frac{|\xi(x)|}{\delta}\Big)\mathbf{1}_{x\in\mathbb{R}^3\setminus\bar{\Omega}}.$$

Therefore without loss of generality we assume that Φ is defined on the whole space $\mathbb{R}^3$.

Definition 2.1 Assume $\Phi = \Phi(x) \in C^1$. Consider the steady linear transport equation

$$v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_x f = g. \tag{2.1.6}$$

The equations of the characteristics for (2.1.6) are

$$\dot{X} = V, \quad \dot{V} = \varepsilon^2\Phi(X), \quad X(t;t;x,v) = x, \quad V(t;t;x,v) = v. \tag{2.1.7}$$

If $X(\tau; t, x, v) \in \Omega$ for all τ in between s and t then

$$\begin{aligned} X(s;t;x,v) &= x + v(s-t) + \varepsilon^2 \int_t^s \int_t^\tau \Phi(X(\tau';t;x,v))\mathrm{d}\tau'\mathrm{d}\tau, \\ V(s;t;x,v) &= v + \varepsilon^2 \int_t^s \Phi(\tau;s;x,v))\mathrm{d}\tau. \end{aligned} \tag{2.1.8}$$

Note that the ODE (2.1.7) is autonomous since Φ is time-independent.

Define

$$\begin{aligned} &t_{\mathbf{b}}(x,v) := \inf\{t \ge 0 : X(-t;0;x,v) \notin \Omega\}, \\ &x_{\mathbf{b}}(x,v) := X(-t_{\mathbf{b}}(x,v);0;x,v,0), \quad v_{\mathbf{b}}(x,v) := V(-t_{\mathbf{b}}(x,v);0;x,v), \end{aligned} \tag{2.1.9}$$

and

$$\begin{aligned} &t_{\mathbf{f}}(x,v) := \inf\{t \ge 0 : X(t;0;x,v) \notin \Omega\}, \\ &x_{\mathbf{f}}(x,v) := X(t_{\mathbf{f}}(x,v);0;x,v,0), \quad v_{\mathbf{f}}(x,v) := V(t_{\mathbf{f}}(x,v);0;x,v). \end{aligned} \tag{2.1.10}$$

Clearly $(x_{\mathbf{b}}(x,v), v_{\mathbf{b}}(x,v)) \in \gamma_-$ and $(x_{\mathbf{f}}(x,v), v_{\mathbf{f}}(x,v)) \in \gamma_+$.

Lemma 2.1 *For any open subset $\Omega \subset \mathbb{R}^3$, $B \subset \partial\Omega$, and $f \in L^1(\Omega \times \mathbb{R}^3)$,*

$$\begin{aligned}&\iint_{\Omega\times\mathbb{R}^3} |f(x,v)|\mathbf{1}_{x_{\mathbf{b}}(x,v)\in B}\mathbf{1}_{t_{\mathbf{b}}(x,v)\le\frac{1}{m}\ln\frac{1}{\varepsilon}}\mathrm{d}x\mathrm{d}v\\ =&\int_B\int_{n(y)\cdot u<0}\int_0^{\min\{t_{\mathbf{f}}(y,u),\frac{1}{m}\ln\frac{1}{\varepsilon}\}} |f(X(s;0,y,u),V(s;0,y,u))|\\ &\times\{|n(y)\cdot u| + O(\varepsilon)(1+|u|)s\}\mathrm{d}s\mathrm{d}u\mathrm{d}S_y,\end{aligned} \tag{2.1.11}$$

and

$$\begin{aligned}&\iint_{\Omega\times\mathbb{R}^3} |f(x,v)|\mathbf{1}_{x_{\mathbf{f}}(x,v)\in B}\mathbf{1}_{t_{\mathbf{f}}(x,v)\le\frac{1}{m}\ln\frac{1}{\varepsilon}}\mathrm{d}x\mathrm{d}v\\ =&\int_B\int_{n(y)\cdot u>0}\int_{-\min\{t_{\mathbf{b}}(y,u),\frac{1}{m}\ln\frac{1}{\varepsilon}\}}^0 |f(X(s;0,y,u),V(s;0,y,u))|\\ &\times\{|n(y)\cdot u| + O(\varepsilon)(1+|u|)|s|\}\mathrm{d}s\mathrm{d}u\mathrm{d}S_y.\end{aligned} \tag{2.1.12}$$

For the proof we refer to Lemma 2.2 in [21]. From (2.1.7), for $\nabla \in \{\nabla_x, \nabla_v\}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\nabla X\\ \nabla V\end{pmatrix} = \mathbb{A}\begin{pmatrix}\nabla X\\ \nabla V\end{pmatrix},\quad \mathbb{A} = \left(\begin{array}{c|c} 0_{3,3} & I_{3,3}\\ \hline \varepsilon^2\nabla_x\Phi & 0_{3,3}\end{array}\right). \tag{2.1.13}$$

Note $\binom{\nabla X}{\nabla V}|_{s=t} = Id$. Since the matrix $\mathbb{A}$ is bounded, there exists a $C_\Phi > 0$ such that

$$\begin{aligned}&|\partial_{x_j}X_i(s;t,x,v)| \le C_\Phi \mathrm{e}^{C_\Phi|t-s|}, && |\partial_{v_j}X_i(s;t,x,v)| \le C_\Phi|t-s|\mathrm{e}^{C_\Phi|t-s|},\\ &|\partial_{x_j}V_i(s;t,x,v)| \le C_\Phi\varepsilon^2|t-s|\mathrm{e}^{C_\Phi|t-s|}, && |\partial_{v_j}V_i(s;t,x,v)| \le C_\Phi \mathrm{e}^{C_\Phi|t-s|}.\end{aligned} \tag{2.1.14}$$

Next lemma extends the Ukai's Lemma ([17]) to the case with external fields.

Lemma 2.2 *Assume Ω is an open bounded subset of $\mathbb{R}^3$ with $\partial\Omega$ is C^3. We define*

$$\gamma_\pm^\delta := \Big\{(x,v)\in\gamma_\pm : |n(x)\cdot v| > \delta,\quad \delta \le |v| \le \frac{1}{\delta}\Big\}. \tag{2.1.15}$$

Then

$$|f\mathbf{1}_{\gamma_\pm^\delta}|_1 \lesssim_{\delta,\Omega} \|f\|_1 + \|v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f\|_1.$$

For the proof we refer to Lemma 2.3 in [21].

Lemma 2.3 *Let $\Phi \in C^1$. Assume that $f(x,v),\ g(x,v) \in L^2(\Omega\times\mathbb{R}^3)$, $\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}f, \{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}g \in L^2(\Omega\times\mathbb{R}^3)$ and $f_\gamma, g_\gamma \in L^2(\partial\Omega\times\mathbb{R}^3)$. Then*

$$\iint_{\Omega\times\mathbb{R}^3}\{v\cdot\nabla_x f+\varepsilon^2\Phi\cdot\nabla_v f\}g+\{v\cdot\nabla_x g+\varepsilon^2\Phi\cdot\nabla_v g\}f = \int_{\gamma_+} fg - \int_{\gamma_-} fg. \tag{2.1.16}$$

Proof It is easy to check that the proof in Chapter 9 of [17], equation (2.18), still holds in the presence of C^1 field.

2.2 Gain of Integrability: L^3_x Estimate

In this section, we prove the crucial result on the gain of integrability for velocity averages of the solution to the transport equation.

First, we define f_δ which represents either the interior, or the non-grazing parts of f near the boundary.

Definition 2.2 We define, for $(x,v)\in\bar{\Omega}\times\mathbb{R}^3$ and $0<\delta\ll 1$,

$$f_\delta(x,v):=\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)f(x,v), \tag{2.2.1}$$

where $n(x)$ is defined in (2.1.5).

At the boundary $(x,v)\in\gamma=\partial\Omega\times\mathbb{R}^3$,

$$f_\delta(x,v)|_\gamma\equiv 0, \quad \text{for } |n(x)\cdot v|\le\delta \quad\text{or}\quad |v|\ge\frac{1}{\delta}. \tag{2.2.2}$$

The main goal of this section is the following:

Proposition 2.1 *Assume $\Phi=\Phi(x)\in C^1$. Let $f(x,v)$ solve* (2.1.6) *in the sense of distribution and $f(x,v)|_\gamma=f_\gamma(x,v)\in L^2(\gamma)$. Then*

$$\begin{aligned}
&|a(x)|+|b(x)|+|c(x)| \ \le\ \mathbf{S}_1f(x)+\mathbf{S}_2f(x),\\
&\mathbf{S}_1f(x):=4\int_{\mathbb{R}^3}|f_\delta(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v,\\
&\mathbf{S}_2f(x):=4\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v,
\end{aligned} \tag{2.2.3}$$

where f_δ is defined in (2.2.1) *and (a,b,c) in* (1.2.4).

Moreover,

$$\begin{aligned}
&\|\mathbf{S}_1f\|_{L^3(\Omega)}\lesssim\|w^{-1}f\|_{L^2(\Omega\times\mathbb{R}^3)}+\|w^{-1}g\|_{L^2(\Omega\times\mathbb{R}^3)}+\|f\|_{L^2(\gamma)},\\
&\|\mathbf{S}_2f\|_{L^3(\Omega)}\lesssim\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)},
\end{aligned} \tag{2.2.4}$$

for $w(v)=\mathrm{e}^{\beta|v|^2}$ with $0<\beta\ll 1$.

Let $\tilde{C}:=\frac{1}{10(1+\|\xi\|_{C^2})}$ and

$$\Omega_{\tilde{C}\delta^4}:=\big\{x\in\mathbb{R}^3:\ \xi(x)<\tilde{C}\delta^4\big\}. \tag{2.2.5}$$

We define, for $(x,v)\in\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}$, with $\bar{\Omega}=\Omega\cup\partial\Omega$,

$$\begin{aligned}
&t^*_{\mathbf{b}}(x,v):=\inf\{s>0:0<\xi(X(s;0,x,v))<\tilde{C}\delta^4 \text{ for all } 0<\tau<s\},\\
&t^*_{\mathbf{f}}(x,v):=t^*_{\mathbf{b}}(x,-v),\\
&(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v)):=(X(-t^*_{\mathbf{b}}(x,v);0,x,v),V(-t^*_{\mathbf{b}}(x,v);0,x,v)),\\
&(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v)):=(X(t^*_{\mathbf{f}}(x,v);0,x,v),V(t^*_{\mathbf{f}}(x,v);0,x,v)).
\end{aligned} \tag{2.2.6}$$

Lemma 2.4 *Let $f \in L^2(\Omega \times \mathbb{R}^3)$ solve* (2.1.6) *in the sense of distribution and $g \in L^2(\Omega \times \mathbb{R}^3)$, and $f(x,v)|_\gamma = f_\gamma(x,v) \in L^2(\gamma)$. Then there exists an $\bar{f}(x,v) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, such that $\bar{f}|_{\bar{\Omega}\times\mathbb{R}^3} \equiv f_\delta$. Moreover, in the sense of distributions,*

$$\{v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v\}\bar{f} = h \equiv h_1 + h_2 + h_3 + h_4, \tag{2.2.7}$$

where

$$\begin{aligned}
h_1(x,v) &= \mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3} g\Big[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|),\\
h_2(x,v) &= \mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3} f\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}\Big\{\Big[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big\},\\
h_3(x,v) &= \mathbf{1}_{(x,v)\in[\Omega_{\tilde{C}\delta^4}\setminus\bar{\Omega}]\times\mathbb{R}^3}\ \frac{1}{\tilde{C}\delta^4} v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\\
&\qquad\times\big[f_\delta(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} + f_\delta(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big],\\
h_4(x,v) &= \mathbf{1}_{(x,v)\in[\Omega_{\tilde{C}\delta^4}\setminus\bar{\Omega}]\times\mathbb{R}^3}\Big[f_\delta(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&\qquad+ f_\delta(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\|h_1\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|g\|_{L^2(\Omega\times\mathbb{R}^3)},\\
&\|h_2\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|f\|_{L^2(\Omega\times\mathbb{R}^3)},\\
&\|h_3\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} + \|h_4\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|f_\delta\|_{L^2(\gamma)}.
\end{aligned}\tag{2.2.8}$$

Proof Step 1 Consider f_δ in (2.2.1). In the sense of distributions on $\Omega \times \mathbb{R}^3$,

$$\begin{aligned}
v\cdot\nabla_x f_\delta + \varepsilon^2\Phi\cdot\nabla_v f_\delta &= \Big[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)g + f\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}\\
&\qquad\times\Big\{\Big[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big\}.
\end{aligned}\tag{2.2.9}$$

Note that,

$$\begin{aligned}
&\Big|\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}\Big[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big|\\
&= \Big| -\frac{1}{\delta}\{v\cdot\nabla_x n(x)\cdot v + \varepsilon^2\Phi\cdot n(x)\}\chi'\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\chi(\delta|v|)\\
&\quad - \frac{1}{\delta} v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\delta}\Big)\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi(\delta|v|)\\
&\quad + \varepsilon^2\delta\Phi\cdot\frac{v}{|v|}\chi'(\delta|v|)\Big[1 - \chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\Big|\\
&\le \frac{4}{\delta}(|v|^2\|\xi\|_{C^2} + \varepsilon^2\|\Phi\|_\infty)\chi(\delta|v|) + \frac{C_\Omega}{\delta}|v|\chi(\delta|v|) + \varepsilon^2\delta\|\Phi\|_\infty\mathbf{1}_{|v|\le 2\delta^{-1}}\\
&\lesssim \delta^{-3}\mathbf{1}_{|v|\le 2\delta^{-1}}.
\end{aligned}\tag{2.2.10}$$

This proves the second line of (2.2.8). Since

$$\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\le 1,$$

we prove the first line of (2.2.8) directly. The third line of (2.2.8) will be proved in Step 5.

Step 2 We claim that for $0\le\xi(x)\le\tilde C_1\delta^4$ and $|v|\le\frac{1}{\delta}$, if $n(x)\cdot v>\delta$ then $\xi(x^*_{\mathbf f}(x,v))=\tilde C_1\delta^4$; and if $n(x)\cdot v<-\delta$ then $\xi(x^*_{\mathbf b}(x,v))=\tilde C_1\delta^4$.

If $v\cdot n(x)\ge\delta$, we take $s>0$, while if $v\cdot n(x)\le-\delta$ then we take $s<0$. From (2.1.8),

$$\begin{aligned}\xi(X(s;0,x,v))&=\xi(x)+\int_0^s V(\tau;0,x,v)\cdot\nabla_x\xi(X(\tau;0,x,v))\mathrm{d}\tau\\ =\xi(x)&+\int_0^s\{v+O(1)\varepsilon^2\|\Phi\|_\infty\tau\}\cdot\{\nabla_x\xi(x)+O(1)\|\xi\|_{C^2}(|v|+\varepsilon^2\|\Phi\|_\infty\tau)\tau\}\\ =\xi(x)&+v\cdot\nabla_x\xi(x)s+O(1)\|\xi\|_{C^2}\big\{|v|^2s^2+\varepsilon^2\|\Phi\|_\infty s^2+\varepsilon^2\|\Phi\|_\infty|v|s^3+\varepsilon^4\|\Phi\|^2_\infty s^4\big\}.\end{aligned}$$

From $\xi(x)\ge0$,

$$\begin{aligned}&\xi(X(s;0,x,v))\\ \ge{}&\delta|s|\Big\{1-\frac{\|\xi\|_{C^2}}{\delta}\big[|v|^2|s|+\varepsilon^2\|\Phi\|_\infty|s|+\varepsilon^2\|\Phi\|_\infty|v||s|^2+\varepsilon^4\|\Phi\|^2_\infty|s|^3\big]\Big\}\\ \ge{}&\delta|s|\Big\{1-\frac{\|\xi\|_{C^2}}{\delta}\Big[\frac{1}{\delta^2}|s|+\varepsilon^2\|\Phi\|_\infty|s|+\varepsilon^2\|\Phi\|_\infty\frac{1}{\delta}|s|^2+\varepsilon^4\|\Phi\|^2_\infty|s|^3\Big]\Big\}\\ \ge{}&\delta|s|\Big\{1-\Big[\frac14+\frac{\varepsilon^2\delta^2\|\Phi\|_\infty}{4}+\frac{\varepsilon^2\delta^4\|\Phi\|_\infty}{16}+\frac{\varepsilon^4\delta^8\|\Phi\|^2_\infty}{64}\Big]\Big\}\ge\frac{\delta|s|}{2},\end{aligned}\tag{2.2.11}$$

for $0\le|s|\le\frac{\delta^3}{4(1+\|\xi\|_{C^2})}$ and $0<\varepsilon\ll1$. Then we choose $s_*=+\frac{\delta^3}{4(1+\|\xi\|_{C^2})}$ for $n(x)\cdot v>\delta$ and $s_*=-\frac{\delta^3}{4(1+\|\xi\|_{C^2})}$ for $n(x)\cdot v<-\delta$, to have

$$\xi(X(s_*;0,y,v))\ge\frac{\delta^4}{8(1+\|\xi\|_{C^2})}>\tilde C_1\delta^4.$$

By the intermediate value theorem, we prove our claim.

Step 3 We define $f_E(x,v)$ for $(x,v)\in[\mathbb{R}^3\backslash\bar\Omega]\times\mathbb{R}^3$:

$$f_E(x,v):=\begin{cases}f_\delta(x^*_{\mathbf b}(x,v),v^*_{\mathbf b}(x,v))\chi\Big(\dfrac{\xi(x)}{\tilde C\delta^4}\Big)\chi(t^*_{\mathbf b}(x,v)), & \text{if } x^*_{\mathbf b}(x,v)\in\partial\Omega,\\ f_\delta(x^*_{\mathbf f}(x,v),v^*_{\mathbf f}(x,v))\chi\Big(\dfrac{\xi(x)}{\tilde C\delta^4}\Big)\chi(t^*_{\mathbf f}(x,v)), & \text{if } x^*_{\mathbf f}(x,v)\in\partial\Omega,\\ 0, & \text{if } x^*_{\mathbf b}(x,v)\notin\partial\Omega \text{ and } x^*_{\mathbf f}(x,v)\notin\partial\Omega.\end{cases}\tag{2.2.12}$$

We check that f_E is well-defined. It suffices to prove the following:

If $x^*_{\mathbf{b}}(x,v)\in\partial\Omega$ and $x^*_{\mathbf{f}}(x,v)\in\partial\Omega$, then $f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\chi(t^*_{\mathbf{b}}(x,v))=0=f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\chi(t^*_{\mathbf{f}}(x,v))$. If $|n(x^*_{\mathbf{b}}(x,v))\cdot v^*_{\mathbf{b}}(x,v)|\le\delta$ or $|v^*_{\mathbf{b}}(x,v)|\ge\frac{1}{\delta}$, then $f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))=0$ due to (2.2.2). If $n(x^*_{\mathbf{b}}(x,v))\cdot v^*_{\mathbf{b}}(x,v)>\delta$ and $|v^*_{\mathbf{b}}(x,v)|\le\frac{1}{\delta}$, then, by Step 2, $\xi(x^*_{\mathbf{f}}(x,v))=\xi(x^*_{\mathbf{f}}(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v)))=\tilde{C}\delta^4$ so that $x^*_{\mathbf{f}}(x,v)\notin\partial\Omega$.

On the other hand, if $|n(x^*_{\mathbf{f}}(x,v))\cdot v^*_{\mathbf{f}}(x,v)|\le\delta$ or $|v^*_{\mathbf{f}}(x,v)|\ge\frac{1}{\delta}$, then $f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))=0$ due to (2.2.2). If $n(x^*_{\mathbf{f}}(x,v))\cdot v^*_{\mathbf{f}}(x,v)<-\delta$ and $|v^*_{\mathbf{f}}(x,v)|\le\frac{1}{\delta}$, then by Step 2, $\xi(x^*_{\mathbf{b}}(x,v))=\xi(x^*_{\mathbf{b}}(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v)))=\tilde{C}\delta^4$ so that $x^*_{\mathbf{b}}(x,v)\notin\partial\Omega$.

Note that

$$f_E(x,v)=f_\delta(x,v)\quad\text{for all } x\in\partial\Omega. \tag{2.2.13}$$

If $x\in\partial\Omega$ and $n(x)\cdot v>\delta$, then $(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))=(x,v)$. From the definition (2.2.12), for those (x,v), we have $f_E(x,v)=f_\delta(x,v)$. If $x\in\partial\Omega$ and $n(x)\cdot v<-\delta$, then $(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x.v))=(x,v)$. From the definition (2.2.12), we conclude (2.2.13) again. Otherwise, if $-\delta<n(x)\cdot v<\delta$, then $f_E|_{\partial\Omega}\equiv0\equiv f_\delta|_{\partial\Omega}$.

Step 4 We claim that $f_E(x,v)\in L^2([\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3)$.

From the definition (2.2.12), we have $f_E(x,v)\equiv0$ if $x^*_{\mathbf{b}}(x,v)\notin\partial\Omega$ and $x^*_{\mathbf{f}}(x,v)\notin\partial\Omega$. Therefore we can decompose the following integration as

$$\begin{aligned}
&\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}|f_E(x,v)|^2\mathrm{d}x\mathrm{d}v\\
=&\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}|f_E(x,v)|^2\mathrm{d}x\mathrm{d}v+\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}|f_E(x,v)|^2\mathrm{d}x\mathrm{d}v\\
=&\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}|f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))|^2\Big|\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big|^2|\chi(t^*_{\mathbf{b}}(x,v))|^2\mathrm{d}x\mathrm{d}v \qquad (2.2.14)\\
&+\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}|f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))|^2\Big|\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big|^2|\chi(t^*_{\mathbf{f}}(x,v))|^2\mathrm{d}x\mathrm{d}v, \qquad (2.2.15)
\end{aligned}$$

where we have used (2.2.12).

From (2.1.11),

$$\begin{aligned}
&(2.2.14)\\
&\le\int_{\partial\Omega}\int_{n(y)\cdot u>0}\int_0^{\min\{t^*_{\mathbf{f}}(y,u),1\}}\big|f_\delta(x^*_{\mathbf{b}}(X(s;0,y,u),V(s;0,y,u)),\\
&\quad v^*_{\mathbf{b}}(X(s;0,y,u),V(s;0,y,u)))\big|^2\{|n(y)\cdot u|+O(\varepsilon)(1+|u|)s\}\mathrm{d}s\mathrm{d}u\mathrm{d}S_y\\
&\le\int_{\partial\Omega}\int_{n(y)\cdot u>0}\int_0^1\big|f_\delta(x_b,u)\big|^2\{|n(y)\cdot u|+O(\varepsilon)(1+|u|)s\}\mathrm{d}s\mathrm{d}u\mathrm{d}S_y\\
&\lesssim\int_{\partial\Omega}\int_{n(y)\cdot u>0}\big|f_\delta(y,u)\big|^2|n(y)\cdot u|\mathrm{d}u\mathrm{d}S_y\lesssim\|f_\delta\|^2_{L^2(\partial\Omega\times\mathbb{R}^3)}\le\|f\|^2_{L^2(\partial\Omega\times\mathbb{R}^3)},
\end{aligned}$$

where we have used the fact, from (2.2.1),

$$O(\varepsilon)(1+|u|)|s| \leq O(\varepsilon)\Big(1+\frac{1}{\delta}\Big) \lesssim \delta \lesssim |n(y)\cdot u| \quad \text{for } (y,u)\in \operatorname{supp}(f_\delta),$$

and the fact

$$(x_{\mathbf{b}}^*(X(s;0,y,u),V(s;0,y,u)),v_{\mathbf{b}}^*(X(s;0,y,u),V(s;0,y,u)))=(x_{\mathbf{b}}^*(y,u),v_{\mathbf{b}}^*(y,u))=(y,u),$$

for $n(y)\cdot u>0$, $y\in\partial\Omega$, and $0\leq s\leq t_{\mathbf{f}}^*(y,u)$.

Similarly we can show that (2.2.15) $\lesssim \|f_\delta\|^2_{L^2(\partial\Omega\times\mathbb{R}^3)} \lesssim \|f\|^2_{L^2(\partial\Omega\times\mathbb{R}^3)}$.

Step 5 We show that, in the sense of distributions on $[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3$,

$$\begin{aligned}
&v\cdot\nabla_x f_E+\varepsilon^2\Phi\cdot\nabla_v f_E\\
=\ &\frac{1}{\tilde{C}\delta^4}v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\big[f_\delta(x_{\mathbf{b}}^*(x,v),v_{\mathbf{b}}^*(x,v))\chi(t_{\mathbf{b}}^*(x,v))\mathbf{1}_{x_{\mathbf{b}}^*(x,v)\in\partial\Omega}\\
&+f_\delta(x_{\mathbf{f}}^*(x,v),v_{\mathbf{f}}^*(x,v))\chi(t_{\mathbf{f}}^*(x,v))\mathbf{1}_{x_{\mathbf{f}}^*(x,v)\in\partial\Omega}\big]\\
&+f_\delta(x_{\mathbf{b}}^*(x,v),v_{\mathbf{b}}^*(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t_{\mathbf{b}}^*(x,v))\mathbf{1}_{x_{\mathbf{b}}^*(x,v)\in\partial\Omega}\\
&-f_\delta(x_{\mathbf{f}}^*(x,v),v_{\mathbf{f}}^*(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t_{\mathbf{f}}^*(x,v))\mathbf{1}_{x_{\mathbf{f}}^*(x,v)\in\partial\Omega}.
\end{aligned}\tag{2.2.16}$$

For $\phi\in C_c^\infty([\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3)$, we choose small $t>0$ such that $X(s;0,x,v)\in\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}$ for all $|s|\leq t$ and all $(x,v)\in\operatorname{supp}(\phi)$. Then, from (2.2.12), for $(X(s),V(s))=(X(s;0,x,v),V(s;0,x,v))$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}f_E(X(s),V(s))\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\big[f_\delta(x_{\mathbf{b}}^*(X(s),V(s)),v_{\mathbf{b}}^*(X(s),V(s)))\chi(t_{\mathbf{b}}^*(X(s),V(s)))\mathbf{1}_{x_{\mathbf{b}}^*(X(s),V(s))\in\partial\Omega}\\
&+f_\delta(x_{\mathbf{f}}^*(X(s),V(s)),v_{\mathbf{f}}^*(X(s),V(s)))\chi(t_{\mathbf{f}}^*(X(s),V(s)))\mathbf{1}_{x_{\mathbf{f}}^*(X(s),V(s))\in\partial\Omega}\big]\chi\Big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\Big)\\
&+\big[f_\delta(x_{\mathbf{b}}^*(X(s),V(s)),v_{\mathbf{b}}^*(X(s),V(s)))\chi(t_{\mathbf{b}}^*(X(s),V(s)))\mathbf{1}_{x_{\mathbf{b}}^*(X(s),V(s))\in\partial\Omega}\\
&+f_\delta(x_{\mathbf{f}}^*(X(s),V(s)),v_{\mathbf{f}}^*(X(s),V(s)))\chi(t_{\mathbf{f}}^*(X(s),V(s)))\mathbf{1}_{x_{\mathbf{f}}^*(X(s),V(s))\in\partial\Omega}\big]\frac{\mathrm{d}}{\mathrm{d}s}\chi\Big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\Big).
\end{aligned}$$

From

$$\begin{aligned}
&(x_{\mathbf{b}}^*(X(s;0,x,v),V(s;0,x,v)),v_{\mathbf{b}}^*(X(s;0,x,v),V(s;0,x,v)))=(x_{\mathbf{b}}^*(x,v),v_{\mathbf{b}}^*(x,v)),\\
&(x_{\mathbf{f}}^*(X(s;0,x,v),V(s;0,x,v)),v_{\mathbf{f}}^*(X(s;0,x,v),V(s;0,x,v)))=(x_{\mathbf{f}}^*(x,v),v_{\mathbf{f}}^*(x,v)),\\
&t_{\mathbf{f}}^*(X(s;0,x,v),V(s;0,x,v))=t_{\mathbf{f}}^*(x,v)-s \quad\text{and}\\
&t_{\mathbf{b}}^*(X(s;0,x,v),V(s;0,x,v))=t_{\mathbf{b}}^*(x,v)+s,
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} f_E(X(s),V(s)) = &\big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi'(t^*_{\mathbf{b}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&-f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi'(t^*_{\mathbf{f}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big]\chi\Big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\Big)\\
&+\big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&+f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big]\\
&\times\frac{1}{\tilde{C}\delta^4}V(s)\cdot\nabla_x\xi(X(s))\chi'\Big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\Big). \qquad (2.2.17)
\end{aligned}
$$

By the change of variables $(x,v)\mapsto(X(s;0,x,v),V(s;0,x,v))$, for sufficiently small s,

$$
\begin{aligned}
&-\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E(x,v)\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(x,v)\mathrm{d}x\mathrm{d}v\\
=&-\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E(X(s),V(s))\{V(s)\cdot\nabla_X+\varepsilon^2\Phi\cdot\nabla_V\}\phi(X(s),V(s))\mathrm{d}x\mathrm{d}v\\
=&-\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E(X(s),V(s))\frac{\mathrm{d}}{\mathrm{d}s}\phi(X(s),V(s))\mathrm{d}x\mathrm{d}v. \qquad (2.2.18)
\end{aligned}
$$

Since the change of variables $(x,v)\mapsto(X(s;0,x,v),V(s;0,x,v))$ has unit Jacobian, it follows that, for s sufficiently small,

$$
\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E((X(s),V(s))\phi(X(s),V(s)))=\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E(x,v)\phi(x,v),
$$

and hence

$$
\frac{\mathrm{d}}{\mathrm{d}s}\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E((X(s),V(s))\phi(X(s),V(s))=0.
$$

Therefore we can move the s-derivative on f_E: By (2.2.17),

$$
\begin{aligned}
&(2.2.18)\\
=&\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}s}f_E(X(s),V(s))\phi(X(s),V(s))\mathrm{d}x\mathrm{d}v\\
=&\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}\big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi'(t^*_{\mathbf{b}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&-f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi'(t^*_{\mathbf{f}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big]\chi\Big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\Big)\phi(X(s),V(s))\\
&+\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}\big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&+f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big]\\
&\times\frac{1}{\tilde{C}\delta^4}V(s)\cdot\nabla_x\xi(X(s))\chi'\Big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\Big)\phi(X(s),V(s)).
\end{aligned}
$$

From the change of variable $(X(s;0,x,v),V(s;0,x,v))\mapsto(x,v)$,

$$\begin{aligned}(2.2.18)=&\iint_{[\Omega_{\tilde C\delta^4}\backslash\bar\Omega]\times\mathbb{R}^3}\big[f_\delta(x^*_{\mathbf b}(x,v),v^*_{\mathbf b}(x,v))\chi'(t^*_{\mathbf b}(x,v))\mathbf{1}_{x^*_{\mathbf b}(x,v)\in\partial\Omega}\\&-f_\delta(x^*_{\mathbf f}(x,v),v^*_{\mathbf f}(x,v))\chi'(t^*_{\mathbf f}(x,v))\mathbf{1}_{x^*_{\mathbf f}(x,v)\in\partial\Omega}\big]\chi\Big(\frac{\xi(x)}{\tilde C\delta^4}\Big)\phi(x,v)\\&+\iint_{[\Omega_{\tilde C\delta^4}\backslash\bar\Omega]\times\mathbb{R}^3}\big[f_\delta(x^*_{\mathbf b}(x,v),v^*_{\mathbf b}(x,v))\chi(t^*_{\mathbf b}(x,v))\mathbf{1}_{x^*_{\mathbf b}(x,v)\in\partial\Omega}\\&+f_\delta(x^*_{\mathbf f}(x,v),v^*_{\mathbf f}(x,v))\chi(t^*_{\mathbf f}(x,v))\mathbf{1}_{x^*_{\mathbf f}(x,v)\in\partial\Omega}\big]\frac{1}{\tilde C\delta^4}v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde C\delta^4}\Big)\phi(x,v).\end{aligned}$$

Hence (2.2.16) is proved.

On the other hand, following the bounds of (2.2.14) and (2.2.15) in Step 4 we prove the third line of (2.2.8).

Step 6 We define $\bar f(x,v)$ for $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$:

$$\bar f(x,v):=f_\delta(x,v)\mathbf{1}_{(x,v)\in\bar\Omega\times\mathbb{R}^3}+f_E(x,v)\mathbf{1}_{(x,v)\in[\mathbb{R}^3\backslash\bar\Omega]\times\mathbb{R}^3}. \tag{2.2.19}$$

For $\phi\in C_c^\infty(\mathbb{R}^3\times\mathbb{R}^3)$, from the Green's identity (Lemma 2.3),

$$\begin{aligned}&-\iint_{\mathbb{R}^3\times\mathbb{R}^3}\bar f\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi\\=&-\iint_{\Omega\times\mathbb{R}^3}f_\delta\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi-\iint_{[\mathbb{R}^3\backslash\bar\Omega]\times\mathbb{R}^3}f_E\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi\\=&\int_\gamma f_\delta\phi+\int_\gamma f_E\phi+\iint_{\Omega\times\mathbb{R}^3}\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f_\delta\phi\\&+\iint_{[\Omega_{\tilde C\delta^4}\backslash\bar\Omega]\times\mathbb{R}^3}\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f_E\phi.\end{aligned}$$

From (2.2.13), the boundary contributions are cancelled:

$$\int_\gamma f_\delta(x,v)\phi(x,v)\mathrm{d}\gamma-\int_\gamma f_E(x,v)\phi(x,v)\mathrm{d}\gamma=0.$$

Further using (2.2.9) and (2.2.16), we prove that $\bar f$ solves (2.2.7) in the sense of distributions on $\mathbb{R}^3\times\mathbb{R}^3$. The proof is completed.

For $(x,v)\in\operatorname{supp}(\bar f)$, we can choose a fixed $T>0$ such that

$$X(T;0,x,v)\notin\operatorname{supp}(\bar f)\ \text{ and }\ X(T;0,x,v)\notin\operatorname{supp}(h), \tag{2.2.20}$$

so that

$$\bar f(X(T;0,x,v),V(T;0,x,v))=0.$$

Directly,

$$|X(T;0,x,v)-x|=|vT+O(\varepsilon^2)\|\Phi\|_\infty T^2|\geq\delta T-O(\varepsilon^2)\|\Phi\|_\infty T^2.$$

We choose $T = \frac{C}{\delta}$ for large but fixed $C \gg 1$ such that $|X(T;0,x,v) - x| \geq C - O(\frac{\varepsilon^2}{\delta^2}) \geq \frac{C}{2} \gg 1$. This proves our claim (2.2.20).

With such $T > 0$ in (2.2.20),

$$\bar{f}(x,v) = -\int_0^T h(X(s;0,x,v), V(s;0,x,v))\mathrm{d}s.$$

Note that, from (2.2.12),

$$f_E(x,v) \equiv 0, \quad \text{for} \quad \xi(x) > 2\tilde{C}\delta^4 \quad \text{or} \quad |v| > 2\delta^{-1} \quad \text{or} \quad |v| < \delta/2. \tag{2.2.21}$$

Therefore,

$$|\bar{f}(x,v)| \leq \int_0^T \mathbf{1}_{\frac{\delta}{2} \leq |v| \leq \frac{2}{\delta}} |h(X(s;0,x,v), V(s;0,x,v))|\mathrm{d}s. \tag{2.2.22}$$

Definition 2.3 For fixed T in (2.2.20) and $\delta > 0$ and a smooth function $\phi \in L^1(\mathbb{R}^3)$, we define the average operator S as

$$Sh(x) := S(h)(x) := \int_0^T \int_{\frac{\delta}{2} \leq |v| \leq \frac{2}{\delta}} h(X(s;0,x,v), V(s;0,x,v))\phi(v)\mathrm{d}v\mathrm{d}s. \tag{2.2.23}$$

Lemma 2.5 *Assume that $\phi \in C^1(\mathbb{R}^3)$ is such that $|\phi(v)| \leq \bar{\phi}(|v|)$ with $\bar{\phi} \in C^1(\mathbb{R})$ where $\bar{\phi}'$ decays exponentially. Then*

$$\|Sh\|_{L^3(\Omega)} \lesssim_\phi \|w^{-1}h\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{2.2.24}$$

Proof We only prove (2.2.24) in the case of $\beta = 0$ since for sufficiently small $0 < \beta \ll 1$, we can always absorb w growth by ϕ, by using $|V(s;0,x,v)| \leq |v| + \varepsilon^2 T\|\Phi\|_\infty$.

We define the dual operator:

$$S^*(g)(x,v) := \int_0^T \mathbf{1}_{\frac{\delta}{2} \leq |V(-s;0,x,v)| \leq \frac{2}{\delta}} g(X(-s;0,x,v))\phi(V(-s;0,x,v))\mathrm{d}s. \tag{2.2.25}$$

By a change of variable $(X(s;0;x,v), V(s;0;x,v)) \mapsto (x,v)$,

$$\begin{aligned}(Sh,g) &= \int_0^T \int_{\mathbb{R}^3} \int_{\frac{\delta}{2} \leq |v| \leq \frac{2}{\delta}} h(X(s;0;x,v), V(s;0;x,v))\phi(v)g(x)\mathrm{d}v\mathrm{d}x\mathrm{d}s \\ &= \int_0^T \int_{\mathbb{R}^3} \int_{\frac{\delta}{2} \leq |V(-s;0;x,v)| \leq \frac{2}{\delta}} h(x,v)\phi(V(-s;0;x,v))g(X(-s;0;x,v))\mathrm{d}v\mathrm{d}x\mathrm{d}s \\ &= (h, S^*g).\end{aligned}$$

Note that, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|Sh\|_{L^p_x} \equiv \sup_{\|g\|_{L^{p'}_x} \leq 1} (Sh,g)_{L^2_x} = \sup_{\|g\|_{L^{p'}_x} \leq 1} (h, S^*g)_{L^2_{x,v}} \leq \|h\|_{L^2_{x,v}} \sup_{\|g\|_{L^{p'}_x} \leq 1} \|S^*g\|_{L^2_{x,v}}.$$

Therefore, in order to show $\|Sh\|_{L^p_x} \lesssim \|h\|_{L^2_{x,v}}$, it suffices to prove $\|S^*g\|_{L^2_{x,v}} \lesssim \|g\|_{L^{p'}_x}$. But

$$\|S^*g\|^2_{L^2_{x,v}} = (S^*g, S^*g)_{L^2_{x,v}} = (SS^*g, g)_{L^2_x} \le \|SS^*g\|_{L^p_x}\|g\|_{L^{p'}_x}.$$

Hence we only need to show

$$\|SS^*g\|_{L^p(\Omega)} \lesssim \|g\|_{L^{p'}(\Omega)}. \tag{2.2.26}$$

Here, the explicit form of SS^* can be written from (2.2.23) and (2.2.25):

$$\begin{aligned} SS^*(g)(x) = &\int_0^T\int_0^T\int_{\mathbb{R}^3} \mathbf{1}_{\frac{\delta}{2}\le|V(-s;0;x,v)|\le\frac{2}{\delta}}\mathbf{1}_{\frac{\delta}{2}\le|v|\le\frac{2}{\delta}} \\ &\times g(X(-s+\tau;0;x,v))\phi(V(-s+\tau;0;x,v))\phi(v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{2.2.27}$$

First we consider the following change of variables:

$$v \mapsto y := X(-s+\tau;0;x,v). \tag{2.2.28}$$

By (2.1.14),

$$\partial_{v_j}X_i(-s+\tau;0;x,v) = (-s+\tau)\delta_{ij} + O(\varepsilon^2)|s-\tau|\mathrm{e}^{CT} = (-s+\tau)(\delta_{ij}+O(\varepsilon^2)),$$

and the Jacobian is

$$\det\nabla_v X(-s+\tau;x,v) = (1+O(\varepsilon^2))(-s+\tau)^3 \gtrsim (-s+\tau)^3.$$

Therefore we have

$$\mathrm{d}v \lesssim \frac{1}{|s-\tau|^3}\mathrm{d}y.$$

We apply this change of variables to bound

$$\begin{aligned} |SS^*(g)(x)| \lesssim &\int_0^T\int_0^T\int_{\mathbb{R}^3} \mathbf{1}_{\frac{\delta}{2}\le|V(-s;0;x,v(y))|\le\frac{2}{\delta}}\mathbf{1}_{\frac{\delta}{2}\le|v(y)|\le\frac{2}{\delta}}\frac{1}{|s-\tau|^3} \\ &\times|g(y)|\,|\phi(V(-s+\tau;0;x,v(y)))|\,|\phi(v(y))|\mathrm{d}y\mathrm{d}s\mathrm{d}\tau. \end{aligned}$$

For $|v| \ge \frac{\delta}{2}$,

$$|V(-s+\tau;x,v)| = |v| + O(\varepsilon^2)T^2\|\Phi\|_\infty = |v| + O(\varepsilon^2) = O(1)|v|.$$

Let $y := X(-s+\tau;s,x)$. Then

$$\begin{aligned} |y-x| &= |X(-s+\tau;x,v)-x| = |v||s-\tau| + O(\varepsilon^2)|s-\tau|^2, \\ |y-x| &= |V(-s+\tau;x,v)\|s-\tau| + O(\varepsilon^2)|s-\tau|^2, \end{aligned}$$

and

$$\begin{aligned} \frac{|y-x|}{|s-\tau|} &= |v| + O(\varepsilon^2)|s-\tau| = O(1)|v|, \\ \frac{|y-x|}{|s-\tau|} &= |V(-s+\tau;x,v)| + O(\varepsilon^2)|s-\tau| = O(1)|V(-s+\tau;x,v)|. \end{aligned}$$

From the monotonicity of ϕ, there exists a $C>0$ such that

$$|\phi(V(-s+\tau;0,x,v(y)))\phi(v(y))| \le \Big|\phi\Big(C\frac{|y-x|}{|s-\tau|}\Big)\Big|^2.$$

We define

$$M(x) := \int_0^T\int_0^T \frac{1}{|s-\tau|^3}\Big|\phi\Big(C\frac{|x|}{|s-\tau|}\Big)\Big|^2 \mathrm{d}s\mathrm{d}\tau. \tag{2.2.29}$$

Then

$$|SS^*(g)(x)| \lesssim \int_{\mathbb{R}^3} M(x-y)|g(y)|\mathrm{d}y. \tag{2.2.30}$$

Now we claim that

$$M \in L^{3/2}_w(\mathbb{R}^3), \quad \text{i.e.} \quad M(y-x) \lesssim \frac{1}{|y-x|^2}. \tag{2.2.31}$$

We use the change of variables $(s,\tau)\mapsto(s,t)$ with $t=|s-\tau|$ to have

$$M(x-y) \lesssim \int_0^T \frac{1}{t^3}\phi^2\Big(C\frac{|y-x|}{t}\Big)\mathrm{d}t.$$

Then letting $w=\frac{|y-x|}{t}$ so that $\mathrm{d}w=\frac{|y-x|}{t^2}\mathrm{d}t$ and $\mathrm{d}t=\frac{|y-x|}{w^2}\mathrm{d}w$,

$$M(x-y)\lesssim\int_0^T\frac{1}{t^3}\phi^2\Big(C\frac{|y-x|}{t}\Big)\mathrm{d}t\lesssim\int_0^\infty\frac{w}{|y-x|^2}\phi^2(Cw)\mathrm{d}w\lesssim\frac{1}{|y-x|^2}\int_0^\infty w\phi^2(Cw)\mathrm{d}w\lesssim\frac{1}{|y-x|^2}.$$

This proves (2.2.31). Then, by the weak Young's equality (see for example [46] page 106) and (2.2.31), we conclude that $SS^*: L^q \to L^{q'}$, with $1+\frac{1}{q'}=\frac{1}{q}+\frac{2}{3}$, and hence $q'=3$. This proves our claim (2.2.26). The proof is completed.

Now we are ready to prove the main result of this section:

Proof of Proposition 2.1 First we note

$$1-\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|) \le \mathbf{1}_{|v|\ge\frac{1}{\delta}}+\mathbf{1}_{|v|\le\frac{1}{\delta},\mathrm{dist}(x,\partial\Omega)<\frac{\delta}{2},|n(x)\cdot v|<\delta}. \tag{2.2.32}$$

For simplicity, we denote $[a_0,a_1,a_2,a_3,a_4]=[a,b_1,b_2,b_3,c]$. Then $a_i := a_i(f) := (f,\zeta_i)_2$ with

$$[\zeta_0(v),\zeta_1(v),\zeta_2(v),\zeta_3(v),\zeta_4(v)] := \Big[\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, \frac{2}{3}\frac{|v|^2-3}{2}\sqrt{\mu}\Big]. \tag{2.2.33}$$

Then from (2.2.1),

$$\begin{aligned}
&\int_{\mathbb{R}^3} f_\delta(x,v)\zeta_i(v)\mathrm{d}v\\
&=\int_{\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)f(x,v)\zeta_i(v)\mathrm{d}v\\
&=\int_{\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big\{\sum_{j=0}^4 a_j(x)\zeta_j(v)+(\mathbf{I}-\mathbf{P})f(x,v)\Big\}\zeta_i(v)\mathrm{d}v\\
&=a_i(x)+O(\delta)\sum_{j=0}^4|a_j(x)|+O_\delta(1)\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(x,v)|\zeta_i(v)\mathrm{d}v,
\end{aligned}$$

where we have used the fact that, for $i \neq j$,

$$\int_{\mathbb{R}^3} \zeta_i(v)\zeta_j(v)\mathbf{1}_{|v|\le 2\delta}\mathrm{d}v \to \int_{\mathbb{R}^3} \zeta_i(v)\zeta_j(v)\mathrm{d}v = 0, \quad \text{as} \quad \delta \downarrow 0. \tag{2.2.34}$$

Therefore

$$\sum_{i=0}^{4} |a_i(x)| \le \sum_{i=0}^{4} \left| \int_{\mathbb{R}^3} f_\delta(x,v)\zeta_i(v)\mathrm{d}v \right| + O(\delta)\sum_{j=0}^{4} |a_j(x)|$$
$$+O_\delta(1)\int_{\mathbb{R}^3} |(\mathbf{I}-\mathbf{P})f(x,v)| \sum_{i=0}^{4} |\zeta_i(v)|\mathrm{d}v.$$

Hence,

$$|a(x)|+|b(x)|+|c(x)| \le 4\int_{\mathbb{R}^3} |f_\delta(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v + 4\int_{\mathbb{R}^3} |(\mathbf{I}-\mathbf{P})f(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v.$$

These prove (2.2.3). The second estimate of (2.2.4) is clear from the definition.

Now we focus on the first estimate of (2.2.4). From Lemma 2.4,

$$\int_{\mathbb{R}^3} |f_\delta(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v \le \int_{\mathbb{R}^3} |\bar{f}(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v.$$

From (2.2.22) and (2.2.23) with $\phi(v)=\langle v\rangle^2\sqrt{\mu(v)}$, we have $\int_{\mathbb{R}^3} |\bar{f}(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v \lesssim |S(h)(t,x)|$.

Finally, from Lemma 2.5 and (2.2.24), and (2.2.8), we conclude (2.2.4). The proof is completed.

2.3 Steady L^2-Coercivity

The main purpose of this section is to prove the following:

Proposition 2.2 *Suppose $\Phi \in L^\infty$, $g \in L^2(\Omega\times\mathbb{R}^3)$, and $r \in L^2(\gamma_-)$ such that*

$$\iint_{\Omega\times\mathbb{R}^3} g(x,v)\sqrt{\mu}\mathrm{d}x\mathrm{d}v = 0 = \int_{\gamma_-} r(x,v)\sqrt{\mu}\mathrm{d}\gamma. \tag{2.3.1}$$

Then, for sufficiently small $\varepsilon > 0$, there exists a unique solution to

$$v\cdot\nabla_x f + \varepsilon^2\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v[\sqrt{\mu}f] + \varepsilon^{-1}Lf = g, \quad f|_{\gamma_-} = P_\gamma f + r, \tag{2.3.2}$$

such that

$$\iint_{\Omega\times\mathbb{R}^3} f(x,v)\sqrt{\mu}\ \mathrm{d}x\mathrm{d}v = 0, \tag{2.3.3}$$

and

$$\|\mathbf{P}f\|_2^2 + \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + |f|_2^2 \lesssim \|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_2^2 + \varepsilon^{-2}\|\mathbf{P}g\|_2^2 + \varepsilon^{-1}|r|_{2,-}^2. \tag{2.3.4}$$

For the proof we refer to the proof in [21] for the details. As the first step of the proof of Proposition 2.3, we consider the following penalized problem:

$$
\begin{aligned}
&\mathcal{L}f := \Big(\lambda + \varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v\Big)f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f = g \quad \text{in } \Omega\times\mathbb{R}^3,\\
&f = P_\gamma f + r \quad \text{on } \gamma_-.
\end{aligned}
\tag{2.3.5}
$$

Lemma 2.6 *Assume that* $g \in L^2(\Omega\times\mathbb{R}^3)$ *and* $r \in L^2(\gamma_-)$ *satisfy* (2.3.1)*. Moreover, let* $\Phi \in L^\infty(\Omega)$ *and* $\lambda > 0$*. Then, if* $\varepsilon > 0$ *is sufficiently small, the solution to* (2.3.5) *exists and is unique. Moreover it satisfies the bounds*

$$
\varepsilon^{-1}\|f\|_\nu^2 + |(1-P_\gamma)f|_{2,+}^2 \lesssim \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2. \tag{2.3.6}
$$

We remark that Lemma 2.6 implies that, for ε sufficiently small, the operator $\mathcal{L}^{-1}$ is well-defined and bounded as a map from L^2 to L^2. For the proof we refer to Lemma 2.10 in [21].

Lemma 2.7 *For any* $\lambda, \varepsilon > 0$*, the operator* $K\mathcal{L}^{-1}$ *is compact in* L^2*. Explicitly, if* $g^n \in L^2$ *and* $\sup_n \|g^n\|_2 < \infty$ *then there exist subsequence* n_k *such that* $Kf^{n_k} \to Kf$ *in* L^2*, where* f^n *solve*

$$
\lambda f^n + v\cdot\nabla_x f^n + \frac{1}{\varepsilon}\nu f^n + \varepsilon^2\Phi\cdot\nabla_v f^n - \frac{1}{2}\varepsilon^2\Phi\cdot v f^n = g^n, \quad f^n|_{\gamma_-} = P_\gamma f^n + r.
$$

For the proof we refer to Lemma 2.11 in [21].

Next we prove the essential bound for $\mathbf{P}f$, where f solves

$$
\begin{aligned}
&\Big[\lambda + (1-\theta)\varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v)\Big]f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f + \varepsilon^{-1}\theta L f = g, \quad \text{in } \Omega\times\mathbb{R}^3\\
&f_- = P_\gamma f + r, \quad \text{on } \gamma_-.
\end{aligned}
\tag{2.3.7}
$$

We denote

$$
\mathring{f} := f - \langle f\rangle\sqrt{\mu}, \quad \langle f\rangle := \Big(\iint_{\Omega\times\mathbb{R}^3} f\sqrt{\mu}\mathrm{d}x\mathrm{d}v\Big)\Big/\Big(\iint_{\Omega\times\mathbb{R}^3}\mu\mathrm{d}x\mathrm{d}v\Big). \tag{2.3.8}
$$

Lemma 2.8 *Assume* (2.3.1)*. Let* f *be a solution to* (2.3.7) *in the sense of distribution. Then, for all* $\lambda \geq 0$ *and all* $\theta \in [0,1]$*,*

$$
\|\mathbf{P}\mathring{f}\|_2^2 \lesssim \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + |(1-P_\gamma)f|_{2,+}^2 + |r|_2^2 + \Big\|\frac{g}{\sqrt{\nu}}\Big\|_2^2 + \varepsilon^2\|\Phi\|_\infty|\langle f\rangle|^2, \tag{2.3.9}
$$

and

$$
\lambda|\langle f\rangle| \lesssim (1-\theta)\varepsilon^{-1}\|f\|_2. \tag{2.3.10}
$$

For the proof we refer to Lemma 2.12 in [21].

2.4 L^∞ Estimate

The main goal of this section is to prove the following lemma.

Proposition 2.3 *Let f satisfy*

$$\big[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v+\varepsilon^{-1}C_0\langle v\rangle\big]|f|\le\varepsilon^{-1}K_\beta|f|+|g|,\qquad \big|f|_{\gamma_-}\big|\le P_\gamma|f|+|r|, \tag{2.4.1}$$

where, for $0<\beta<\frac14$, $K_\beta|f|=\int_{\mathbb{R}^3}\mathbf{k}_\beta(v,u)|f(u)|\mathrm{d}u$ and

$$\mathbf{k}_\beta(v,u):=\big\{|v-u|+|v-u|^{-1}\big\}\exp\Big[-\beta|v-u|^2-\beta\frac{[|v|^2-|u|^2]^2}{|v-u|^2}\Big]. \tag{2.4.2}$$

Then, for $w(v)=\mathrm{e}^{\beta'|v|^2}$ with $0<\beta'\ll\beta$,

$$\begin{aligned}\|\varepsilon wf\|_\infty\lesssim{}& o(1)\|\varepsilon wf\|_\infty+\|\varepsilon wr(s)\|_\infty+\varepsilon^2\|\langle v\rangle^{-1}wg\|_\infty\\&+\|\mathbf{S}_1f\|_{L^3(\Omega)}+\frac{1}{\varepsilon^{1/2}}\|\mathbf{S}_2f\|_{L^2(\Omega)}+\frac{1}{\varepsilon^{1/2}}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)}.\end{aligned} \tag{2.4.3}$$

For the proof we refer to the proof in [21] page 27.

We define the stochastic cycles for the steady case.

Definition 2.3 Define, for free variables $v_k\in\mathbb{R}^3$, from (2.1.9)

$$\begin{aligned}&t_1=t-t_{\mathbf{b}}(x,v),\quad x_1=X(t_1;t,x,v)=x_{\mathbf{b}}(x,v),\\&t_2=t_1-t_{\mathbf{b}}(x_1,v_1),\quad x_2=X(t_2;t_1,x_1,v_1)=x_{\mathbf{b}}(x_1,v_1),\\&\qquad\vdots\\&t_{k+1}=t_k-t_{\mathbf{b}}(x_k,v_k),\quad x_{k+1}=X(t_{k+1};t_k,x_k,v_k)=x_{\mathbf{b}}(x_k,v_k).\end{aligned}$$

Set

$$X_{\mathbf{cl}}(s;t,x,v):=\sum_k\mathbf{1}_{[t_{k+1},t_k)}(s)X(s;t_k,x_k,v_k),\quad V_{\mathbf{cl}}(s;t,x,v):=\sum_k\mathbf{1}_{[t_{k+1},t_k)}(s)v_k.$$

For $x\in\partial\Omega$, we define

$$\mathcal{V}(x):=\{v\in\mathbb{R}^3:n(x)\cdot v>0\},\quad \mathrm{d}\sigma(x,v):=\sqrt{2\pi}\mu(v)\{n(x)\cdot v\}\mathrm{d}v. \tag{2.4.4}$$

For $j\in\mathbb{N}$, we denote

$$\mathcal{V}_j:=\{v_j\in\mathbb{R}^3:n(x_j)\cdot v_j>0\},\quad \mathrm{d}\sigma_j:=\sqrt{2\pi}\mu(v_j)\{n(x_j)\cdot v_j\}\mathrm{d}v_j. \tag{2.4.5}$$

The following lemma is a generalized version of Lemma 23 of [32].

Lemma 2.9[32] *Assume $\Phi=\Phi(x)\in C^1$. For sufficiently large $T_0>0$, there exist constants $C_1,C_2>0$, independent of T_0, such that for $k=C_1T_0^{5/4}$,*

$$\sup_{(t,x,v)\in[0,T_0]\times\bar\Omega\times\mathbb{R}^3}\int_{\prod_{\ell=1}^{k-1}\mathcal{V}_\ell}\mathbf{1}_{t_k(t,x,v_1,v_2,\cdots,v_{k-1})>0}\prod_{\ell=1}^{k-1}\mathrm{d}\sigma_\ell<\Big\{\frac12\Big\}^{C_2T_0^{5/4}}. \tag{2.4.6}$$

Proof For $0<\delta\ll1$, we define

$$\mathcal{V}_\ell^\delta:=\Big\{v_\ell\in\mathcal{V}_\ell:\ |v_\ell\cdot n(x_\ell)|>\delta\ \text{ and }\ \delta<|v_\ell|<\frac1\delta\Big\}.$$

Clearly, $\int_{\mathcal{V}_\ell \setminus \mathcal{V}_\ell^\delta} \mathrm{d}\sigma_\ell \leq C\delta$, where C is independent of ℓ. We claim that

$$|t_\ell - t_{\ell+1}| \geq \frac{\delta^3}{C_\Omega}, \quad \text{for} \quad v_\ell \in \mathcal{V}_\ell^\delta. \tag{2.4.7}$$

It suffices to prove, for $(x,v) \in \gamma_-^\delta$ and $0 < \varepsilon \ll 1$,

$$t_{\mathbf{b}}(x,v) \gtrsim |v|^{-2}|n(x)\cdot v|.$$

Note that $\frac{|n(x)\cdot v|}{|v|^2} \leq \delta^2$. Therefore we only need to consider the case of $t_{\mathbf{b}}(x,v) < \delta^2$.

From $|v| > \delta$ and $x_{\mathbf{b}} = x + t_{\mathbf{b}} v + O(\varepsilon^2)(t_{\mathbf{b}})^2$,

$$t_{\mathbf{b}} = |x_{\mathbf{b}} - x||v|^{-1} + O(\varepsilon^2)(t_{\mathbf{b}})^2|v|^{-1} = |x_{\mathbf{b}} - x||v|^{-1} + t_{\mathbf{b}}O(\varepsilon^2)\delta.$$

For fixed $\delta > 0$ and $\varepsilon < \varepsilon_0 \ll_\delta 1$,

$$t_{\mathbf{b}}(x,v) \gtrsim |x_{\mathbf{b}}(x,v) - x||v|^{-1}.$$

From the fact $|x_{\mathbf{b}} - x| \gtrsim |n(x)\cdot \frac{x - x_{\mathbf{b}}}{|x-x_{\mathbf{b}}|}|$ for $x_{\mathbf{b}}, x \in \partial\Omega$ from [32], we have

$$t_{\mathbf{b}}(x,v) \gtrsim \big|n(x)\cdot[x - x_{\mathbf{b}}(x,v)]\big|^{1/2}|v|^{-1}.$$

On the other hand, for $(x,v) \in \gamma_-^\delta$ and $\varepsilon \ll 1$

$$|n(x)\cdot(x_{\mathbf{b}} - x)| = \big|n(x)\cdot[t_{\mathbf{b}}v + O(\varepsilon^2)(t_{\mathbf{b}})^2]\big| = t_{\mathbf{b}}|n(x)\cdot v| + O(\varepsilon^2)(t_{\mathbf{b}})^2 \gtrsim t_{\mathbf{b}}|n(x)\cdot v|.$$

Therefore we prove our claim. The rest of proof of (2.4.6) is identical to the proof of Lemma 23 on [32].

Now we are ready to prove the main result of this section:

Proof of Proposition 2.3 Define, for $w(v) = \mathrm{e}^{\beta'|v|^2}$,

$$h(t,x,v) := w(v)f(t,x,v). \tag{2.4.8}$$

From Lemma 3 of [32], there exists a $\tilde\beta = \tilde\beta(\beta,\beta') > 0$ such that $\mathbf{k}_\beta(v,u)\frac{w(v)}{w(u)} \lesssim \mathbf{k}_{\tilde\beta}(v,u)$.

Then, from (2.4.1),

$$\Big[v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v + \varepsilon^{-1}C_0\langle v\rangle + \frac{\varepsilon^4\Phi\cdot\nabla_v w}{w}\Big]|\varepsilon h| \leq \varepsilon^{-1}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\beta}(v,u)|\varepsilon h(u)|\mathrm{d}u + \varepsilon|wg|. \tag{2.4.9}$$

Clearly $\varepsilon^{-1}C_0\langle v\rangle + \frac{\varepsilon^4\Phi\cdot\nabla_v w}{w} \sim \varepsilon^{-1}C_0\langle v\rangle$.

From (2.4.1), on $(x,v) \in \gamma_-$,

$$\begin{aligned}\varepsilon|h(x,v)| &\leq \sqrt{2\pi}w(v)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\varepsilon|h(x,u)|\frac{\sqrt{\mu(u)}}{w(u)}\{n(x)\cdot u\}\mathrm{d}u + \varepsilon w(v)|r(x,v)| \\ &\leq \frac{1}{\tilde w(v)}\int_{n(x)\cdot u>0}\varepsilon|h(x,u)|\tilde w(u)\mathrm{d}\sigma + \varepsilon w(v)|r(x,v)|, \end{aligned} \tag{2.4.10}$$

where we define

$$\tilde{w}(v) := \frac{1}{w(v)\sqrt{\mu(v)}}.$$

We claim, for $t = T_0$ where T_0 in Lemma 2.9 (does not depends on ε),

$$\begin{aligned}
&|\varepsilon h(x,v)| \\
\le& \Big[CT_0^{5/4} \Big\{ \frac{1}{2} \Big\}^{C_2 T_0^{5/4}} + o(1) CT_0^{5/2} \Big] \|\varepsilon h\|_\infty + \varepsilon \|wr\|_\infty + CT_0^{5/2} \varepsilon^2 \|\langle v\rangle^{-1} wg\|_\infty \\
&+ CT_0^{5/2} \Big[\|\mathbf{S}_1 f\|_{L^3(\Omega)} + \frac{1}{\varepsilon^{1/2}} \|\mathbf{S}_2 f\|_{L^2(\Omega)} + \frac{1}{\varepsilon^{1/2}} \|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)} \Big]. \quad (2.4.11)
\end{aligned}$$

Once (2.4.11) holds, Proposition 2.3 is a direct consequence.

We first prove (2.4.11). From (2.4.9), for $t_1(t,x,v) < s \le t$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} \Big[\exp\Big(-\int_s^t \frac{C_0}{\varepsilon} \langle V(\tau;t,x,v)\rangle \mathrm{d}\tau \Big) \varepsilon h(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) \Big] \\
\le& \exp\Big(-\int_s^t \frac{C_0}{\varepsilon} \langle V(\tau;t,x,v)\rangle \mathrm{d}\tau \Big) \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \mathbf{k}_{\tilde\beta}(V_{\mathbf{cl}}(s;t,x,v),v') |\varepsilon h(X_{\mathbf{cl}}(s;t,x,v),v')| \mathrm{d}v' \\
&+ \exp\Big(-\int_s^t \frac{C_0}{\varepsilon} \langle V(\tau;t,x,v)\rangle \mathrm{d}\tau \Big) |\varepsilon wg(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))|.
\end{aligned}$$

Along the stochastic cycles, for $k = C_1 T_0^{5/4}$, we deduce the following estimate:

$$\begin{aligned}
&|\varepsilon h(x,v)| \\
\le& \mathbf{1}_{\{t_1(t,x,v)<0\}} \exp\Big(-\int_0^t \frac{C_0 \langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau \Big) |\varepsilon h(X_{\mathbf{cl}}(0;t,x,v), V_{\mathbf{cl}}(0;t,x,v))| \quad (2.4.12) \\
&+ \int_{\max\{0,t_1(t,x,v)\}}^t \mathrm{d}s \; \frac{\exp\big(-\int_s^t \frac{C_0 \langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau \big)}{\varepsilon} \\
&\times \int_{\mathbb{R}^3} \mathrm{d}v' \; \mathbf{k}_{\tilde\beta}(V_{\mathbf{cl}}(s;t,x,v),v') |\varepsilon h^\ell(X_{\mathbf{cl}}(s;t,x,v),v')| \quad (2.4.13) \\
&+ \int_{\max\{0,t_1(t,x,v)\}}^t \mathrm{d}s \; \frac{\exp\big(-\int_s^t \frac{C_0 \langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau \big)}{\varepsilon} |\varepsilon^2 wg(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))| \\
&\quad (2.4.14) \\
&+ \mathbf{1}_{\{t_1(t,x,v)\ge 0\}} \exp\Big(-\int_{t_1(t,x,v)}^t \frac{C_0 \langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau \Big) |\varepsilon wr(x_1(x,v), v_1(x,v))| \quad (2.4.15) \\
&+ \mathbf{1}_{\{t_1(t,x,v)\ge 0\}} \frac{\exp\big(-\int_{t_1(t,x,v)}^t \frac{C_0 \langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau \big)}{\tilde{w}(v)} \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} H,
\end{aligned}$$

where H is given by

$$\sum_{l=1}^{k-1}\mathbf{1}_{t_{l+1}\leq 0<t_l}\big|\varepsilon h(X_{\mathbf{cl}}(0;t_l,x_l,v_l),V_{\mathbf{cl}}(0;\ t_l,x_l,v_l))\big|\mathrm{d}\Sigma_l(0) \tag{2.4.16}$$

$$+\sum_{l=1}^{k-1}\int_{\max\{0,t_{l+1}\}}^{t_l}\mathbf{1}_{t_l>0}\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\beta}}(V_{\mathbf{cl}}(\tau;t_l,x_l,v_l),u)\big|\varepsilon h(X_{\mathbf{cl}}(\tau;t_l,x_l,v_l),u)\big|\mathrm{d}u\mathrm{d}\Sigma_l(\tau)\mathrm{d}\tau \tag{2.4.17}$$

$$+\sum_{l=1}^{k-1}\int_{\max\{0,t_{l+1}\}}^{t_l}\mathbf{1}_{t_l>0}\frac{1}{\varepsilon}\big|\varepsilon^2wg(X_{\mathbf{cl}}(\tau;t_l,x_l,v_l),V_{\mathbf{cl}}(\tau;t_l,x_l,v_l))\big|\mathrm{d}\Sigma_l(\tau)\mathrm{d}\tau \tag{2.4.18}$$

$$+\sum_{l=1}^{k-1}\mathbf{1}_{t_l>0}|\varepsilon w(v_l)r(x_{l+1},v_l)|\mathrm{d}\Sigma_l(t_{l+1}) \tag{2.4.19}$$

$$+\mathbf{1}_{t_k>0}|\varepsilon h(x_k,v_{k-1})|\mathrm{d}\Sigma_{k-1}(t_k), \tag{2.4.20}$$

and $\mathrm{d}\Sigma_{k-1}(t_k)$ is evaluated at $s=t_k$ of

$$\begin{aligned}\mathrm{d}\Sigma_l(s):=&\Big\{\prod_{j=l+1}^{k-1}\mathrm{d}\sigma_j\Big\}\Big\{\exp\Big(-\int_s^{t_l}\frac{C_0\langle V_{\mathbf{cl}}(\tau;t_l,x_l,v_l)\rangle}{\varepsilon}\mathrm{d}\tau\Big)w(v_l)\mathrm{d}\sigma_l\Big\}\\&\times\prod_{j=1}^{l-1}\Big\{\exp\Big(-\int_{t_{j+1}}^{t_j}\frac{C_0\langle V_{\mathbf{cl}}(\tau;t_j,x_j,v_j)\rangle}{\varepsilon}\mathrm{d}\tau\Big)\mathrm{d}\sigma_j\Big\}.\end{aligned} \tag{2.4.21}$$

Directly, from our choice $k=C_1T_0^{5/4}$,

$$(2.4.12)+(2.4.16)\lesssim C_1T_0^{5/4}\mathrm{e}^{-\frac{C_0}{\varepsilon}t}\|\varepsilon h\|_\infty,\quad (2.4.15)+(2.4.19)\lesssim C_1T_0^{5/4}\|\varepsilon wr\|_\infty$$

and

$$\begin{aligned}&(2.4.14)+(2.4.18)\\ \lesssim&\big\|\varepsilon^2\langle v\rangle^{-1}wg\big\|_\infty\times\Big\{\int_0^t\frac{\langle V_{\mathbf{cl}}(s;t,x,v)\rangle}{\varepsilon}\exp\Big(-\int_s^t\frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon}\mathrm{d}\tau\Big)\mathrm{d}s\\&+C_1T_0^{5/4}\sup_l\int_0^{t_l}\frac{\langle V_{\mathbf{cl}}(\tau;t_l,x_l,v_l)\rangle}{\varepsilon}\exp\Big(-\int_s^{t_l}\frac{C_0\langle V_{\mathbf{cl}}(\tau;t_l,x_l,v_l)\rangle}{\varepsilon}\mathrm{d}\tau\Big)\mathrm{d}\tau\Big\}\\ \lesssim&C_1T_0^{5/4}\big\|\varepsilon^2\langle v\rangle^{-1}wg\big\|_\infty\times\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\exp\Big(-\int_s^t\frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon}\mathrm{d}\tau\Big)\mathrm{d}s\\ \lesssim&C_1T_0^{5/4}\big\|\varepsilon^2\langle v\rangle^{-1}wg\big\|_\infty,\end{aligned}$$

where we have used the fact that $\mathrm{d}\sigma_j$ is a probability measure of $\mathcal{V}_j$.

Now we focus on (2.4.13) and (2.4.17). For $N>1$, we can choose $m=m(N)\gg 1$ such that

$$\mathbf{k}_m(v,u):=\mathbf{1}_{|v-u|\geq\frac{1}{m}}\mathbf{1}_{|u|\leq m}\mathbf{1}_{|v|\leq m}\mathbf{k}_{\tilde{\beta}}(v,u),\quad\sup_v\int_{\mathbb{R}^3}|\mathbf{k}_m(v,u)-\mathbf{k}_{\tilde{\beta}}(v,u)|\mathrm{d}u\leq\frac{1}{N}. \tag{2.4.22}$$

We split $\mathbf{k}_{\tilde{\beta}}(v,u) = [\mathbf{k}_{\tilde{\beta}}(v,u) - \mathbf{k}_m(v,u)] + \mathbf{k}_m(v,u)$, and the first difference would lead to a small contribution in (2.4.13) and (2.4.17) as, for $N \gg_{T_0} 1$,

$$\frac{k}{N}\|\varepsilon h\|_\infty = \frac{C_1 T_0^{5/4}}{N}\|\varepsilon h\|_\infty.$$

We further split the time integrations in (2.4.13) and (2.4.17) as $[t_l - \kappa\varepsilon, t_l]$ and $[\max\{0, t_{l+1}\}, t_l - \kappa\varepsilon]$:

$$(2.4.13) = \underbrace{\int_{t-\kappa\varepsilon}^{t}} + \int_{\max\{0,t_1\}}^{t-\kappa\varepsilon}, \quad (2.4.17) = \mathbf{1}_{\{t_1\ge 0\}} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \Big\{ \underbrace{\int_{t_l-\kappa\varepsilon}^{t_l}} + \int_{\max\{0,t_{l+1}\}}^{t_l-\kappa\varepsilon} \Big\}.$$

The small-in-time contributions of both (2.4.13) and (2.4.17), underbraced terms, are bounded by

$$\begin{aligned}
&\kappa\varepsilon\frac{1}{\varepsilon}\sup_v \int_{|v'|\le N} \mathbf{k}_m(v,v')\mathrm{d}v'\|\varepsilon h\|_\infty \lesssim \kappa\|\varepsilon h\|_\infty,\\
&C_1 T_0^{5/4}\kappa\varepsilon\frac{1}{\varepsilon}\sup_v \int_{|v'|\le N} \mathbf{k}_m(v,v')\mathrm{d}v'\|\varepsilon h\|_\infty \lesssim \kappa C_1 T_0^{5/4}\|\varepsilon h\|_\infty.
\end{aligned}$$

For (2.4.20), by Lemma 2.9,

$$\begin{aligned}
(2.4.20) &\lesssim \sup_{(t,x,v)\in[0,T_0]\times\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{t_k(t,x,v,v_1,v_2,\cdots,v_{k-1})>0} \prod_{j=1}^{k-1} \mathrm{d}\sigma_j \|\varepsilon h\|_\infty\\
&\lesssim \Big\{\frac{1}{2}\Big\}^{C_2 T_0^{5/4}} \|\varepsilon h\|_\infty.
\end{aligned}$$

Overall, for $(t,x,v) \in [0,T_0]\times\bar{\Omega}\times\mathbb{R}^3$,

$$\begin{aligned}
|\varepsilon h(x,v)| \lesssim & \int_{\max\{0,t_1(x,v)\}}^{t-\kappa\varepsilon} \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\le m} \underbrace{|\varepsilon h(X_{\mathbf{cl}}(s;t,x,v),v')|}\mathrm{d}v'\mathrm{d}s\\
&+\mathbf{1}_{\{t_1\ge 0\}} \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-t_1)}}{\tilde{w}(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0,t_{\ell+1}\}}^{t_\ell-\kappa\varepsilon} \frac{\mathbf{1}_{t_\ell>0}}{\varepsilon}\\
&\times \int_{|v''|\le m} \underbrace{|\varepsilon h(X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')|}\mathrm{d}v''\mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau\\
&+C T_0^{5/4}\big\{\mathrm{e}^{-\frac{C_0}{\varepsilon}t}\|\varepsilon h\|_\infty + \|\varepsilon w r(s)\|_\infty + \|\varepsilon^2\langle v\rangle^{-1} w g\|_\infty\big\}\\
&+o(1) C T_0^{5/4}\|\varepsilon h\|_\infty + \Big\{\frac{1}{2}\Big\}^{C_2 T_0^{5/4}}\|\varepsilon h\|_\infty. \qquad (2.4.23)
\end{aligned}$$

Note that the same estimate holds for the underbraced terms in (2.4.23). We plug these estimates into the underbraced terms of (2.4.23) to have a bound as

$$|\varepsilon h^{\ell+1}(t,x,v)| \le \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.$$

Here, using $w(u) \lesssim_m 1$ for $|u| \le m$,

$$
\begin{aligned}
\mathbf{I}_1 \lesssim_m & \int_{\max\{0,t_1\}}^{t-\kappa\varepsilon} \mathrm{d}s \, \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\le m} \mathrm{d}v' \int_{\max\{0,t_1'\}}^{s-\kappa\varepsilon} \mathrm{d}s' \frac{\mathrm{e}^{-\frac{C_0(s-s')}{\varepsilon}}}{\varepsilon} \\
& \times \int_{|u|\le m} \mathrm{d}u \big|\varepsilon h(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)\big| \\
& + \int_{\max\{0,t_1\}}^{t-\kappa\varepsilon} \mathrm{d}s \, \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\le m} \mathrm{d}v' \mathbf{1}_{\{t_1'\ge 0\}} \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(s-t_1')}}{\tilde{w}(v)} \\
& \times \int_{\prod_{j=1}^{k-1} \mathcal{V}_j'} \sum_{\ell'=1}^{k-1} \int_{\max\{0,t_{\ell'+1}'\}}^{t_{\ell'}'-\kappa\varepsilon} \mathbf{1}_{t_{\ell'}'>0} \frac{1}{\varepsilon} \big|\varepsilon h(\tau, X_{\mathbf{cl}}(\tau;t_{\ell'}',x_{\ell'}',v_{\ell'}'),u)\big| \mathrm{d}u \mathrm{d}\Sigma_{\ell'}(\tau)\mathrm{d}\tau,
\end{aligned}
$$

where $t_{\ell'}' := \tilde{t}_{\ell'}(s, X_{\mathbf{cl}}(s;t,x,v),v')$, $x_{\ell'}' := x_{\ell'}(X_{\mathbf{cl}}(s;t,x,v),v')$, $v_{\ell'}' := v_{\ell'}(X_{\mathbf{cl}}(s;t,x,v),v')$. Moreover

$$
\begin{aligned}
\mathbf{I}_2 \lesssim_m & \; \mathbf{1}_{\{t_1\ge 0\}} \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-t_1)}}{\tilde{w}(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0,t_{\ell+1}\}}^{t_\ell-\kappa\varepsilon} \mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau \; \mathbf{1}_{t_\ell>0} \frac{1}{\varepsilon} \int_{|v''|\le m} \mathrm{d}v'' \\
& \times \int_{\max\{0,t_1''\}}^{\tau-\kappa\varepsilon} \mathrm{d}s'' \, \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon^2}(\tau-s'')}}{\varepsilon} \int_{|u|\le m} \mathrm{d}u \big|\varepsilon h\big(X_{\mathbf{cl}}(s'';\tau,X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v''),u\big)\big| \\
& + \mathbf{1}_{\{t_1\ge 0\}} \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-t_1)}}{\tilde{w}(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0,t_{\ell+1}\}}^{t_\ell-\kappa\varepsilon} \mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau \; \mathbf{1}_{t_\ell>0} \frac{1}{\varepsilon} \int_{|v''|\le m} \mathrm{d}v'' \\
& \times \mathbf{1}_{t_1''\ge 0} \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(\tau-t_1'')}}{\tilde{w}(v'')} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j''} \sum_{\ell''=1}^{k-1} \int_{\max\{0,t_{\ell''+1}''\}}^{t_{\ell''}''-\kappa\varepsilon} \mathbf{1}_{t_{\ell''}''>0} \frac{1}{\varepsilon} \\
& \times \int_{|u|\le m} \big|\varepsilon h\big(\tau'', X_{\mathbf{cl}}(\tau'';t_{\ell''}'',x_{\ell''}'',v_{\ell''}''),u\big)\big| \mathrm{d}u \mathrm{d}\Sigma_{\ell''}''(\tau'')\mathrm{d}\tau'',
\end{aligned}
$$

where $t_{\ell''}'' := t_{\ell''}(\tau, X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$, $x_{\ell''}'' := x_{\ell''}(X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$, $v_{\ell''}'' := v_{\ell''}(X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$. Furthermore

$$
\begin{aligned}
\mathbf{I}_3 \lesssim & \; CT_0^{5/2}\big\{\mathrm{e}^{-\frac{C_0}{\varepsilon}t}\|\varepsilon h\|_\infty + \|\varepsilon w r\|_\infty + \|\varepsilon^2\langle v\rangle^{-1} w g\|_\infty\big\} \\
& + o(1)CT_0^{5/2}\|\varepsilon h\|_\infty + T_0^{5/4}\Big\{\frac{1}{2}\Big\}^{C_2 T_0^{5/4}}\|\varepsilon h\|_\infty.
\end{aligned}
$$

This bound of $\mathbf{I}_3$ is already included in the RHS of (2.4.11).

Now we focus on $\mathbf{I}_1$ and $\mathbf{I}_2$. Consider the change of variables

$$
v' \;\mapsto\; y := X(s';s,X_{\mathbf{cl}}(s;t,x,v),v'). \tag{2.4.24}
$$

By a direct computation and (2.1.14), for $\max\{0,t_1'\} \le s' \le s-\kappa\varepsilon \le T_0$,

$$\begin{aligned}\frac{\partial X_i(s';s)}{\partial v_j'} &= -(s-s')\delta_{ij} + \int_s^{s'} \mathrm{d}\tau' \int_s^{\tau'} \mathrm{d}\tau'' \varepsilon^2 \sum_m \partial_m \Phi_i(X(\tau'';s)) \frac{\partial X_m}{\partial v_j'}(\tau'';s)\\ &= -(s-s')\big[\delta_{ij} + O(\varepsilon^2)\|\Phi\|_{C^1} T_0^2 \mathrm{e}^{C_\Phi T_0}\big].\end{aligned}$$

By the lower bound of $|s-s'| \geq \kappa\varepsilon$,

$$\det \nabla_{v'} X(s';s) = |s-s'|^3 \det\big(\delta_{ij} + O(\varepsilon^2)\|\Phi\|_{C^1} T_0^2 \mathrm{e}^{C_\Phi T_0}\big) \gtrsim \kappa^3\varepsilon^3.$$

Now integrating over time first

$$\begin{aligned}&\int_{\max\{0,t_1\}}^{t-\kappa\varepsilon} \mathrm{d}s\, \frac{\mathrm{e}^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\leq m} \mathrm{d}v' \int_{\max\{0,t_1'\}}^{s-\kappa\varepsilon} \mathrm{d}s' \frac{\mathrm{e}^{-\frac{C_0(s-s')}{\varepsilon}}}{\varepsilon}\\ &\times \int_{|u|\leq m} \mathrm{d}u \big|\varepsilon h(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)\big|\\ \lesssim& \sup_{0\leq s'\leq s-\kappa\varepsilon\leq s\leq t-\kappa\varepsilon} \int_{|v'|\leq m} \mathrm{d}v' \int_{|u|\leq m} \mathrm{d}u\ |\varepsilon h(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)|\\ \lesssim& \sup_{0\leq s'\leq s-\kappa\varepsilon\leq s\leq t-\kappa\varepsilon} \varepsilon \int_{|v'|\leq m}\int_{|u|\leq m} |f(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)|\mathrm{d}u\mathrm{d}v'\\ \lesssim& \sup_{0\leq s'\leq s-\kappa\varepsilon\leq s\leq t-\kappa\varepsilon} \varepsilon \int_{|v'|\leq m}\int_{|u|\leq m} \sum_{i=1,2} |\mathbf{S}_i f(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'))|\langle u\rangle^2 \sqrt{\mu(u)}\mathrm{d}u\mathrm{d}v'\\ &+ \sup_{0\leq s'\leq s-\kappa\varepsilon\leq s\leq t-\kappa\varepsilon} \varepsilon \int_{|v'|\leq m}\int_{|u|\leq m} |(\mathbf{I}-\mathbf{P})f(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'))|\mathrm{d}u\mathrm{d}v'.\end{aligned}$$

where we have used $|h(u)| = w(u)|f(u)| \lesssim_m |f(u)|$ for $|u| \leq m$ and the decomposition (2.2.3). For $\mathbf{S}_1 f-$contribution,

$$\begin{aligned}&\varepsilon \int_{v'}\int_u \big|\mathbf{S}_1 f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v')\big)\langle u\rangle^2\sqrt{\mu(u)}\big|\mathrm{d}u\mathrm{d}v'\\ \lesssim_m& \varepsilon\Big[\int_{v'} \big|\mathbf{S}_1 f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v')\big)\big|^3 \mathrm{d}v'\Big]^{1/3} \lesssim_m \varepsilon\Big[\int_\Omega \big|\mathbf{S}_1 f\big(y\big)\big|^3 \frac{1}{\kappa^3\varepsilon^3}\mathrm{d}y\Big]^{1/3}\\ \lesssim_m& \|\mathbf{S}_1 f\|_{L^3(\Omega)}.\end{aligned}$$

For $\mathbf{S}_2 f$ and $(\mathbf{I}-\mathbf{P})f$ contributions,

$$\begin{aligned}&\varepsilon \int_{v'}\int_u \big[\mathbf{S}_2 f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v')\big)\langle u\rangle^2\sqrt{\mu(u)}\\ &+|(\mathbf{I}-\mathbf{P})f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u\big)|\big]\mathrm{d}u\mathrm{d}v'\\ \lesssim_m& \varepsilon\Big[\int_{v'} \big|\mathbf{S}_2 f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v')\big)\big|^2 \mathrm{d}v'\Big]^{1/2}\\ &+\varepsilon\Big[\iint \big|(\mathbf{I}-\mathbf{P})f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u\big)\big|^2 \mathrm{d}v'\mathrm{d}u\Big]^{1/2}\end{aligned}$$

$$\lesssim_m \varepsilon\Big[\int_\Omega |\mathbf{S}_1 f(y)|^2 \frac{1}{\kappa^3\varepsilon^3}\mathrm{d}y\Big]^{1/2} + \varepsilon\Big[\iint_{\Omega\times\mathbb{R}^3} |(\mathbf{I}-\mathbf{P})f(y,u)|^2 \frac{1}{\kappa^3\varepsilon^3}\mathrm{d}y\mathrm{d}u\Big]^{1/2}$$
$$\lesssim_m \frac{1}{\varepsilon^{1/2}}\|\mathbf{S}_1 f\|_{L^2(\Omega)} + \frac{1}{\varepsilon^{1/2}}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)}.$$

We have the similar change of variables for $v'_{\ell'} \mapsto X_{\mathbf{cl}}(\tau; t'_{\ell'}, x'_{\ell'}, v'_{\ell'})$, $v''_{\ell''} \mapsto X_{\mathbf{cl}}(-\tau''; t''_{\ell''}, x''_{\ell''}, v''_{\ell''})$, and $v'' \mapsto X_{\mathbf{cl}}(s''; \tau, X_{\mathbf{cl}}(\tau; t_\ell, x_\ell, v_\ell), v'')$.

Following the same proof, we conclude

$$\mathbf{I}_1 + \mathbf{I}_2 \lesssim T_0^{5/2}\|\mathbf{S}_1 f\|_{L^3(\Omega)} + T_0^{5/2}\frac{1}{\varepsilon^{1/2}}\|\mathbf{S}_2 f\|_{L^2(\Omega)} + T_0^{5/2}\frac{1}{\varepsilon^{1/2}}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{2.4.25}$$

All together we prove our claims (2.4.11). The proof is completed.

2.5 Validity of the Steady Problem

The main purpose of this section is to prove Theorem 1.1. We need several estimates before the proof of the main theorem.

Lemma 2.10 *Assume*

$$|a(f)|+|b(f)|+|c(f)| \le \mathbf{S}_1 f(x)+\mathbf{S}_2 f(x), \quad |a(g)|+|b(g)|+|c(g)| \le \mathbf{S}_1 g(x)+\mathbf{S}_2 g(x),$$

where $[a,b,c]$ *is defined in* (1.2.4) *and* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are defined in* (2.2.3)*. Then, for* $w = \mathrm{e}^{\beta|v|^2}$, $0<\beta\ll 1$,

$$\begin{aligned}&\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(f,g)\|_{L^2_{x,v}} + \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(g,f)\|_{L^2_{x,v}}\\ \lesssim\, & [\varepsilon^{3/2}\|wg\|_{L^\infty_{x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}] + [\varepsilon^{-1}\|\mathbf{S}_2 f\|_{L^2_x}]\big\}\\ &+[\varepsilon^{3/2}\|wf\|_{L^\infty_{x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^2_{x,v}}] + [\varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_x}]\big\}\\ &+\|\mathbf{S}_1 f\|_{L^3_x}^{1/2}[\varepsilon\|wf\|_{L^\infty_{x,v}}]^{1/2}\|\mathbf{S}_1 g\|_{L^3_x},\end{aligned} \tag{2.5.1}$$

and

$$\begin{aligned}&\|\nu^{-\frac12}\Gamma_\pm(f,g)\|_{L^2_{x,v}} + \|\nu^{-\frac12}\Gamma_\pm(g,f)\|_{L^2_{x,v}}\\ \lesssim\, & \|f\|_{L^6_xL^2_v}\|\mathbf{S}_1 g\|_{L^3_x} + \varepsilon\|wf\|_{L^\infty_{x,v}}\big\{[\varepsilon^{-1}\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^2_{x,v}}] + [\varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_x}]\big\}.\end{aligned} \tag{2.5.2}$$

Proof By the decomposition

$$\begin{aligned}&\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(f,g)\|_{L^2_{x,v}} + \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(g,f)\|_{L^2_{x,v}}\\ \lesssim\, & \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})f|,|g|)\|_{L^2_{x,v}} + \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(|g|,|(\mathbf{I}-\mathbf{P})f|)\|_{L^2_{x,v}}\\ &+\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(|f|,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{x,v}} + \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})g|,|f|)\|_{L^2_{x,v}}\\ &+\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(\mathbf{S}_2 f\langle v\rangle^2\sqrt{\mu},|g|)\|_{L^2_{x,v}} + \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(|f|,\mathbf{S}_2 g\langle v\rangle^2\sqrt{\mu})\|_{L^2_{x,v}}\\ &+\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(|g|,\mathbf{S}_2 f\langle v\rangle^2\sqrt{\mu})\|_{L^2_{x,v}} + \|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(\mathbf{S}_2 g\langle v\rangle^2\sqrt{\mu},|f|)\|_{L^2_{x,v}}\\ &+\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(\mathbf{S}_1 f\langle v\rangle^2\sqrt{\mu},\mathbf{S}_1 g\langle v\rangle^2\sqrt{\mu})\|_{L^2_{x,v}}\\ &+\|\nu^{-\frac12}\varepsilon^{1/2}\Gamma_\pm(\mathbf{S}_1 g\langle v\rangle^2\sqrt{\mu},\mathbf{S}_1 f\langle v\rangle^2\sqrt{\mu})\|_{L^2_{x,v}}.\end{aligned} \tag{2.5.3}$$

The first two lines of the RHS of (2.5.3) are bounded by

$$\varepsilon^{3/2}\|wg\|_{L^\infty_{x,v}}\{\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})\|_{L^2_{x,v}}+\|\nu^{-1/2}\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|)\|_{L^2_{x,v}}\}$$
$$+\varepsilon^{3/2}\|wf\|_{L^\infty_{x,v}}\{\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|,w^{-1})\|_{L^2_{x,v}}+\|\nu^{-1/2}\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{x,v}}\}.$$

From $|v|^2+|u|^2=|v'|^2+|u'|^2$ and $\nu^{-1/2}|(v-u)\cdot\omega|\sqrt{\mu(u)}\lesssim\nu^{-1/2}[|v|+|u|]\sqrt{\mu(u)}\lesssim[1+|v|+|u|]^{\frac12}\mu(u)^{\frac12-}$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\nu^{-1}\big|\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})(v)\big|^2\mathrm{d}v+\int_{\mathbb{R}^3}\nu^{-1}\big|\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|)(v)\big|^2\mathrm{d}v\\
\lesssim&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v'|+|u'|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v')|^2w(u')^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v'|+|u'|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(u')|^2w(v')^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v|+|u|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v)|^2w(u)^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v|+|u|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(u)|^2w(v)^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v.\end{aligned}\tag{2.5.4}$$

Now by the change of variables $(v,u)\leftrightarrow(v',u')$ for the first term, $(v,u)\leftrightarrow(u',v')$ for the second term and $(v,u)\leftrightarrow(u,v)$ for the last term, we bound all the above terms as

$$\begin{aligned}&\int_{\mathbb{R}^3}\nu^{-1}\big|\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})\big|^2+\int_{\mathbb{R}^3}\nu^{-1}\big|\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|)\big|^2\\
\lesssim&\int_{\mathbb{R}^3}\Big[\iint_{\mathbb{R}^3\times\mathbb{S}^2}[1+|v|+|u|]w(u)^{-1}\mathrm{d}\omega\mathrm{d}u\Big]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v)|^2\mathrm{d}v\\
\lesssim&\int_{\mathbb{R}^3}\nu^{-1}|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v)|^2\mathrm{d}v.\end{aligned}\tag{2.5.5}$$

Similarly,

$$\begin{aligned}&\int_{\mathbb{R}^3}\nu^{-1}\big|\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|,w^{-1})(v)\big|^2\mathrm{d}v+\int_{\mathbb{R}^3}\nu^{-1}\big|\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|)(v)\big|^2\mathrm{d}v\\
\lesssim&\int_{\mathbb{R}^3}\nu^{-1}|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})g(v)|^2\mathrm{d}v.\end{aligned}\tag{2.5.6}$$

Therefore, the first two lines of the RHS of (2.5.3) are bounded by

$$\varepsilon^{3/2}\|wg\|_\infty\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\|_\nu+\varepsilon^{3/2}\|wf\|_\infty\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})g\|_\nu.$$

The third and fourth lines of the RHS of (2.5.3) are bounded by

$$\begin{aligned}&\varepsilon^{3/2}\|wg\|_{L^\infty_{x,v}}\|\varepsilon^{-1}\mathbf{S}_2f\|_{L^2_x}\|\nu^{-1/2}\Gamma_\pm(\nu^2\sqrt{\mu},w^{-1})\|_{L^2_v}\\
&+\varepsilon^{3/2}\|wf\|_{L^\infty_{x,v}}\|\varepsilon^{-1}\mathbf{S}_2g\|_{L^2_x}\|\nu^{-1/2}\Gamma_\pm(w^{-1},\nu^2\sqrt{\mu})\|_{L^2_v}\\
\lesssim&\,\varepsilon^{3/2}\|wg\|_{L^\infty_{x,v}}\|\varepsilon^{-1}\mathbf{S}_2f\|_{L^2_x}+\varepsilon^{3/2}\|wf\|_{L^\infty_{x,v}}\|\varepsilon^{-1}\mathbf{S}_2g\|_{L^2_x},\end{aligned}$$

where we have used

$$
\begin{aligned}
&\|\nu^{-1/2}\Gamma_{\pm}(\nu^2\sqrt{\mu},w^{-1})\|_{L^2_v}+\|\nu^{-1/2}\Gamma_{\pm}(w^{-1},\nu^2\sqrt{\mu})\|_{L^2_v}\\
\lesssim&\Big\|\nu^{-1/2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-u)\cdot\omega|w(v)^{-1}w(u)^{-1}\mathrm{d}\omega\mathrm{d}u\Big\|_{L^2_v}\lesssim 1.
\end{aligned}\tag{2.5.7}
$$

The last line of (2.5.3) is bounded as, by $\|\nu^{-1/2}\Gamma(\nu^2\sqrt{\mu},\nu^2\sqrt{\mu})\|_{L^2_v}<\infty$,

$$
\big\|\varepsilon^{1/2}\mathbf{S}_1f(x)\mathbf{S}_1g(x)\big\|_{L^2_x}\lesssim[\varepsilon^{1/2}\|\mathbf{S}_1f\|_{L^6_x}]\|\mathbf{S}_1g\|_{L^3_x}\lesssim[\varepsilon\|f\|_\infty]^{1/2}\|\mathbf{S}_1f\|^{1/2}_{L^3_x}\|\mathbf{S}_1g\|_{L^3_x}.
$$

All together we prove (2.5.1).

Now we prove (2.5.2). Using the decomposition of g, we conclude

$$
\begin{aligned}
&\|\nu^{-1/2}\Gamma(f,g)\|_{L^2_{x,v}}+\|\nu^{-1/2}\Gamma(g,f)\|_{L^2_{x,v}}\\
\le&\|\nu^{-1/2}\Gamma(f,\mathbf{S}_1g\langle v\rangle^2\sqrt{\mu})\|_{L^2_{x,v}}+\|\nu^{-1/2}\Gamma(f,\mathbf{S}_2g\langle v\rangle^2\sqrt{\mu})\|_{L^2_{x,v}}\\
&+\|\nu^{-1/2}\Gamma(f,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{x,v}}+\|\nu^{-1/2}\Gamma(\mathbf{S}_1g\langle v\rangle^2\sqrt{\mu},f)\|_{L^2_{x,v}}\\
&+\|\nu^{-1/2}\Gamma(\mathbf{S}_2g\langle v\rangle^2\sqrt{\mu},f)\|_{L^2_{x,v}}+\|\nu^{-1/2}\Gamma(|(\mathbf{I}-\mathbf{P})g|,f)\|_{L^2_{x,v}}\\
\lesssim&\big\|\|f\|_{L^2_v}|\mathbf{S}_1g|\big\|_{L^2_x}+\|wf\|_{L^\infty_{x,v}}\big\{\|\mathbf{S}_2g\|_{L^2_x}+\|(\mathbf{I}-\mathbf{P})g\|_\nu\big\}\\
\lesssim&\big\|\|f\|_{L^2_v}|\mathbf{S}_1g|\big\|_{L^2_x}+\varepsilon\|wf\|_{L^\infty_{x,v}}\big\{\varepsilon^{-1}\|\mathbf{S}_2g\|_{L^2_x}+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})g\|_\nu\big\}\\
\lesssim&\|f\|_{L^6_xL^2_v}\|\mathbf{S}_1g\|_{L^3_x}+\varepsilon\|wf\|_{L^\infty_{x,v}}\big\{\varepsilon^{-1}\|\mathbf{S}_2g\|_{L^2_x}+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})g\|_\nu\big\}.
\end{aligned}
$$

The proof is completed.

Lemma 2.11 *For some polynomial $P(s)=O(s)$ for $|s|\ll 1$,*

$$
\|r_s\|_{L^2(\gamma_-)}\le P\Big(\|u_s\|_{W^{2,\frac{3}{2}+}(\Omega)}+\|\vartheta_s\|_{H^{\frac{3}{2}+}(\Omega)}+\Big\|p_s-\fint p_s\Big\|_{H^{\frac{1}{2}+}(\Omega)}\Big).
$$

Moreover,

$$
\begin{aligned}
&\|f_1+\varepsilon f_2\|_{L^6_xL^2_v}\lesssim P_0+\varepsilon P_1,\qquad \|w[f_1+\varepsilon f_2]\|_\infty\lesssim P_1+\varepsilon P_2,\\
&\|(\mathbf{I}-\mathbf{P})A_s\|_{L^2(\Omega\times\mathbb{R}^3)}\lesssim P_1,\qquad \|w\mathbf{P}A_s\|_{L^2(\Omega\times\mathbb{R}^3)}\lesssim\varepsilon[1+\|\Phi\|_\infty]P_2,\\
&\|wA_s\|_{L^\infty(\Omega\times\mathbb{R}^3)}\lesssim P_3+\varepsilon[1+\|\Phi\|_\infty]P_2,\qquad \|wr_s\|_{L^\infty(\gamma_-)}\lesssim P_2,\\
&\|\mathcal{Q}R\|_{L^2(\gamma_-)}\lesssim\|\vartheta^w\|_{L^\infty(\partial\Omega)}\big[1+\varepsilon\|\vartheta^w\|_{L^\infty(\partial\Omega)}\big]\|\sqrt{\mu}R\|_{L^2(\gamma_+)},\\
&\|w\mathcal{Q}R\|_{L^\infty(\gamma_-)}\lesssim\|\vartheta^w\|_{L^\infty(\partial\Omega)}\big[1+\varepsilon\|\vartheta^w\|_{L^\infty(\partial\Omega)}\big]\|\sqrt{\mu}R\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)},
\end{aligned}
$$

where we have used notations, for some polynomial $P(s)=O(s)$ for $|s|\ll 1$,

$$
P_i:=\Big[P\Big(\|u_s\|_{H^{i+1}_x}+\|\vartheta_s\|_{H^{i+1}_x}+\Big\|p_s-\fint p_s\Big\|_{H^i_x}\Big)\Big]^2.\tag{2.5.8}
$$

Proof Note that from [13],

$$
|\mathcal{A}_{ij}|+|\mathcal{B}_i|\lesssim\langle v\rangle^{10}\sqrt{\mu}.
$$

Moreover, by $L^{-1}\{\Gamma(\mathbf{P}g_1,\mathbf{P}g_2)+\Gamma(\mathbf{P}g_2,\mathbf{P}g_1)\} = (\mathbf{I}-\mathbf{P})\big[(\mathbf{P}g_1\cdot\mathbf{P}g_2)\mu^{-1/2}\big]$, from [7],

$$|L^{-1}[\Gamma(f_1,f_1)]| \lesssim \big[|\rho|+|u|+|\vartheta|\big]\langle v\rangle^4\mu^{-1/2}.$$

From (1.3.3), (1.2.8), (1.2.9), and (1.3.1),

$$\begin{aligned}|r_s(x,v)| \lesssim& \big|\nabla_x u_s|_{\partial\Omega}\big| + \big|\nabla_x\vartheta_s|_{\partial\Omega}\big| + \left|\Big[p_s - \fint p_s\Big]\Big|_{\partial\Omega}\right| \\ &+O\bigg(\Big\|\vartheta^w - \frac{1}{|\Omega|}\int_\Omega\vartheta\Big\|_{L^\infty(\partial\Omega)}\|\vartheta^w\|_{L^\infty(\partial\Omega)}\bigg)\big[1+O(\varepsilon\|\vartheta^w\|_{L^\infty(\partial\Omega)})\big]\langle v\rangle^4\sqrt{\mu(v)}.\end{aligned}$$

By the trace theorem $H^{\frac12+}(\Omega)\hookrightarrow L^2(\partial\Omega)$ and the Sobolev embedding $W^{2,\frac32+}(\Omega)\hookrightarrow H^{\frac32+}(\Omega)$, we conclude the first estimate.

From (1.2.5) and (1.2.8),

$$|f_1|\lesssim\left[|\vartheta|+|u|+\left|\fint\vartheta\right|\right]\langle v\rangle^2\sqrt{\mu(v)},\quad |f_2|\lesssim P\bigg(|\nabla_x u|+|\nabla_x\vartheta|+|\vartheta|+\left|\fint\vartheta\right|\bigg)\mu(v)^{\frac12-}.$$

Due to our choices (1.2.9), (1.2.8),

$$\begin{aligned}|\mathbf{P}A_s| \lesssim& \,\varepsilon\|\Phi\|_\infty\big[|\rho_s|+|u_s|+|\vartheta_s|\big] \\ &+\varepsilon^2\|\Phi\|_\infty\left[|\nabla_x u_s|+|\nabla_x\vartheta_s|+\big(|\rho_s|+|u_s|+|\vartheta_s|\big)^2+\left|p_s-\fint p_s\right|+|\rho_s||\vartheta_s|\right] \\ \lesssim& \,\varepsilon P\bigg(|u_s|+|\nabla_x u_s|+|\vartheta_s|+|\nabla_x\vartheta_s|+\int_\Omega|\vartheta_s|+\left|p_s-\fint p_s\right|\bigg)\mu(v)^{\frac{1}{2+}},\end{aligned}$$

$$\begin{aligned}|(\mathbf{I}-\mathbf{P})A_s| \lesssim& \, P\bigg(\sum_{i=0}^{2}\big[|\nabla_x^i u_s|+|\nabla_x^i\vartheta_s|\big]+|\nabla_x p_s|+\left|p_s-\fint p_s\right|+\int_\Omega|\vartheta_s|\bigg) \\ &+P\bigg(|u_s|+|\nabla_x u_s|+|\vartheta_s|+|\nabla_x\vartheta_s|+\int_\Omega|\vartheta_s|+\left|p_s-\fint p_s\right|\bigg)\mu(v)^{\frac{1}{2+}} \\ &+\varepsilon P\bigg(\sum_{i=0}^{1}|\nabla_x^i u_s|+|\nabla_x^i\vartheta_s|+\int_\Omega|\theta_s|+\left|p_s-\fint p_s\right|\bigg).\end{aligned}$$

From (1.2.9), (1.2.8), and (1.3.4) we have

$$\begin{aligned}\mathcal{Q}R(x,v) =& \sqrt{2\pi}\Big(\frac{|v|^2}{2}-2\Big)\sqrt{\mu(v)}\vartheta^w(x)\int_{n(x)\cdot u>0}R(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &+\varepsilon O(|\vartheta^w|^2)\langle v\rangle^4\sqrt{\mu(v)}\int_{n(x)\cdot u>0}R(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned}$$

By the standard Sobolev embedding we prove the estimates. The proof is completed.

Now we are ready to prove the main theorem for the steady case:

Proof of Theorem 1.1 We prove Theorem 1.1 by considering a sequence R^ℓ, for $\ell \geq 0$,

$$v\cdot\nabla_x R^{\ell+1}+\varepsilon^2\Phi\cdot\nabla_v R^{\ell+1}+\frac{1}{\varepsilon}LR^{\ell+1}=\varepsilon^{1/2}\Gamma(R^\ell,R^\ell)+\frac{\varepsilon^2}{2}\Phi\cdot vR^{\ell+1}+L_1R^\ell+\varepsilon^{1/2}A_s,$$
$$R^{\ell+1}|_{\gamma_-}=P_\gamma R^{\ell+1}+\varepsilon\mathcal{Q}R^\ell+\varepsilon^{1/2}r_s,\quad R^0\equiv 0, \tag{2.5.9}$$

where L_1 is defined at (1.2.13), A_s at (1.2.14), $\mathcal{Q}$ at (1.3.6), and r_s at (1.3.3) with ρ,u,ϑ replaced by ρ_s,u_s,ϑ_s. Note that Proposition 2.2, with (1.2.15), (1.3.8), guarantees the solvability of such linear problem (2.5.9).

Step 1 From (1.4.4), (.1), and Lemma 2.11,

$$\|r_s\|_{L^2(\gamma_-)}+\|f_1+\varepsilon f_2\|_{L^6_xL^2_v}\lesssim\|\vartheta^w\|_{H^{1+}(\partial\Omega)}+\|\Phi\|_{L^{\frac{3}{2}+}(\Omega)}+\varepsilon P_1,$$

where P_1 is defined at (2.5.8).

For $0<\eta_0\ll 1$, we assume that (induction hypothesis), $\|\vartheta^w\|^2_{H^{1+}(\partial\Omega)}+\|\Phi\|^2_{L^{\frac{3}{2}+}(\Omega)}$ $<c_0\eta_0$ for $0<c_0\ll 1$ and $P_2,P_3<\infty$,

$$\sup_{0\leq j\leq\ell}\big\{\|\mathbf{P}R^j\|_2^2+\big[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R^j\|_\nu\big]^2+|R^j|_2^2+\|\mathbf{S}_1R^j\|^2_{L^3_x}+\big[\varepsilon\|wR^j\|_\infty\big]^2\big\}<\eta_0. \tag{2.5.10}$$

Now we claim the same bound for $j=\ell+1$.

By Proposition 2.2, (1.4.10), (2.2.4), and (2.5.1), for $\varepsilon\ll 1$,

$$\begin{aligned}
&\|\mathbf{P}R^{\ell+1}\|_2^2+\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})R^{\ell+1}\|_\nu^2+|R^{\ell+1}|_2^2\\
\lesssim&\big\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})\big[\varepsilon^{1/2}\Gamma(R^\ell,R^\ell)+L_1R^\ell+\varepsilon^{1/2}A_s\big]\big\|_2^2\\
&+\varepsilon^{-2}\big\|\mathbf{P}\big[\varepsilon^{1/2}\Gamma(R^\ell,R^\ell)+L_1R^\ell+\varepsilon^{1/2}A_s\big]\big\|_2^2+\varepsilon|\mathcal{Q}R^\ell|^2_{2,-}+|r_s|^2_{2,-}\\
\lesssim&\varepsilon\|\nu^{-\frac12}\Gamma(R^\ell,R^\ell)\|_2^2+\|\nu^{-\frac12}L_1R^\ell\|_2^2+\varepsilon\|(\mathbf{I}-\mathbf{P})A_s\|_2^2+\varepsilon^{-1}\|\mathbf{P}A_s\|_2^2+\varepsilon|R^\ell|^2_{2,+}+c_0\eta_0\\
\lesssim&\varepsilon[\varepsilon\|wR^\ell\|_\infty]^2[\varepsilon^{-1}\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})R^\ell\|_2]^2+[\varepsilon\|wR^\ell\|_\infty]\|\mathbf{S}_1R^\ell\|^3_{L^3_x}+\|f_1+\varepsilon f_2\|^2_{L^6_xL^2_v}\|\mathbf{S}_1\tilde R^\ell\|^2_{L^3_x}\\
&+\varepsilon^2\|w[f_1+\varepsilon f_2]\|^2_\infty[\varepsilon^{-1}\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})\tilde R^\ell\|_2]^2+\varepsilon P_1^2+\varepsilon P_2^2+\varepsilon\eta_0+c_0\eta_0\\
\lesssim&[1+O(\varepsilon^{1/2})]\eta_0^2+c_0\eta_0+\varepsilon P_1^2+\varepsilon P_2^2<\frac{\eta_0}{10}.
\end{aligned}$$

From Proposition 2.1, for $\varepsilon\ll 1$,

$$\begin{aligned}
\|\mathbf{S}_1R^{\ell+1}\|_{L^3(\Omega)}\lesssim&\|R^{\ell+1}\|_2^2+|R^{\ell+1}|_2^2+\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})R^{\ell+1}\|_\nu^2+\varepsilon\|\nu^{-\frac12}\Gamma(R^\ell,R^\ell)\|_2^2\\
&+\varepsilon^4\|\Phi\|^2_\infty\|R^{\ell+1}\|_2^2+\|\nu^{-\frac12}L_1R^\ell\|_2^2+\varepsilon\|A_s\|_2^2\\
\lesssim&\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})R^{\ell+1}\|_\nu^2+\varepsilon\|\nu^{-\frac12}\Gamma(R^\ell,R^\ell)\|_2^2+\|\nu^{-\frac12}L_1R^\ell\|_2^2\\
&+\varepsilon\|A_s\|_2^2+\varepsilon|R^\ell|^2_{2,+}+|r_s|^2_{2,-}<\frac{\eta_0}{10}.
\end{aligned}$$

From Proposition 2.3 and Lemma 2.11, for $\varepsilon\ll 1$,

$$
\begin{aligned}
\varepsilon\|wR^{\ell+1}\|_\infty \lesssim\,& \varepsilon^2 \sup_{0\le j\le \ell}\|w\mathcal{Q}R^j\|_\infty + \varepsilon^{3/2}\|wr_s\|_\infty + \varepsilon^{5/2}\|\nu^{-\frac12}w\Gamma(R^\ell,R^\ell)\|_\infty\\
&+\varepsilon^4\|\Phi\|_\infty\|R^{\ell+1}\|_\infty + \varepsilon^2\|\nu^{-\frac12}wL_1R^\ell\|_\infty + \varepsilon^{5/2}\|\nu^{-\frac12}wA_s\|_\infty\\
\lesssim\,& \varepsilon^2 \sup_{0\le j\le \ell}\|wR^j\|_\infty + \varepsilon^{3/2}P_2 + \varepsilon^{1/2}\big[\varepsilon\|wR^\ell\|_\infty\big]^2 + \varepsilon^4\|R^{\ell+1}\|_\infty\\
&+\varepsilon^2\|w^{-1}[f_1+\varepsilon f_2]\|_\infty\|wR^\ell\|_\infty + \varepsilon^{5/2}P_3 + \varepsilon^{7/2}P_2\\
\lesssim\,& \varepsilon(\eta_0)^{\frac12} + \varepsilon^{\frac12}\eta_0 + \varepsilon^3(\eta_0)^{\frac12} + \varepsilon(\eta_0)^{\frac12}(P_1+\varepsilon P_2) + O(\varepsilon^{3/2})P_2 + O(\varepsilon^{5/2})P_3\\
<\,& \sqrt{\frac{\eta_0}{10}}.
\end{aligned}
$$

Altogether we prove the uniform bound, (2.5.10) for all ℓ.

Step 2 We repeat Step 1 for $R^{\ell+1}-R^\ell$ to show that R^ℓ is Cauchy sequence in $L^\infty\cap L^2$ for fixed ε. Now it is standard to conclude that the limiting $R^\ell\to R$ solves the equation. The uniqueness is standard. (See [20] for the details)

The positivity $F_s\ge 0$ is left for the unsteady case in Section 3.7. The proof is completed.

3 Unsteady Problems

3.1 Trace and Green's Identity

Definition 3.1 Assume $\Phi=\Phi(x)\in C^1$. Consider a unsteady linear transport equation

$$\varepsilon\partial_t f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f = g. \tag{3.1.1}$$

The equations of the characteristics for (3.1.1) are

$$\dot Y = \varepsilon^{-1}W,\quad \dot W = \varepsilon\Phi(Y),\quad Y(t;t,x,v)=x,\quad W(t;t,x,v)=v. \tag{3.1.2}$$

By the uniqueness of ODE

$$
\begin{aligned}
[Y(s;t,x,v),W(s;t,x,v)] &= \Big[X\Big(t-\frac{t-s}{\varepsilon};t,x,v\Big),V\Big(t-\frac{t-s}{\varepsilon};t,x,v\Big)\Big]\\
&= [X(\varepsilon^{-1}s;0,x,v),V(\varepsilon^{-1}s;0,x,v)],
\end{aligned}\tag{3.1.3}
$$

where (X,V) is defined in (2.1.7).

Define

$$
\begin{aligned}
\tilde t_{\mathbf b}(x,v) &:= \sup\{t>0 : Y(-s;0,x,v)\in\Omega \text{ for all } 0<s<t\}\\
&= \varepsilon\sup\Big\{\frac{t}{\varepsilon}>0 : X\Big(-\frac{s}{\varepsilon};0,x,v\Big)\in\Omega \text{ for all } 0<\frac{s}{\varepsilon}<\frac{t}{\varepsilon}\Big\} = \varepsilon t_{\mathbf b}(x,v),\\
\tilde t_{\mathbf f}(x,v) &:= \sup\{t>0 : Y(s;0,x,v)\in\Omega \text{ for all } 0<s<t\}\\
&= \varepsilon\sup\Big\{\frac{t}{\varepsilon}>0 : X\Big(\frac{s}{\varepsilon};0,x,v\Big)\in\Omega \text{ for all } 0<\frac{s}{\varepsilon}<\frac{t}{\varepsilon}\Big\} = \varepsilon t_{\mathbf f}(x,v).
\end{aligned}\tag{3.1.4}
$$

Moreover

$$
\begin{aligned}
\tilde{x}_{\mathbf{b}}(x,v)&=Y(-\tilde{t}_{\mathbf{b}}(x,v);0,x,v)=X\Big(-\frac{\tilde{t}_{\mathbf{b}}(x,v)}{\varepsilon};0,x,v\Big)=X(-t_{\mathbf{b}}(x,v);0,x,v)=x_{\mathbf{b}}(x,v),\\
\tilde{x}_{\mathbf{f}}(x,v)&=Y(-\tilde{t}_{\mathbf{f}}(x,v);0,x,v)=X\Big(-\frac{\tilde{t}_{\mathbf{f}}(x,v)}{\varepsilon};0,x,v\Big)=X(-t_{\mathbf{f}}(x,v);0,x,v)=x_{\mathbf{f}}(x,v),\\
\tilde{v}_{\mathbf{b}}(x,v)&=W(-\tilde{t}_{\mathbf{b}}(x,v);0,x,v)=V\Big(-\frac{\tilde{t}_{\mathbf{b}}(x,v)}{\varepsilon};0,x,v\Big)=V(-t_{\mathbf{b}}(x,v);0,x,v)=v_{\mathbf{b}}(x,v),\\
\tilde{v}_{\mathbf{f}}(x,v)&=W(-\tilde{t}_{\mathbf{f}}(x,v);0,x,v)=V\Big(-\frac{\tilde{t}_{\mathbf{f}}(x,v)}{\varepsilon};0,x,v\Big)=V(-t_{\mathbf{f}}(x,v);0,x,v)=v_{\mathbf{f}}(x,v).
\end{aligned}
\tag{3.1.5}
$$

Lemma 3.1 *For* $f \in L^1([0,T]\times\Omega\times\mathbb{R}^3)$,

$$
\begin{aligned}
&\int_0^T\int_{\gamma_+^\delta}|f(t,x,v)|\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim \varepsilon\iint_{\Omega\times\mathbb{R}^3}|f(0,x,v)|\mathrm{d}v\mathrm{d}x+\varepsilon\int_0^T\iint_{\Omega\times\mathbb{R}^3}|f(t,x,v)|\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^T\iint_{\Omega\times\mathbb{R}^3}\big|[\varepsilon\partial_t f+v\cdot\nabla_x f+\varepsilon^2\Phi\cdot\nabla_v f](t,x,v)\big|\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.1.6}
$$

We refer to the proof of Lemma 3.2 in [21].

Lemma 3.2 *Assume* $\Phi\in C^1$. *Assume that* $f(t,x,v),\ g(t,x,v)\in L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$, $\{\partial_t+\varepsilon^{-1}v\cdot\nabla_x+\varepsilon\Phi\cdot\nabla_v\}f, \{\partial_t+\varepsilon^{-1}v\cdot\nabla_x+\varepsilon\Phi\cdot\nabla_v\}g\in L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$ *and* $f_\gamma, g_\gamma\in L^2(\mathbb{R}_+\times\gamma)$. *Then*

$$
\begin{aligned}
&\int_s^t\iint_{\Omega\times\mathbb{R}^3}\{\varepsilon\partial_t+v\cdot\nabla_x f+\varepsilon^2\Phi\cdot\nabla_v f\}g+\{\varepsilon\partial_t+v\cdot\nabla_x g+\varepsilon^2\Phi\cdot\nabla_v g\}f\,\mathrm{d}v\mathrm{d}x\mathrm{d}\tau\\
&=\varepsilon\iint_{\Omega\times\mathbb{R}^3}f(s,x,v)g(s,x,v)\mathrm{d}v\mathrm{d}x-\varepsilon\iint_{\Omega\times\mathbb{R}^3}f(t,x,v)g(t,x,v)\mathrm{d}v\mathrm{d}x\\
&\quad+\int_s^t\Big[\int_{\gamma_+}fg\mathrm{d}\gamma-\int_{\gamma_-}fg\mathrm{d}\gamma\Big]\mathrm{d}\tau.
\end{aligned}
$$

Proof The proof is from Chapter 9 of [17] with the same modification as Lemma 2.3.

3.2 Gain of Integrability: $L_x^3L_t^2$ Estimate

Definition 3.2 We define, for $(t,x,v)\in\mathbb{R}\times\bar{\Omega}\times\mathbb{R}^3$ and for $0<\delta\ll 1$,

$$
\begin{aligned}
f_\delta(t,x,v):=&\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\\
&\times\big\{\mathbf{1}_{t\in[0,\infty)}f(t,x,v)+\mathbf{1}_{t\in(-\infty,0]}\chi(t)f_0(x,v)\big\}.
\end{aligned}
\tag{3.2.1}
$$

Here $n(x)$ is defined in (2.1.5).

We extend f_δ to the negative time so that we are able to take the time-derivative. Clearly,

$$\|f_\delta\|_{L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)} + \|f_0\|_{L^2(\Omega\times\mathbb{R}^3)},$$
$$\|f_\delta\|_{L^2(\mathbb{R}\times\gamma)} \lesssim \|f_\gamma\|_{L^2(\mathbb{R}_+\times\gamma)} + \|f_0\|_{L^2(\gamma)}.$$

Note that, at the boundary $(x,v)\in\gamma := \partial\Omega\times\mathbb{R}^3$,

$$f_\delta(t,x,v)|_\gamma \equiv 0, \quad \text{for } |n(x)\cdot v|\le\delta \ \text{ or } \ |v|\ge\frac{1}{\delta}. \tag{3.2.2}$$

The main goal of this section is the following:

Proposition 3.1 *Assume $g\in L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$, $f_0\in L^2(\Omega\times\mathbb{R}^3)$, and $f_\gamma\in L^2(\mathbb{R}_+\times\gamma)$. Let $f\in L^\infty(\mathbb{R}_+;L^2(\Omega\times\mathbb{R}^3))$ solves* (3.1.1) *in the sense of distribution and satisfies $f(t,x,v)=f_\gamma(t,x,v)$ on $\mathbb{R}_+\times\gamma$ and $f(0,x,v)=f_0(x,v)$ on $\Omega\times\mathbb{R}^3$. Then*

$$\begin{aligned}
&|a(t,x)|+|b(t,x)|+|c(t,x)| \le \mathbf{S}_1 f(t,x)+\mathbf{S}_2 f(t,x),\\
&\mathbf{S}_1 f(t,x) := 4\int_{\mathbb{R}^3} |f_\delta(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v,\\
&\mathbf{S}_s f(t,x) := 4\int_{\mathbb{R}^3} |(\mathbf{I}-\mathbf{P})f(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v+2\chi(t)\int_{\mathbb{R}^3}|f_0(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v,
\end{aligned}\tag{3.2.3}$$

where f_δ is defined in (3.2.1).

Moreover

$$\begin{aligned}
&\|\mathbf{S}_1 f\|_{L^3_x L^2_t} \lesssim \|w^{-1}f\|_{L^2_{t,x,v}} + \|w^{-1}g\|_{L^2_{t,x,v}} + \|f\|_{L^2(\mathbb{R}_+\times\gamma)},\\
&\|\mathbf{S}_2 f\|_{L^2_{t,x}} \lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}} + \|f_0\|_{L^2_{x,v}},
\end{aligned}\tag{3.2.4}$$

for $w=\mathrm{e}^{\beta|v|^2}$ with $0<\beta\ll 1$.

We need several lemmas to prove Proposition 3.1.

Lemma 3.3 *Assume the same hypothesis of Proposition* 3.1. *Then there exist an $\bar f(t,x,v)\in L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ and an extension of f_δ in* (3.2.1), *such that*

$$\bar f|_{\Omega\times\mathbb{R}^3}\equiv f_\delta \quad \text{and} \quad \bar f|_\gamma \equiv f_\delta|_\gamma \quad \text{and} \quad \bar f|_{t=0}=f_\delta|_{t=0}.$$

Moreover, in the sense of distributions on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$,

$$\{\varepsilon\partial_t + v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\bar f = h = h_1+h_2+h_3+h_4, \tag{3.2.5}$$

where

$$\begin{aligned}
h_1(t,x,v) &= \mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\\
&\quad\times\Big[\mathbf{1}_{t\in[0,\infty)}g(t,x,v)+\mathbf{1}_{t\in(-\infty,0]}\chi(t)\Big\{\varepsilon\frac{\chi'(t)}{\chi(t)}+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\Big\}f_0(x,v)\Big],\\
h_2(t,x,v) &= \mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3}\big[\mathbf{1}_{t\in[0,\infty)}f(t,x,v)+\mathbf{1}_{t\in(-\infty,0]}\chi(t)f_0(x,v)\big]\\
&\quad\times\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\Big\{\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big\},
\end{aligned}$$

$$\begin{aligned} h_3(t,x,v) &= \mathbf{1}_{(x,v)\in[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}\ \frac{1}{\tilde{C}\delta^4} v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big) \\ &\quad\times\big[f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\ &\quad+f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big], \\ h_4(t,x,v) &= \mathbf{1}_{(x,v)\in[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) \\ &\quad\times\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}+\mathbf{1}_{(x,v)\in[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} \\ &\quad\times f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}, \end{aligned}$$

where $\Omega_{\tilde{C}\delta^4}, t^*_{\mathbf{b}}, x^*_{\mathbf{b}}, v^*_{\mathbf{b}}, t^*_{\mathbf{f}}, x^*_{\mathbf{f}}, v^*_{\mathbf{f}}$ *are defined in* (2.2.5), (2.2.6).

Moreover,

$$\begin{aligned} \|h_1\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} &\lesssim \|g\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)}+\varepsilon\|f_0\|_{L^2(\Omega\times\mathbb{R}^3)} \\ &\quad+\|[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v]f_0\|_{L^2(\Omega\times\mathbb{R}^3)}, \\ \|h_2\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} &\lesssim_\delta \|f\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)}+\|f_0\|_{L^2(\Omega\times\mathbb{R}^3)}, \\ \|h_3\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)}+\|h_4\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} &\lesssim_\delta \|f_\gamma\|_{L^2(\mathbb{R}_+\times\gamma)}+\|f_0\|_{L^2(\gamma)}. \end{aligned} \tag{3.2.6}$$

Proof Step 1 In the sense of distributions on $[0,\infty)\times\Omega\times\mathbb{R}^3$,

$$\begin{aligned} &\varepsilon\partial_t f_\delta+v\cdot\nabla_x f_\delta+\varepsilon^2\Phi\cdot\nabla_v f_\delta \\ &=\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|) \\ &\quad\times\Big[\mathbf{1}_{t\in[0,\infty)}g+\mathbf{1}_{t\in(-\infty,0]}\chi(t)\Big\{\varepsilon\frac{\chi'(t)}{\chi(t)}+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\Big\}f_0(x,v)\Big] \\ &\quad+\big[\mathbf{1}_{t\in[0,\infty)}f+\mathbf{1}_{t\in(-\infty,0]}\chi(t)f_0(x,v)\big]\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\} \\ &\quad\times\Big\{\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big\}. \end{aligned} \tag{3.2.7}$$

From Step 1 of the proof of Lemma 2.4 and (2.2.10), we prove the first and the second line of (3.2.6).

Step 2 We claim that if $0\le\xi(x)\le\tilde{C}\delta^4$, $|n(x)\cdot v|>\delta$ and $|v|\le\frac{1}{\delta}$ then either $\xi(\tilde{x}_{\mathbf{f}}(x,v))=\tilde{C}\delta^4$ or $\xi(\tilde{x}_{\mathbf{b}}(x,v))=\tilde{C}\delta^4$.

From Step 2 of the proof of Lemma 2.4,

$$\xi(Y(s;0,x,v))=\xi\Big(X\Big(\frac{s}{\varepsilon};0,x,v\Big)\Big)\ge\delta\frac{|s|}{2\varepsilon},$$

for all $0\le|s|\le\frac{\varepsilon\delta^3}{4(1+\|\xi\|_{C^2})}$ with $0<\varepsilon\ll1$. Especially with $\varepsilon s_*=+\frac{\varepsilon\delta^3}{4(1+\|\xi\|_{C^2})}$ for $n(x)\cdot v>\delta$ and $\varepsilon s_*=-\frac{\varepsilon\delta^3}{4(1+\|\xi\|_{C^2})}$ for $n(x)\cdot v<\delta$,

$$\xi(Y(s_*;0,x,v))>\tilde{C}\delta^4.$$

Therefore, by the intermediate value theorem, we prove our claim.

Step 3 We define $f_E(t,x,v)$ for $(x,v)\in[\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3$:

$$f_E(t,x,v):=\begin{cases} f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v),x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi\big(t^*_{\mathbf{b}}(x,v)\big), & \text{if } x^*_{\mathbf{b}}(x,v)\in\partial\Omega,\\ f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v),x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi\big(t^*_{\mathbf{f}}(x,v)\big), & \text{if } x^*_{\mathbf{f}}(x,v)\in\partial\Omega,\\ 0, & \text{if } x^*_{\mathbf{b}}(x,v)\notin\partial\Omega \text{ and } x^*_{\mathbf{f}}(x,v)\notin\partial\Omega. \end{cases} \tag{3.2.8}$$

We check that f_E is well-defined. It suffices to prove the following:

If $x^*_{\mathbf{b}}(x,v)\in\partial\Omega$ and $x^*_{\mathbf{f}}(x,v)\in\partial\Omega$, then

$$\begin{aligned}&f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v),x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\\ =&\,0=f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v),x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big).\end{aligned}$$

If $|n(x^*_{\mathbf{b}}(x,v))\cdot v^*_{\mathbf{b}}(x,v)|\le\delta$ or $|v^*_{\mathbf{b}}(x,v)|\ge\frac{1}{\delta}$ then $f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v),x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))=0$ due to (3.2.2). If $n(x^*_{\mathbf{b}}(x,v))\cdot v^*_{\mathbf{b}}(x,v)>\delta$ and $|v^*_{\mathbf{b}}(x,v)|\le\frac{1}{\delta}$ then, due to Step 2, $\xi(x^*_{\mathbf{f}}(x,v))=\xi(x^*_{\mathbf{f}}(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v)))=\tilde{C}\delta^4$ so that $x^*_{\mathbf{f}}(x,v)\notin\partial\Omega$.

On the other hand, if $|n(x^*_{\mathbf{f}}(x,v))\cdot v^*_{\mathbf{f}}(x,v)|\le\delta$ or $|v^*_{\mathbf{f}}(x,v)|\ge\frac{1}{\delta}$ then $f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v),x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))=0$ due to (3.2.2). If $n(x^*_{\mathbf{f}}(x,v))\cdot v^*_{\mathbf{f}}(x,v)<-\delta$ and $|v^*_{\mathbf{f}}(x,v)|\le\frac{1}{\delta}$ then, due to Step 2, $\xi(x^*_{\mathbf{b}}(x,v))=\xi(x^*_{\mathbf{b}}(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v)))=\tilde{C}\delta^4$ so that $x^*_{\mathbf{b}}(x,v)\notin\partial\Omega$.

Note that

$$f_E(t,x,v)=f_\delta(t,x,v)\quad\text{for all } x\in\partial\Omega. \tag{3.2.9}$$

If $x\in\partial\Omega$ and $n(x)\cdot v>\delta$ then $(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))=(x,v)$. From the definition (3.2.8), for those (x,v), we have $f_E(t,x,v)=f_\delta(t,x,v)$. If $x\in\partial\Omega$ and $n(x)\cdot v<-\delta$ then $(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x.v))=(x,v)$. From the definition (3.2.8), we conclude (3.2.9) again. Otherwise, if $-\delta<n(x)\cdot v<\delta$ then $f_E|_{\partial\Omega}\equiv0\equiv f_\delta|_{\partial\Omega}$.

Step 4 We claim that $f_E(x,v)\in L^2([\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3)$.

From the definition of (3.2.8), we have $f_E(x,v)\equiv0$ if $x^*_{\mathbf{b}}(x,v)\notin\partial\Omega$ and $x^*_{\mathbf{f}}(x,v)\notin\partial\Omega$. Therefore, from (3.2.8),

$$\begin{aligned}&\int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}|f_E(t,x,v)|^2\mathrm{d}x\mathrm{d}v\mathrm{d}t\\ =&\int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}|f_E|^2+\int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{f}}\in\partial\Omega}|f_E|^2\\ =&\int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}|f_\delta(t-\varepsilon t^*_{\mathbf{b}},x^*_{\mathbf{b}},v^*_{\mathbf{b}})|^2\Big|\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big|^2|\chi(t^*_{\mathbf{b}})|^2\mathrm{d}x\mathrm{d}v\mathrm{d}t \quad (3.2.10)\\ &+\int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3}\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}|f_\delta(t+\varepsilon t^*_{\mathbf{f}},x^*_{\mathbf{f}},v^*_{\mathbf{f}})|^2\Big|\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big|^2|\chi(t^*_{\mathbf{f}})|^2\mathrm{d}x\mathrm{d}v\mathrm{d}t, \quad (3.2.11)\end{aligned}$$

where $(t^*_{\mathbf{b}}, x^*_{\mathbf{b}}, v^*_{\mathbf{b}})$ and $(t^*_{\mathbf{f}}, x^*_{\mathbf{f}}, v^*_{\mathbf{f}})$ are evaluated at (x,v).

By (2.1.11),

$$\begin{aligned}
(3.2.10) &\leq \int_{-\infty}^{\infty} \mathrm{d}t \int_{\partial\Omega}\int_{n(x)\cdot v>0}\int_0^{\min\{t^*_{\mathbf{f}}(x,v),1\}} \mathrm{d}S_x \mathrm{d}v\mathrm{d}s \{|n(x)\cdot v| + O(\varepsilon)(1+|v|)s\} \\
&\quad \times \big|f_\delta\big(t-\varepsilon s, x^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v)), v^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v))\big)\big|^2 \\
&\leq \int_{-\infty}^{\infty} \mathrm{d}t \int_{\partial\Omega}\int_{n(x)\cdot v>0}\int_0^1 \big|f_\delta(t,x,v)\big|^2 \{|n(x)\cdot v| + O(\varepsilon)(1+|v|)s\}\mathrm{d}s\mathrm{d}v\mathrm{d}S_x \\
&\lesssim \int_{-\infty}^{\infty} \mathrm{d}t \int_{\partial\Omega}\int_{n(x)\cdot v>0} \big|f_\delta(t,x,v)\big|^2 |n(x)\cdot v|\mathrm{d}v\mathrm{d}S_x \lesssim \|f_\delta\|^2_{L^2(\mathbb{R}\times\partial\Omega\times\mathbb{R}^3)},
\end{aligned}$$

where we have used the fact, from (3.2.1), $O(\varepsilon)(1+|v|)|s| \leq O(\varepsilon)(1+\frac{1}{\delta}) \lesssim \delta \lesssim |n(x)\cdot v|$ for $(x,v)\in \mathrm{supp}(f_\delta)$, and, for $n(x)\cdot v>0$, $x\in\partial\Omega$, and $0\leq s\leq t^*_{\mathbf{f}}(x,v)$,

$$(x^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v)), v^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v))) = (x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) = (x,v),$$

and $t^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v)) = s$ and the change of variables $t-\varepsilon s\mapsto t$. Similarly we can show $(3.2.11) \lesssim \|f_\delta\|^2_{L^2(\mathbb{R}\times\partial\Omega\times\mathbb{R}^3)}$.

Step 5 We claim that, in the sense of distributions on $\mathbb{R}\times[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3$,

$$\begin{aligned}
&\varepsilon\partial_t f_E + v\cdot\nabla_x f_E + \varepsilon^2\Phi\cdot\nabla_v f_E \\
=\ & \frac{1}{\tilde{C}\delta^4} v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\big[f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&+ f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big] \\
&+ f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&- f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}.
\end{aligned} \tag{3.2.12}$$

Note that

$$\begin{aligned}
&[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) \\
=\ & \underbrace{[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v](t-\varepsilon t^*_{\mathbf{b}}(x,v))} \times \partial_t f(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) \\
&+ [v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(s, x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))|_{s=t-\varepsilon t^*_{\mathbf{b}}(x,v)}, \\
&[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) \\
=\ & \underbrace{[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v](t+\varepsilon t^*_{\mathbf{f}}(x,v))} \times \partial_t f(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) \\
&+ [v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(s, x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))|_{s=t+\varepsilon t^*_{\mathbf{f}}(x,v)}.
\end{aligned}$$

If the underbraced term vanishes then we can apply the Step 5 of the proof of Lemma 2.4 to conclude (3.2.12).

This is true because

$$\begin{aligned}[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v](t-\varepsilon t^*_{\mathbf{b}}(x.v)) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(t-\varepsilon t^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v)))\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(t-\varepsilon s)=-\varepsilon,\end{aligned}$$

and

$$\begin{aligned}[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v](t+\varepsilon t^*_{\mathbf{f}}(x.v)) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(t+\varepsilon t^*_{\mathbf{f}}(X(s;0,x,v),V(s;0,x,v)))\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(t-\varepsilon s+\varepsilon t^*_{\mathbf{f}}(x,v))=-\varepsilon.\end{aligned}$$

On the other hand, following the bounds of (3.2.10) and (3.2.11) in Step 4 we prove the third line of (3.2.6).

Step 6 We define $\bar f(t,x,v)$ for $(t,x,v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$:

$$\bar f(t,x,v):=f_\delta(t,x,v)\mathbf{1}_{(x,v)\in\bar\Omega\times\mathbb{R}^3}+f_E(t,x,v)\mathbf{1}_{(x,v)\in[\mathbb{R}^3\backslash\bar\Omega]\times\mathbb{R}^3}. \tag{3.2.13}$$

For $\phi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$, by Lemma 3.2,

$$\begin{aligned}&-\int_{-\infty}^{\infty}\mathrm{d}t\iint_{\mathbb{R}^3\times\mathbb{R}^3}\bar f(t,x,v)\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\ =&-\int_{-\infty}^{\infty}\mathrm{d}t\iint_{\Omega\times\mathbb{R}^3}f_\delta(t,x,v)\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\ &-\int_{-\infty}^{\infty}\mathrm{d}t\iint_{[\mathbb{R}^3\backslash\bar\Omega]\times\mathbb{R}^3}f_E(t,x,v)\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\ =&\int_{-\infty}^{\infty}\mathrm{d}t\int_\gamma f_\delta(t,x,v)\phi(t,x,v)\{n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\\ &+\int_{-\infty}^{\infty}\mathrm{d}t\int_\gamma f_E(t,x,v)\phi(t,x,v)\{-n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\\ &+\int_{-\infty}^{\infty}\mathrm{d}t\iint_{\Omega\times\mathbb{R}^3}\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f_\delta(t,x,v)\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\ &+\int_{-\infty}^{\infty}\mathrm{d}t\iint_{[\Omega_{\tilde C\delta^4}\backslash\bar\Omega]\times\mathbb{R}^3}\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f_E(x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v,\end{aligned}$$

where the contributions of $\{t=\infty\}$ and $\{t=-\infty\}$ vanish since $\phi(t)\in C_c^\infty(\mathbb{R})$.

From (3.2.9), the boundary contributions are cancelled:

$$\int_{-\infty}^{\infty}\int_\gamma f_\delta(t,x,v)\phi(t,x,v)\mathrm{d}\gamma\mathrm{d}t-\int_{-\infty}^{\infty}\int_\gamma f_E(t,x,v)\phi(t,x,v)\mathrm{d}\gamma\mathrm{d}t=0.$$

Further from (3.2.7) and (3.2.12), we prove that $\bar f$ solves (3.2.5) in the sense of distributions on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$. The proof is completed.

Recall $T > 0$ in (2.2.20). With such choice $T > 0$,

$$\bar{f}(t,x,v) = -\int_0^{\varepsilon T} \frac{1}{\varepsilon} h(t+s, Y(t+s;t,x,v), W(t+s;t,x,v))\mathrm{d}s$$
$$= -\int_0^{\varepsilon T} \frac{1}{\varepsilon} h\Big(t+s, X\Big(t+\frac{s}{\varepsilon};t,x,v\Big), V\Big(t+\frac{s}{\varepsilon};t,x,v\Big)\Big)\mathrm{d}s.$$

Note that, from (3.2.8),

$$\bar{f}(x,v) \equiv 0, \quad \text{for } \ \xi(x) > 2\tilde{C}\delta^4 \ \text{ or } \ |v| > 2\delta^{-1} \ \text{ or } \ |v| < \frac{\delta}{2}. \tag{3.2.14}$$

Therefore

$$|\bar{f}(t,x,v)| \le \frac{1}{\varepsilon}\int_0^{\varepsilon T} \mathbf{1}_{\frac{\delta}{2}\le|v|\le\frac{2}{\delta}} \Big| h\Big(t+s, X\Big(t+\frac{s}{\varepsilon};t,x,v\Big), V\Big(t+\frac{s}{\varepsilon};t,x,v\Big)\Big)\Big| \mathrm{d}s. \tag{3.2.15}$$

Definition 3.3 For fixed T in (2.2.20), $\delta > 0$ and a smooth function $\phi \in L^1(\mathbb{R}^3)$, we define the average operator S as

$$S(h)(t,x) = \frac{1}{\varepsilon}\int_0^{\varepsilon T}\int_{\frac{\delta}{2}\le|v|\le\frac{2}{\delta}} h\Big(t+s, X\Big(t+\frac{s}{\varepsilon};t,x,v\Big), V\Big(t+\frac{s}{\varepsilon};t,x,v\Big)\Big)\phi(v)\mathrm{d}v\mathrm{d}s$$
$$= \frac{1}{\varepsilon}\int_t^{t+\varepsilon T}\int_{\frac{\delta}{2}\le|v|\le\frac{2}{\delta}} h\Big(s, X\Big(\frac{s-t}{\varepsilon};0,x,v\Big), V\Big(\frac{s-t}{\varepsilon};0,x,v\Big)\Big)\phi(v)\mathrm{d}v\mathrm{d}s. \tag{3.2.16}$$

Lemma 3.4 *Assume that $\phi \in C^1(\mathbb{R}^3)$ is such that $|\phi(v)| \le \bar{\phi}(|v|)$ with $\bar{\phi} \in C^1(\mathbb{R})$ such that $\bar{\phi}'$ decays exponentially. Then*

$$\|Sh\|_{L^3_x L^2_t} \lesssim_{\phi,w} \|w^{-1}h\|_{L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)}, \tag{3.2.17}$$

where $w(v) = \mathrm{e}^{\beta|v|^2}$ with $0 \le \beta \ll 1$.

Proof We only prove (3.2.17) in the case of $\beta = 0$. For sufficiently small $0 < \beta \ll 1$, we can always absorb w growth by ϕ using $|V(\frac{s-t}{\varepsilon};0,x,v)| \lesssim |v| + \varepsilon T\|\Phi\|_\infty$.

We define the dual operator:

$$S^*(g)(t,x,v) := \frac{1}{\varepsilon}\int_{t-\varepsilon T}^{t} \mathbf{1}_{\frac{\delta}{2}\le|V(\frac{-t+s}{\varepsilon};0,x,v)|\le\frac{2}{\delta}}\, g\Big(s, X\Big(\frac{-t+s}{\varepsilon};0,x,v\Big)\Big)$$
$$\times\phi\Big(V\Big(\frac{-t+s}{\varepsilon};0,x,v\Big)\Big)\mathrm{d}s. \tag{3.2.18}$$

Consider the time-space inner product $(\cdot,\cdot)_{t,x}$,

$$
\begin{aligned}
(Sh,g)_{t,x} :=& \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} S(h)(t,x)g(t,x)\mathrm{d}x\mathrm{d}t \\
=& \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\int_t^{t+\varepsilon T}\mathrm{d}s\int_{\frac{\delta}{2}\le|v|\le\frac{2}{\delta}}\mathrm{d}v \\
&\times h\Big(s, X\Big(\frac{s-t}{\varepsilon};0,x,v\Big), V\Big(\frac{s-t}{\varepsilon};0,x,v\Big)\Big)\phi(v)g(t,x).
\end{aligned}
$$

From the change of variables $(X(\frac{s-t}{\varepsilon};0,x,v), V(\frac{s-t}{\varepsilon};0,x,v)) \mapsto (x,v)$ for all fixed $t,s\in\mathbb{R}$ and then $t\mapsto s$ and $s\mapsto t$,

$$
\begin{aligned}
(Sh,g)_{t,x} =& \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}s\int_{\mathbb{R}^3}\mathrm{d}x\int_{\frac{\delta}{2}\le|V(\frac{-s+t}{\varepsilon};0,x,v)|\le\frac{2}{\delta}}\mathrm{d}v \\
&\times \mathbf{1}_{t\le s\le t+\varepsilon T}\ h(s,x,v)\phi\Big(V\Big(\frac{-s+t}{\varepsilon};0,x,v\Big)\Big)g\Big(t, X\Big(\frac{-s+t}{\varepsilon};0,x,v\Big)\Big) \\
=& \frac{1}{\varepsilon}\iiint_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}v\mathrm{d}x\mathrm{d}t\ h(t,x,v) \\
&\times\int_{t-\varepsilon T}^{t}\mathbf{1}_{\frac{\delta}{2}\le|V(\frac{-t+s}{\varepsilon};0,x,v)|\le\frac{2}{\delta}}g\Big(s, X\Big(\frac{-t+s}{\varepsilon};0,x,v\Big)\Big)\phi\Big(V\Big(\frac{-t+s}{\varepsilon};0,x,v\Big)\Big)\mathrm{d}s \\
=& (h, S^*g)_{t,x,v}.
\end{aligned}
$$

As the steady case, note that

$$
\|Sh\|_{L^p_xL^2_t}\equiv\sup_{\|g\|_{L^{p'}_xL^2_t}\le1}(Sh,g)_{t,x}=\sup_{\|g\|_{L^{p'}_xL^2_t}\le1}(h,S^*g)_{t,x}\le\|h\|_{L^2_{t,x,v}}\sup_{\|g\|_{L^{p'}_xL^2_t}\le1}\|S^*g\|_{L^2_{t,x,v}}.
$$

Therefore in order to show $\|Sh\|_{L^p_xL^2_t}\le\|h\|_{L^2_{t,x,v}}$, we only need to show

$$
\|S^*g\|_{L^2_{t,x,v}}\le\|g\|_{L^{p'}_xL^2_t}.
$$

But $\|S^*g\|^2_{L^2_{t,x,v}} = (S^*g, S^*g) = (SS^*g, g) \le \|SS^*g\|_{L^p_x(L^2_t)}\|g\|_{L^{p'}_x(L^2_t)}$. Therefore it suffices to show that, SS^*h is bounded from $L^{3/2}_xL^2_t$ to $L^3_xL^2_t$:

$$
\|SS^*g\|_{L^3_xL^2_t}\lesssim\|g\|_{L^{3/2}_xL^2_t}. \tag{3.2.19}
$$

From the definition of S and S^*, for any $g\in L^{3/2}_xL^2_t$,

$$
\begin{aligned}
&SS^*(g)(t,x) \\
=& \frac{1}{\varepsilon^2}\int_t^{t+\varepsilon T}\mathrm{d}s\int_{s-\varepsilon T}^{s}\mathrm{d}\tau\int_{\frac{\delta}{2}\le|v|\le\frac{2}{\delta}}\mathrm{d}v\ \mathbf{1}_{\frac{\delta}{2}\le|V(-\frac{s-\tau}{\varepsilon};0,X(\frac{s-t}{\varepsilon};0,x,v),V(\frac{s-t}{\varepsilon};0,x,v))|\le\frac{2}{\delta}} \\
&\times g\Big(\tau, X\Big(\frac{s-\tau}{\varepsilon};0,X\Big(\frac{s-t}{\varepsilon};0,x,v\Big),V\Big(\frac{s-t}{\varepsilon};0,x,v\Big)\Big)\Big) \\
&\times\phi(v)\phi\Big(V\Big(\frac{s-\tau}{\varepsilon};0,X\Big(\frac{s-t}{\varepsilon};0,x,v\Big),V\Big(\frac{s-t}{\varepsilon};0,x,v\Big)\Big)\Big)
\end{aligned}
$$

$$= \frac{1}{\varepsilon^2}\int_t^{t+\varepsilon T} \mathrm{d}s \int_{s-\varepsilon T}^{s} \mathrm{d}\tau \int_{\frac{\delta}{2}\le |v|\le \frac{2}{\delta}} \mathrm{d}v$$
$$\times \mathbf{1}_{\frac{\delta}{2}\le |V(\frac{-\tau+t}{\varepsilon};0,x,v)|\le\frac{2}{\delta}} g\Big(\tau, X\Big(\frac{-\tau+t}{\varepsilon};0,x,v\Big)\Big)\phi(v)\phi\Big(V\Big(\frac{-\tau+t}{\varepsilon};0,x,v\Big)\Big), \tag{3.2.20}$$

where we have used the fact that since the characteristics equation is autonomous,

$$X\Big(\frac{s-\tau}{\varepsilon};0,X\Big(\frac{s-t}{\varepsilon};0,x,v\Big),V\Big(\frac{s-t}{\varepsilon};0,x,v\Big)\Big) = X\Big(\frac{-\tau+t}{\varepsilon};0,x,v\Big),$$
$$V\Big(\frac{s-\tau}{\varepsilon};0,X\Big(\frac{s-t}{\varepsilon};0,x,v\Big),V\Big(\frac{s-t}{\varepsilon};0,x,v\Big)\Big) = V\Big(\frac{-\tau+t}{\varepsilon};0,x,v\Big).$$

Recall the change of variables from (2.2.28)

$$v \mapsto y \equiv X\Big(\frac{-\tau+t}{\varepsilon};0,x,v\Big), \quad \mathrm{d}v \lesssim \frac{\varepsilon^3}{|t-\tau|^3}\mathrm{d}y.$$

On the other hand, $|v| = O(1)\frac{\varepsilon|y-x|}{|t-\tau|}$, $|V(s;t,x,v)| = O(1)\frac{\varepsilon|y-x|}{|t-\tau|}$. As the steady case we can reduce to $\phi(v)\phi(V(\frac{-\tau+t}{\varepsilon};0,x,v)) \sim \phi^2(C\varepsilon\frac{|x-y|}{|t-\tau|})$ for some $C>0$.

We define

$$M(t-\tau,x-y) := \frac{1}{\varepsilon}\frac{\varepsilon^3}{|t-\tau|^3}\phi^2\Big(C\varepsilon\frac{|y-x|}{|t-\tau|}\Big). \tag{3.2.21}$$

Then,

$$\begin{aligned}|SS^*g(t,x)| &\lesssim \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}\mathrm{d}y\int_t^{t+\varepsilon T}\mathrm{d}s\int_{s-\varepsilon T}^{s}\mathrm{d}\tau|g(\tau,y)|\phi^2\Big(O(\varepsilon)\frac{|y-x|}{|t-\tau|}\Big)\frac{\varepsilon^3}{|t-\tau|^3}\\ &\lesssim \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}\mathrm{d}y\int_{t-\varepsilon T}^{t+\varepsilon T}\mathrm{d}\tau\int_{\tau}^{\tau+\varepsilon T}\mathrm{d}s|g(\tau,y)|\phi^2\Big(O(\varepsilon)\frac{|y-x|}{|t-\tau|}\Big)\frac{\varepsilon^3}{|t-\tau|^3}\\ &\lesssim \int_{\mathbb{R}^3}\int_{t-\varepsilon T}^{t+\varepsilon T}|g(\tau,y)|M(t-\tau,x-y)\mathrm{d}\tau\mathrm{d}y.\end{aligned}$$

From (3.2.21),

$$\begin{aligned}\int_{-\varepsilon T}^{+\varepsilon T} M(\tau,x-y)\mathrm{d}\tau &\le \int_{-\varepsilon T}^{\varepsilon T}\frac{1}{\varepsilon}\frac{\varepsilon^3}{|\tau|^3}\phi^2\Big(\frac{\varepsilon|x-y|}{|\tau|}\Big)\mathrm{d}\tau\\ &\lesssim \int_0^\infty \frac{w}{|x-y|^2}\phi^2(w)\mathrm{d}w \lesssim \frac{1}{|x-y|^2},\end{aligned}$$

where we have used the change of variables $w=\frac{\varepsilon|x-y|}{|\tau|}$ with $\mathrm{d}\tau = \frac{\varepsilon|x-y|}{w^2}\mathrm{d}w$.

By weak Young's inequality

$$\begin{aligned}\big\|\|SS^*g\|_{L^2_t}\big\|_{L^3_x} &\lesssim \left\|\left\|\int_{\mathbb{R}^3}\int_{t-\varepsilon T}^{t+\varepsilon T}|g(\tau,y)|M(t-\tau,x-y)\mathrm{d}\tau\mathrm{d}y\right\|_{L^2_t(\mathbb{R})}\right\|_{L^3_x}\\ &=\left\|\left\|\int_{\mathbb{R}^3}\int_{-\varepsilon T}^{+\varepsilon T}|g(t-\tau,y)|M(\tau,x-y)\mathrm{d}\tau\mathrm{d}y\right\|_{L^2_t(\mathbb{R})}\right\|_{L^3_x}\\ &\lesssim\left\|\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\|g(\cdot,y)\|_{L^2_t}\mathrm{d}y\right\|_{L^3_x}\\ &\lesssim\left\|\frac{1}{|x|^2}\right\|_{L^{3/2}_w(\mathbb{R}^3)}\big\|\|g\|_{L^2_t(\mathbb{R})}\big\|_{L^{3/2}_x(\mathbb{R}^3)}\lesssim\|g\|_{L^{3/2}_xL^2_t}.\end{aligned}$$

The proof is completed.

Now we are ready to prove the main result of this section:

Proof of Proposition 3.1 Recall (1.2.4), (2.2.33) and the temporary notation above (2.2.33). From (3.2.1) and (2.2.34),

$$\begin{aligned}&\int_{\mathbb{R}^3}f_\delta(t,x,v)\zeta_i(v)\mathrm{d}v\\ =&\int_{\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\big\{\mathbf{1}_{t\geq 0}f(t,x,v)+\mathbf{1}_{t\leq 0}\chi(t)f_0(x,v)\big\}\zeta_i(v)\mathrm{d}v\\ =&\mathbf{1}_{t\geq0}\int_{\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\Big\{\sum_{j=0}^4 a_j(t,x)\zeta_j(v)+(\mathbf{I}-\mathbf{P})f(t,x,v)\Big\}\zeta_i(v)\mathrm{d}v\\ &+\mathbf{1}_{t\leq0}\int_{\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\chi(t)f_0(x,v)\zeta_i(v)\mathrm{d}v\\ =&\mathbf{1}_{t\geq0}\Big\{a_i(t,x)+O(\delta)\sum_{j=0}^4|a_j(t,x)|+O_\delta(1)\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(t,x,v)|\zeta_i(v)\mathrm{d}v\Big\}\\ &+\mathbf{1}_{t\leq0}\chi(t)\int_{\mathbb{R}^3}|f_0(x,v)|\zeta_i(v)\mathrm{d}v.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=0}^4\mathbf{1}_{t\geq0}|a_i(t,x)|\leq&\sum_{i=0}^4\left|\int_{\mathbb{R}^3}f_\delta(t,x,v)\zeta_i(v)\mathrm{d}v\right|+\mathbf{1}_{t\leq0}\chi(t)\int_{\mathbb{R}^3}|f_0(x,v)|\sum_{i=0}^4|\zeta_i(v)|\mathrm{d}v\\ &+\mathbf{1}_{t\geq0}\Big\{O(\delta)\sum_{j=0}^4|a_j(t,x)|+O_\delta(1)\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(t,x,v)|\sum_{i=0}^4|\zeta_i(v)|\mathrm{d}v\Big\}.\end{aligned}$$

Hence for all $i=0,1,2,3,4$,

$$\begin{aligned}|a_i(t,x)|\leq& 4\int_{\mathbb{R}^3}|f_\delta(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v\\ &+4\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v+4\chi(t)\int_{\mathbb{R}^3}|f_0(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v.\end{aligned}$$

Clearly, we can conclude the second estimate of (3.2.4).

Now we focus on the first estimate of (3.2.4). From Lemma 3.3,

$$\int_{\mathbb{R}^3} |f_\delta(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v \le \int_{\mathbb{R}^3} |\bar f(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v.$$

From (3.2.15) and (3.2.16) with $\phi(v) = \langle v\rangle^2\sqrt{\mu(v)}$, $\int_{\mathbb{R}^3} |\bar f(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v \lesssim S(h)(t,x)$. Finally, from Lemma 3.4 and (3.2.17), (3.2.6), we conclude the first estimate in (3.2.4). The proof is completed.

3.3 Unsteady L^2-Coercivity Estimate

The main purpose of this section is to prove the following:

Proposition 3.2 *Suppose* $\Phi = \Phi(x) \in C^1$, $g \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$, *and* $r \in L^2(\mathbb{R}_+ \times \gamma_-)$ *such that, for all* $t > 0$,

$$\iint_{\Omega\times\mathbb{R}^3} g(t,x,v)\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0 = \int_{\gamma_-} r(t,x,v)\sqrt{\mu}\mathrm{d}\gamma. \tag{3.3.1}$$

Then, for any sufficiently small ε, *there exists a unique solution to the problem*

$$\varepsilon\partial_t f + v\cdot\nabla_x f + \frac{1}{\sqrt{\mu}}\varepsilon^2\Phi\cdot\nabla_v(\sqrt{\mu}f) + \varepsilon^{-1}Lf = g, \tag{3.3.2}$$

with $f|_{t=0} = f_0$ *and* $f_- = P_\gamma f + r$ *on* $\mathbb{R}_+ \times \gamma_-$ *such that*

$$\iint_{\Omega\times\mathbb{R}^3} f(t,x,v)\sqrt{\mu}\mathrm{d}x\mathrm{d}v = 0, \quad \textit{for all } t \ge 0. \tag{3.3.3}$$

Moreover, there is $0 < \lambda \ll 1$ *such that for* $0 \le s \le t$,

$$\begin{aligned}&\|\mathrm{e}^{\lambda t}f(t)\|_2^2+\varepsilon^{-2}\int_s^t \|\mathrm{e}^{\lambda\tau}(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2\mathrm{d}\tau+\int_s^t \|\mathrm{e}^{\lambda\tau}\mathbf{P}f(\tau)\|_2^2\mathrm{d}\tau+\int_s^t |\mathrm{e}^{\lambda\tau}f|_2^2\\ &\lesssim \|\mathrm{e}^{\lambda s}f(s)\|_2^2+\varepsilon^{-1}\int_s^t |\mathrm{e}^{\lambda\tau}r|_{2,-}^2+\int_s^t \|\nu^{-\frac12}\mathrm{e}^{\lambda\tau}(\mathbf{I}-\mathbf{P})g\|_2^2+\varepsilon^{-2}\int_s^t \|\mathrm{e}^{\lambda\tau}\mathbf{P}g\|_2^2.\end{aligned} \tag{3.3.4}$$

We refer to Proposition 3.8 in [21].

3.4 L^∞ Estimate

The main goal of this section is to prove the following:

Proposition 3.3 *Let* f *satisfy*

$$\begin{aligned}&\big[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v + \varepsilon^{-1}C_0\langle v\rangle\big]|f| \le \varepsilon^{-1}K_\beta|f| + |g|,\\ &\big|f|_{\gamma_-}\big| \le P_\gamma|f| + |r|, \quad \big|f|_{t=0}\big| \le |f_0|.\end{aligned} \tag{3.4.1}$$

Then, for $w(v) = \mathrm{e}^{\beta'|v|^2}$ *with* $0 < \beta' \ll \beta$,

$$\begin{aligned}\|\varepsilon wf(t)\|_\infty \lesssim \|\varepsilon wf_0\|_\infty &+ \sup_{0\le s\le\infty}\|\varepsilon wr(s)\|_\infty + \varepsilon^2\sup_{0\le s\le\infty}\|\langle v\rangle^{-1}wg(s)\|_\infty\\ &+\sup_{0\le s\le t}\|\mathbf{S}_1 f(s)\|_{L^3(\Omega)} + \frac{1}{\varepsilon^{1/2}}\sup_{0\le s\le t}\|\mathbf{S}_2 f(s)\|_{L^2(\Omega)}\\ &+\frac{1}{\varepsilon^{1/2}}\sup_{0\le s\le t}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2(\Omega\times\mathbb{R}^3)},\end{aligned}\tag{3.4.2}$$

and

$$\begin{aligned}\|\varepsilon wf(t)\|_\infty \lesssim \|\varepsilon wf_0\|_\infty &+ \sup_{0\le s\le\infty}\|\varepsilon wr(s)\|_\infty + \varepsilon^2\sup_{0\le s\le\infty}\|\langle v\rangle^{-1}wg(s)\|_\infty\\ &+\frac{1}{\varepsilon^{1/2}}\sup_{0\le s\le t}\|f(s)\|_{L^2(\Omega\times\mathbb{R}^3)}.\end{aligned}\tag{3.4.3}$$

We define the stochastic cycles for the unsteady case. Note that from (3.1.5), $\tilde{x}_{\mathbf{b}}(x,v)=x_{\mathbf{b}}(x,v)$.

Definition 3.4 Define, for free variables $v_k\in\mathbb{R}^3$, from (3.1.5)

$$\begin{aligned}&\tilde{t}_1 = t-\tilde{t}_{\mathbf{b}}(x,v) = t-\varepsilon t_{\mathbf{b}}(x,v),\\ &\tilde{x}_1 = Y(\tilde{t}_1;t,x,v)=\tilde{x}_{\mathbf{b}}(x,v)=x_{\mathbf{b}}(x,v)=x_1,\\ &\tilde{t}_2 = t-\tilde{t}_{\mathbf{b}}(x,v)-\tilde{t}_{\mathbf{b}}(x_1,v_1) = t-\varepsilon t_{\mathbf{b}}(x,v)-\varepsilon t_{\mathbf{b}}(x_1,v_1),\\ &\tilde{x}_2 = Y(\tilde{t}_2;\tilde{t}_1,x_1,v_1)=\tilde{x}_{\mathbf{b}}(x_1,v_1)=x_{\mathbf{b}}(x_1,v_1)=x_2,\\ &\qquad\vdots\\ &\tilde{t}_{k+1} = \tilde{t}_k-\tilde{t}_{\mathbf{b}}(x_k,v_k)=\tilde{t}_k-\varepsilon t_{\mathbf{b}}(x_k,v_k),\\ &\tilde{x}_{k+1} = Y(\tilde{t}_{k+1};\tilde{t}_k,x_k,v_k)=\tilde{x}_{\mathbf{b}}(x_k,v_k)=x_{\mathbf{b}}(x_k,v_k)=x_{k+1}.\end{aligned}$$

and

$$\begin{aligned}t-\tilde{t}_1 &= \varepsilon t_{\mathbf{b}}(x,v)=\varepsilon(t-t_1),\\ t-\tilde{t}_2 &= \varepsilon t_{\mathbf{b}}(x,v)+\varepsilon t_{\mathbf{b}}(x_1,v_1)=\varepsilon(t-t_2),\\ &\;\;\vdots\\ t-\tilde{t}_k &= \varepsilon(t-t_k).\end{aligned}$$

Set

$$Y_{\mathbf{cl}}(s;t,x,v) := \sum_k \mathbf{1}_{[\tilde{t}_{k+1},\tilde{t}_k)}(s)Y(s;\tilde{t}_k,x_k,v_k),\quad W_{\mathbf{cl}}(s;t,x,v) := \sum_k \mathbf{1}_{[\tilde{t}_{k+1},\tilde{t}_k)}(s)v_k.$$

Clearly

$$[Y_{\mathbf{cl}}(s;t,x,v),W_{\mathbf{cl}}(s;t,x,v)] = \Big[X_{\mathbf{cl}}\Big(t-\frac{t-s}{\varepsilon};t,x,v\Big),V_{\mathbf{cl}}\Big(t-\frac{t-s}{\varepsilon};t,x,v\Big)\Big].\tag{3.4.4}$$

The following lemma is a generalized version of Lemma 23 of [32].

Lemma 3.5[32] *Assume $\Phi = \Phi(x) \in C^1$. For sufficiently large $T_0 > 0$, there exist constants $C_1, C_2 > 0$, independent of T_0, such that for $k = C_1 T_0^{5/4}$,*

$$\sup_{(t,x,v)\in[0,\varepsilon T_0]\times\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{\ell=1}^{k-1}\mathcal{V}_\ell} \mathbf{1}_{\tilde{t}_k(t,x,v_1,v_2,\cdots,v_{k-1})>0} \prod_{\ell=1}^{k-1} \mathrm{d}\sigma_\ell < \Big\{\frac{1}{2}\Big\}^{C_2 T_0^{5/4}}. \tag{3.4.5}$$

We refer to Lemma 3.12 in [21] for the proof.

Now we are ready to prove the main result of this section. We refer to the proof of Proposition 3.10 in [21] in page 80.

3.5 $L^p_x L^\infty_t$ and $L^\infty_t L^p_x$ Bounds and Estimates of the Collision Operators

Lemma 3.6 *For $g(t,x)$, $g(t,x)|_{t=0} = g_0(x)$, for $0 < \delta \ll 1$, and for all $T \in [0,\infty]$,*

$$\begin{aligned}
&\|g\|_{L^p_x L^\infty_t([0,T])} \lesssim \delta \|g_t\|_{L^p_x L^2_t([0,T])} + C_\delta \|g\|_{L^p_x L^2_t([0,T])} + \|g_0\|_{L^p_x},\\
&\|g\|_{L^\infty_t([0,T]) L^p_x} \lesssim \delta \|g_t\|_{L^p_x L^2_t([0,T])} + C_\delta \|g\|_{L^p_x L^2_t([0,T])} + \|g_0\|_{L^p_x}.
\end{aligned} \tag{3.5.1}$$

Note that this crucial estimate follows essentially by Sobolev imbedding in $L^\infty_t \subset C^0_t \subset H^1_t$.

Proof Note that

$$\begin{aligned}
|g(t,x)|^2 &\le |g(0,x)|^2 + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}|g(s,x)|^2 \mathrm{d}s\\
&\le |g(0,x)|^2 + 2\int_0^t |g(s,x) g_t(s,x)| \mathrm{d}s\\
&\le |g(0,x)|^2 + 2\Big[\int_0^t |g(s,x)|^2 \mathrm{d}s\Big]^{1/2}\Big[\int_0^t |\partial_t g(s,x)|^2 \mathrm{d}s\Big]^{1/2}\\
&\lesssim |g(0,x)|^2 + \Big[\int_0^T |g(s,x)|^2 \mathrm{d}s\Big] + o(1)\Big[\int_0^T |\partial_t g(s,x)|^2 \mathrm{d}s\Big]\\
&\lesssim |g(0,x)|^2 + \|g(\cdot,x)\|^2_{L^2([0,T])} + o(1)\|\partial_t g(\cdot,x)\|^2_{L^2([0,T])}.
\end{aligned} \tag{3.5.2}$$

We prove the first estimate of (3.5.1): Taking $L^\infty_t([0,T])$ and taking $\{\cdots\}^{1/2}$,

$$\|g(\cdot,x)\|_{L^\infty_t([0,T])} \lesssim |g_0(x)| + \|g(\cdot,x)\|_{L^2_t([0,T])} + o(1)\|g_t(\cdot,x)\|_{L^2_t([0,T])}.$$

By taking L^p_x-norm, we conclude the first bound.

Now we prove the second estimate of (3.5.1): From (3.5.2), for all $t \in [0,T]$,

$$|g(t,x)|^p \lesssim |g(0,x)|^p + \|g(\cdot,x)\|^p_{L^2_t([0,T])} + o(1)\|\partial_t g(\cdot,x)\|^p_{L^2_t([0,T])}.$$

Taking the integration over x and taking $\{\cdots\}^{1/p}$, for all $t \in [0,T]$,

$$\begin{aligned}\|g(t,\cdot)\|_{L^p_x} &\lesssim \|g(0,\cdot)\|_{L^p_x} + \big\|\|g\|_{L^2_t([0,T])}\big\|_{L^p_x} + o(1)\big\|\|g_t\|_{L^2_t([0,T])}\big\|_{L^p_x}\\ &\lesssim \|g_0\|_{L^p_x} + \|g\|_{L^p_x L^2_t([0,T])} + o(1)\|g_t\|_{L^p_x L^2_t([0,T])}.\end{aligned}$$

Finally taking $L^\infty-$norm in $t\in[0,T]$,

$$\|g\|_{L^\infty_t([0,T])L^3_x} = \big\|\|g(t,\cdot)\|_{L^p_x}\big\|_{L^\infty_t([0,T])} \lesssim \|g_0\|_{L^p_x} + \|g\|_{L^p_x L^2_t([0,T])} + o(1)\|g_t\|_{L^p_x L^2_t([0,T])}.$$

The proof is completed.

Lemma 3.7 *Assume*

$$|a_i(f)| \le \mathbf{S}_1 f(t,x) + \mathbf{S}_2 f(t,x), \quad |a_i(g)| \le \mathbf{S}_1 g(t,x) + \mathbf{S}_2 g(t,x),$$

where $a_i(f)$ *and* $a_i(g)$ *are defined in* (2.2.33). *Then*

$$\begin{aligned}&\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(g,f)\|_{L^2_{t,x,v}}\\ \lesssim&[\varepsilon^{3/2}\|wg\|_{L^\infty_{t,x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-1/2}(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}] + [\varepsilon^{-1}\|\mathbf{S}_2 f\|_{L^2_{t,x}}]\big\}\\ &+[\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}] + [\varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_{t,x}}]\big\}\\ &+\big\{\|\mathbf{S}_1 f\|^{1/2}_{L^3_x L^2_t} + \|\mathbf{S}_1\partial_t f\|^{1/2}_{L^3_x L^2_t} + \|\mathbf{S}_1 f(0)\|^{1/2}_{L^3_x}\big\}[\varepsilon\|wf\|_{L^\infty_{t,x,v}}]^{1/2}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}, \qquad (3.5.3)\\ &\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(g,f)\|_{L^2_{t,x,v}}\\ \le&\|\mathbf{S}_1 f\|^{1/2}_{L^3_x L^\infty_t}[\varepsilon\|wf\|_{L^\infty_{t,x,v}}]^{1/2}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}\\ &+[\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}]\big\{\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}} + \varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_{t,x}}\big\}\\ &+\varepsilon^{1/6}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}\big\{[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}L^\infty_t}]^{1/3} + [\varepsilon^{-1}\|\mathbf{S}_2 f\|_{L^2_x L^\infty_t}]^{1/3}\big\}[\varepsilon\|wf\|_{L^\infty_{t,x,v}}]^{2/3},\\ &\qquad (3.5.4)\end{aligned}$$

and

$$\begin{aligned}&\|\nu^{-1/2}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\Gamma_\pm(g,f)\|_{L^2_{t,x,v}}\\ \lesssim&\|f\|_{L^6_x L^\infty_t L^2_v}\|\mathbf{S}_1 g\|_{L^3_x L^2_t} + \varepsilon\|wf\|_{L^\infty_{t,x,v}}\big\{\varepsilon^{-1}\|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}} + \varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_{t,x}}\big\},\\ &\|\nu^{-1/2}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\Gamma_\pm(g,f)\|_{L^2_{t,x,v}}\\ \lesssim&\|f\|_{L^6_x L^2_{t,v}}\|\mathbf{S}_1 g\|_{L^3_x L^\infty_t} + \varepsilon\|wf\|_{L^\infty_{x,v}L^2_t}\big\{\varepsilon^{-1}\|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{x,v}L^\infty_t} + \varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_x L^\infty_t}\big\}.\\ &\qquad (3.5.5)\end{aligned}$$

Proof First we prove (3.5.3). We decompose

$$\begin{aligned}|f(t,x,v)| &\le |\mathbf{P}f(t,x,v)| + |(\mathbf{I}-\mathbf{P})f(t,x,v)|\\ &\le \mathbf{S}_1 f(t,x)\langle v\rangle^2\sqrt{\mu(v)} + \mathbf{S}_2 f(t,x)\langle v\rangle^2\sqrt{\mu(v)} + |(\mathbf{I}-\mathbf{P})f(t,x,v)|, \quad (3.5.6)\end{aligned}$$

and $|g(t,x,v)|$ in the same way. We use the same decomposition of (2.5.3) replacing the $L^2_{x,v}$ norm with the $L^2_{t,x,v}$ norm.

The first two lines of the RHS of (2.5.3) are bounded by

$$\varepsilon^{3/2}\|wg\|_{L^\infty_{t,x,v}}\{\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|)\|_{L^2_{t,x,v}}\}$$
$$+\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}\{\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|,w^{-1})\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma_\pm(w^{-1},\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{t,x,v}}\}.$$

From (2.5.4), (2.5.5), (2.5.6), and (2.5.7), the third and fourth line of the RHS of (2.5.3) are bounded by

$$\begin{aligned}&\varepsilon^{3/2}\|wg\|_{L^\infty_{t,x,v}}\|\varepsilon^{-1}\mathbf{S}_2 f\|_{L^2_{t,x}}\|\nu^{-1/2}\Gamma_\pm(\nu^2\sqrt{\mu},w^{-1})\|_{L^2_v}\\&+\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}\|\varepsilon^{-1}\mathbf{S}_2 g\|_{L^2_{t,x}}\|\nu^{-1/2}\Gamma_\pm(w^{-1},\nu^2\sqrt{\mu})\|_{L^2_v}\\ \lesssim{}&\varepsilon^{3/2}\|wg\|_{L^\infty_{t,x,v}}\|\varepsilon^{-1}\mathbf{S}_2 f\|_{L^2_{t,x}}+\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}\|\varepsilon^{-1}\mathbf{S}_2 g\|_{L^2_{t,x}}.\end{aligned}$$

The last line of (2.5.3) is bounded by, from $\|\nu^{-1/2}\Gamma(\nu^2\sqrt{\mu},\nu^2\sqrt{\mu})\|_{L^2_v}<\infty$,

$$\begin{aligned}\big\|\|\varepsilon^{1/2}\mathbf{S}_1 f(t,x)\mathbf{S}_1 g(t,x)\|_{L^2_t}\big\|_{L^2_x}&\lesssim\big\|\|\varepsilon^{1/2}\mathbf{S}_1 f\|_{L^\infty_t}\|\mathbf{S}_1 g\|_{L^2_t}\big\|_{L^2_x}\\&\lesssim[\varepsilon^{1/2}\|\mathbf{S}_1 f\|_{L^6_xL^\infty_t}]\|\mathbf{S}_1 g\|_{L^3_xL^2_t}.\end{aligned}$$

From (3.5.1),

$$\begin{aligned}\|\mathbf{S}_1 f\|_{L^6_xL^\infty_t}&=\big\|\|\mathbf{S}_1 f\|_{L^\infty_t}\big\|_{L^6_x}\\&\lesssim\big\|\|\mathbf{S}_1 f\|_{L^\infty_t}\big\|^{1/2}_{L^\infty_x}\big\|\|\mathbf{S}_1 f\|_{L^\infty_t}\big\|^{1/2}_{L^3_x}\lesssim\|\mathbf{S}_1 f\|^{1/2}_{L^\infty_{t,x}}\|\mathbf{S}_1 f\|^{1/2}_{L^3_xL^\infty_t}\\&\lesssim\|\mathbf{S}_1 f\|^{1/2}_{L^\infty_{t,x}}\{\|\mathbf{S}_1 f\|_{L^3_xL^2_t}+\|\partial_t[\mathbf{S}_1 f]\|_{L^3_xL^2_t}+\|\mathbf{S}_1 f(0)\|_{L^3_x}\}^{1/2}\\&\lesssim\|wf\|^{1/2}_{L^\infty_{t,x,v}}\{\|\mathbf{S}_1 f\|_{L^3_xL^2_t}+\|\partial_t[\mathbf{S}_1 f]\|_{L^3_xL^2_t}+\|\mathbf{S}_1 f(0)\|_{L^3_x}\}^{1/2}.\end{aligned}$$

We only need to show $\partial_t[\mathbf{S}_1 f]\lesssim\mathbf{S}_1\partial_t f$ for $t\geq 0$. From the definition of $\mathbf{S}_1 f(t,x)$ in (3.2.3),

$$\partial_t[\mathbf{S}_1 f(t,x)]=2\int_{\mathbb{R}^3}\mathrm{sgn}(f_\delta(t,x,v))\partial_t f_\delta(t,x,v)\nu^2\sqrt{\mu(v)}\mathrm{d}v.$$

Now from the definition of f_δ in (3.2.1), for $t\geq 0$

$$\begin{aligned}&\partial_t f_\delta(t,x,v)|_{t\geq 0}\\&=\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\big\{\mathbf{1}_{t\in[0,\infty)}\partial_t f(t,x,v)+\mathbf{1}_{t\in(-\infty,0]}\chi'(t)f_0(x,v)\big\}\big|_{t\geq 0}\\&=\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\mathbf{1}_{t\in[0,\infty)}\partial_t f(t,x,v)\\&=\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\big\{\mathbf{1}_{t\in[0,\infty)}\partial_t f(t,x,v)+\mathbf{1}_{t\in(-\infty,0]}\chi(t)\partial_t f_0(x,v)\big\}\big|_{t\geq 0}\\&=[\partial_t f]_\delta(t,x,v)|_{t\geq 0}.\end{aligned}$$

Therefore, for $t\geq 0$,

$$\begin{aligned}\partial_t[\mathbf{S}_1 f(t,x)] &\le 2\int_{\mathbb{R}^3} |\partial_t f_\delta(t,x,v)|\nu^2\sqrt{\mu(v)}\mathrm{d}v \\ &\le 2\int_{\mathbb{R}^3} |[\partial_t f]_\delta(t,x,v)|\nu^2\sqrt{\mu(v)}\mathrm{d}v = \mathbf{S}_1\partial_t f(t,x).\end{aligned}$$

All together we prove (3.5.3).

Now we prove (3.5.4). Using (3.5.6), we again decompose

$$\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(g,f)\|_{L^2_{t,x,v}} \lesssim (2.5.3).$$

We use the same upper bound as in the proof of (3.5.3) except the first line of RHS in (2.5.3) and the first term of the third and fourth lines of RHS in (2.5.3):

$$\begin{aligned}&\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|f|,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})g|,|f|)\|_{L^2_{t,x,v}} \\ &+\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|f|,\mathbf{S}_2 g\nu^2\sqrt{\mu})\|_{L^2_{t,x,v}} + \|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(\mathbf{S}_1 f\nu^2\sqrt{\mu},\mathbf{S}_1 g\nu^2\sqrt{\mu})\|_{L^2_{t,x,v}} \\ &+\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(\mathbf{S}_1 g\nu^2\sqrt{\mu},\mathbf{S}_1 f\nu^2\sqrt{\mu})\|_{L^2_{t,x,v}} \\ \lesssim\, &\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}\|\varepsilon^{-1}\mathbf{S}_2 g\|_{L^2_{t,x}} + \varepsilon^{1/2}\|\mathbf{S}_1 f\|_{L^6_x L^\infty_t}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}.\end{aligned}$$

By Hölder inequality,

$$\|\mathbf{S}_1 f\|_{L^6_x L^\infty_t} \lesssim \|\mathbf{S}_1 f\|^{1/2}_{L^3_x L^\infty_t}\|\mathbf{S}_1 f\|^{1/2}_{L^\infty_{t,x}} \lesssim \|\mathbf{S}_1 f\|^{1/2}_{L^3_x L^\infty_t}\|wf\|^{1/2}_{L^\infty_{t,x,v}}.$$

First we focus on the first line of the RHS of (2.5.3). Using the decomposition of g in (3.5.6), these terms are bounded by

$$\begin{aligned}&\big\|\mathbf{S}_1 g\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})f|,\nu^2\sqrt{\mu})\|_{L^2_v}\big\|_{L^2_{t,x}} + \big\|\mathbf{S}_1 g\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(\nu^2\sqrt{\mu},|(\mathbf{I}-\mathbf{P})f|)\|_{L^2_v}\big\|_{L^2_{t,x}} \\ &+\big\|\mathbf{S}_2 g\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})f|,\nu^2\sqrt{\mu})\|_{L^2_v}\big\|_{L^2_{t,x}} + \big\|\mathbf{S}_2 g\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(\nu^2\sqrt{\mu},|(\mathbf{I}-\mathbf{P})f|)\|_{L^2_v}\big\|_{L^2_{t,x}} \\ &+\big\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})f|,|(\mathbf{I}-\mathbf{P})g|)\big\|_{L^2_{t,x,v}} + \big\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})g|,|(\mathbf{I}-\mathbf{P})f|)\big\|_{L^2_{t,x,v}}.\end{aligned} \tag{3.5.7}$$

Note, for $0 \le \beta < \frac{1}{4}$,

$$|\mathrm{e}^{\beta|v|^2}\mathbf{P}f(t,x,v)| \lesssim \|\mathrm{e}^{\beta|v|^2}f(t,x,\cdot)\|_{L^\infty_v}\mathrm{e}^{\beta|v|^2}\nu^2\sqrt{\mu(v)}$$

and

$$\begin{aligned}|\mathrm{e}^{\beta|v|^2}(\mathbf{I}-\mathbf{P})f(t,x,v)| &= \mathrm{e}^{\beta|v|^2}|f(t,x,v)-\mathbf{P}f(t,x,v)| \\ &\lesssim \|\mathrm{e}^{\beta|v|^2}f\|_{L^\infty_{t,x,v}}[1+\mathrm{e}^{\beta|v|^2}\nu^2\sqrt{\mu(v)}] \lesssim \|\mathrm{e}^{\beta|v|^2}f\|_{L^\infty_{t,x,v}}.\end{aligned}$$

From the above estimate and (2.5.4), the last two lines of (3.5.7) are bounded by

$$\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}} \times \big\{\varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_{t,x}} + \varepsilon^{-1}\|\nu^{-1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big\}.$$

Focus on the first line of (3.5.7). For $0 < a \ll 1$,

$$
\begin{aligned}
&\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_{\pm}(|(\mathbf{I}-\mathbf{P})f|,\nu^2\sqrt{\mu})\|_{L^2_v}^2 + \|\nu^{-1/2}\varepsilon^{1/2}\Gamma_{\pm}(\nu^2\sqrt{\mu},|(\mathbf{I}-\mathbf{P})f|)\|_{L^2_v}^2 \\
\lesssim\ & \varepsilon\int_v\int_u\int_\omega |v-u|^2|w^a(\mathbf{I}-\mathbf{P})f(v')|^2 w(v')^{-2a}w(u')^{-2a}\sqrt{\mu(u)} \\
&+\varepsilon\int_v\int_u\int_\omega |v-u|^2|w^a(\mathbf{I}-\mathbf{P})f(v)|^2 w(v)^{-2a}w(u)^{-2a}\sqrt{\mu(u)} \\
&+\varepsilon\int_v\int_u\int_\omega |v-u|^2|w^a(\mathbf{I}-\mathbf{P})f(u')|^2 w(u')^{-2a}w(v')^{-2a}\sqrt{\mu(u)} \\
&+\varepsilon\int_v\int_u\int_\omega |v-u|^2|w^a(\mathbf{I}-\mathbf{P})f(u)|^2 w(u)^{-2a}w(v)^{-2a}\sqrt{\mu(u)}.
\end{aligned}
$$

Now by the change of variables $(v,u)\leftrightarrow(v',u')$ for the first term, $(v,u)\leftrightarrow(u',v')$ for the third term, the above terms are bounded by

$$
\begin{aligned}
&\varepsilon\int_v\int_u\int_\omega [1+|v|^2+|u|^2]|w^a(\mathbf{I}-\mathbf{P})f(v)|^2 w(v)^{-2a}w(u)^{-2a}\sqrt{\mu(u)} \\
\lesssim\ &\varepsilon\Big[\int_v |w^a(\mathbf{I}-\mathbf{P})f(v)|^6\Big]^{1/3}.
\end{aligned}
$$

Then, by the Hölder inequality ($\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$), the first line of (3.5.7) is bounded by

$$
\varepsilon^{1/2}\big\|\mathbf{S}_1 g\|w^a(\mathbf{I}-\mathbf{P})f\|_{L^6_v}\big\|_{L^2_{t,x}} \lesssim \varepsilon^{1/2}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}\|w^a(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}}.
$$

Here, for $0 < a \ll 1$,

$$
\begin{aligned}
\|w^a(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}} &\lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}^{1/3}\|w^{\frac{3a}{2}}f\|_{L^\infty_{x,v}}^{2/3} \\
&\lesssim \varepsilon^{-1/3}[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}]^{1/3}[\varepsilon\|wf\|_{L^\infty_{x,v}}]^{2/3}.
\end{aligned}
$$

Hence the first line of (3.5.7) is bounded by

$$
\varepsilon^{1/6}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}]^{1/3}[\varepsilon\|wf\|_{L^\infty_{x,v}}]^{2/3}.
$$

Therefore, altogether, the first line of RHS of (2.5.3) is bounded by

$$
\begin{aligned}
&\varepsilon^{3/2}\|wf\|_{L^\infty_{t,x,v}}\times\big\{\varepsilon^{-1}\|\mathbf{S}_2 g\|_{L^2_{t,x}}+\varepsilon^{-1}\|\nu^{-1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big\} \\
&+\varepsilon^{1/6}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}L^\infty_t}]^{1/3}[\varepsilon\|wf\|_{L^\infty_{t,x,v}}]^{2/3}.
\end{aligned}
$$

Similarly the first term of the third and fourth lines of RHS in (2.5.3) is bounded as

$$
\|\nu^{-1/2}\varepsilon^{1/2}\Gamma_{\pm}(\mathbf{S}_2 f\nu^2\sqrt{\mu},|g|)\|_{L^2_{t,x,v}} \lesssim \varepsilon^{1/6}[\varepsilon^{-1}\|\mathbf{S}_2 f\|_{L^2_x L^\infty_t}]^{1/3}[\varepsilon\|wf\|_{L^\infty_{t,x,v}}]^{2/3}\|\mathbf{S}_1 g\|_{L^3_x L^2_t}.
$$

All together we prove (3.5.4).

Now we prove (3.5.5). Using the decomposition of g,

$$
\begin{aligned}
&\|\nu^{-1/2}\Gamma(f,g)\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma(g,f)\|_{L^2_{t,x,v}}\\
\le&\|\nu^{-1/2}\Gamma(f,\mathbf{S}_1g\nu^2\sqrt{\mu})\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma(f,\mathbf{S}_2g\nu^2\sqrt{\mu})\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma(f,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{t,x,v}}\\
&+\|\nu^{-1/2}\Gamma(\mathbf{S}_1g\nu^2\sqrt{\mu},f)\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma(\mathbf{S}_2g\nu^2\sqrt{\mu},f)\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma(|(\mathbf{I}-\mathbf{P})g|,f)\|_{L^2_{t,x,v}}\\
\lesssim&\big\|\|f\|_{L^2_v}|\mathbf{S}_1g|\big\|_{L^2_{t,x}}+\|wf\|_{L^\infty_{t,x,v}}\big\{\|\mathbf{S}_2g\|_{L^2_{t,x}}+\|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big\}\\
\lesssim&\big\|\|f\|_{L^\infty_tL^2_v}\|\mathbf{S}_1g\|_{L^2_t}\big\|_{L^2_x}+\varepsilon\|wf\|_{L^\infty_{t,x,v}}\big\{\varepsilon^{-1}\|\mathbf{S}_2g\|_{L^2_{t,x}}+\varepsilon^{-1}\|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big\}\\
\lesssim&\|f\|_{L^6_xL^\infty_tL^2_v}\|\mathbf{S}_1g\|_{L^3_xL^2_t}+\varepsilon\|wf\|_{L^\infty_{t,x,v}}\big\{\varepsilon^{-1}\|\mathbf{S}_2g\|_{L^2_{t,x}}+\varepsilon^{-1}\|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big\},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\|\nu^{-1/2}\Gamma(f,g)\|_{L^2_{t,x,v}}+\|\nu^{-1/2}\Gamma(g,f)\|_{L^2_{t,x,v}}\\
\lesssim&\|f\|_{L^6_xL^2_{t,v}}\|\mathbf{S}_1g\|_{L^3_xL^\infty_t}+\varepsilon\|wf\|_{L^\infty_{x,v}L^2_t}\big\{\varepsilon^{-1}\|\mathbf{S}_2g\|_{L^2_xL^\infty_t}+\varepsilon^{-1}\|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_{x,v}L^\infty_t}\big\}.
\end{aligned}
$$

The proof is completed.

3.6 Global-in-Time Validity

The main purpose of this section is to prove Theorem 1.2. To that we need the following:

Lemma 3.8 *For $0<\lambda<\lambda'$ and $w=\mathrm{e}^{\beta|v|^2}$ with $0<\beta\ll 1$,*

$$
\begin{aligned}
&\|\mathrm{e}^{\lambda t}w[\tilde f_1+\varepsilon\tilde f_2]\|_{L^6_xL^\infty_tL^\infty_v}+\|\mathrm{e}^{\lambda t}w[\partial_t\tilde f_1+\varepsilon\partial_t\tilde f_2]\|_{L^6_xL^2_tL^\infty_v}\lesssim P_{1,0}+\varepsilon P_1,\\
&||\mathrm{e}^{\lambda t}w[\tilde f_1+\varepsilon\tilde f_2]||_{L^\infty_{x,v}L^2_t\cap L^\infty_{t,x,v}}+||\mathrm{e}^{\lambda t}w[\partial_t\tilde f_1+\varepsilon\partial_t\tilde f_2]||_{L^\infty_{x,v}L^2_t\cap L^\infty_{t,x,v}}\lesssim P_1+\varepsilon P_2,
\end{aligned}\tag{3.6.1}
$$

and

$$
\begin{aligned}
&\|\mathrm{e}^{\lambda t}\tilde r\|_{L^2_tL^2_{\gamma_-}}\lesssim P_{0,1},\quad \|\mathrm{e}^{\lambda t}\partial_t\tilde r\|_{L^2_tL^2_{\gamma_-}}\lesssim P_{1,1},\quad \|w\mathrm{e}^{\lambda t}\tilde A_1\|_{L^2_{t,x,v}}\lesssim P_1\times P_{0,1},\\
&\|w\mathrm{e}^{\lambda t}\partial_t\tilde A_1\|_{L^2_{t,x,v}}\lesssim P_2\times P_{1,1},\quad \|w\mathrm{e}^{\lambda t}\tilde A_2\|_{L^2_{t,x,v}}\lesssim P_1\times P_{1,0},\quad \|w\mathrm{e}^{\lambda t}\partial_t\tilde A_2\|_{L^2_{t,x,v}}\lesssim P_2\times P_{2,0},
\end{aligned}\tag{3.6.2}
$$

where

$$
\begin{aligned}
P_1:=&P\big(\|\mathrm{e}^{\lambda' t}\tilde u\|_{L^\infty_tH^4_x}+\|\mathrm{e}^{\lambda' t}\tilde\vartheta\|_{L^\infty_tH^4_x}+\|\mathrm{e}^{\lambda' t}\tilde p\|_{L^\infty_tH^3_x}+\|\mathrm{e}^{\lambda' t}\partial_t\tilde u\|_{L^\infty_tH^3_x}\\
&+\|\mathrm{e}^{\lambda' t}\partial_t\tilde\vartheta\|_{L^\infty_tH^3_x}+\|\mathrm{e}^{\lambda' t}\partial_t\tilde p\|_{L^\infty_tH^2_x}+\|\tilde u_s\|_{H^3_x}+\|\tilde\vartheta_s\|_{H^3_x}\big),\\
P_2:=&P\big(\|\mathrm{e}^{\lambda' t}\tilde u\|_{C^1_tH^4_x}+\|\mathrm{e}^{\lambda' t}\tilde\vartheta\|_{C^1_tH^4_x}+\|\mathrm{e}^{\lambda' t}\tilde p\|_{C^1_tH^3_x}+\|\mathrm{e}^{\lambda' t}\partial_t\tilde u\|_{C^1_tH^3_x}\\
&+\|\mathrm{e}^{\lambda' t}\partial_t\tilde\vartheta\|_{C^1_tH^3_x}+\|\mathrm{e}^{\lambda' t}\partial_t\tilde p\|_{C^1_tH^2_x}+\|\tilde u_s\|_{H^3_x}+\|\tilde\vartheta_s\|_{H^3_x}\big),\\
P_{i,j}:=&P\Big(\|\mathrm{e}^{\lambda' t}\tilde u\|_{H^i_tH^{j+1}_x}+\|\mathrm{e}^{\lambda' t}\tilde\vartheta\|_{H^i_tH^{j+1}_x}+\|\mathrm{e}^{\lambda' t}\Big[\tilde p-\fint\tilde p\Big]\|_{H^i_tH^j_x}+\|\vartheta^w\|_{L^2(\partial\Omega)}\Big),
\end{aligned}\tag{3.6.3}
$$

for some polynomial P with $P(s)=O(s)$.

Moreover,

$$\|w\langle v\rangle^{-1}\mathrm{e}^{\lambda t}\tilde{A}\|_{L^\infty_{t,x,v}},\quad \|w\mathrm{e}^{\lambda t}\tilde{r}\|_{L^\infty_{t,x,v}},\quad \|w\mathrm{e}^{\lambda t}[\tilde{f}_1+\varepsilon\tilde{f}_2]\|_{L^\infty_{t,x,v}}\lesssim P_1,$$
$$\|w\langle v\rangle^{-1}\mathrm{e}^{\lambda t}\partial_t\tilde{A}\|_{L^\infty_{t,x,v}},\quad \|w\mathrm{e}^{\lambda t}\partial_t\tilde{r}\|_{L^\infty_{t,x,v}},\quad \|w\mathrm{e}^{\lambda t}[\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2]\|_{L^\infty_{t,x,v}}\lesssim P_2. \tag{3.6.4}$$

Proof From (1.3.3), (1.3.4), and our choice (1.2.9),

$$\begin{aligned}|\tilde{r}|\lesssim&\big|\mu^{-\frac12}\mathcal{P}^w_\gamma(\tilde{f}_2\sqrt{\mu})-\tilde{f}_2\big|+\varepsilon\langle v\rangle^4\mu(v)^{\frac12}\fint|\vartheta|\\
\lesssim&P\bigg(\frac12\sum_{i,j}\mathcal{A}_{ij}\big(\partial_{x_i}\tilde{u}_j|_{\partial\Omega}+\partial_{x_j}\tilde{u}_i|_{\partial\Omega}\big)+\sum_i\mathcal{B}_i\partial_{x_i}\tilde{\vartheta}|_{\partial\Omega}-\frac{\sqrt{\mu}}{2}\Big(\fint\tilde{\vartheta}(t)\Big)^2\\
&-\sqrt{\mu}\Big(\frac{|v|^2-5}{2}\vartheta_w+\fint\vartheta_s\Big)\fint\tilde{\vartheta}+\frac{|v|^2-3}{2}\sqrt{\mu}\Big(\tilde{p}|_{\partial\Omega}-\fint\tilde{p}-\theta_w\fint\tilde{\vartheta}\Big)+\langle v\rangle^{10}\mu(v)^{\frac12}\fint|\tilde{\vartheta}|\bigg)\\
\lesssim&\langle v\rangle^{10}\mu^{\frac12}P\Big(|\nabla_x\tilde{u}|+|\nabla_x\tilde{\vartheta}|+\Big|\tilde{p}-\fint\tilde{p}\Big|+|\tilde{\vartheta}|+\fint|\tilde{\vartheta}|+|\vartheta^w|+|\vartheta_s|+\fint|\vartheta_s|\Big).\end{aligned}$$

By the definitions of $\tilde{f}_1$ and $\tilde{f}_2$ from (1.4.7) and our choice (1.2.9),

$$\begin{aligned}&|\tilde{f}_1|\lesssim\langle v\rangle^4\mu^{\frac12}\big[|\tilde{u}|+|\tilde{\vartheta}|+\fint|\tilde{\vartheta}|\big],\\
&|\tilde{f}_2|\lesssim\langle v\rangle^4\mu^{\frac12}\Big\{|\nabla_x\tilde{u}|+|\nabla_x\tilde{\vartheta}|\Big|\tilde{p}-\fint\tilde{p}\Big|+\big[1+|\tilde{u}|+|\tilde{\vartheta}|+\fint|\tilde{\vartheta}|\big]\big[|u_s|+|\vartheta_s|+\fint|\vartheta_s|\big]\Big\}.\end{aligned}\tag{3.6.5}$$

Then from (.1) we conclude (3.6.1).

From (1.2.14) and (1.2.9),

$$\begin{aligned}|(\mathbf{I}-\mathbf{P})\tilde{A}|\lesssim&|\nabla_x^2\tilde{u}|+|\nabla_x^2\tilde{\vartheta}|+\big[|\nabla_x\tilde{u}|+|\nabla_x\tilde{\vartheta}||\nabla_xu_s|+|\nabla_x\vartheta_s|\big]\Big[|\tilde{u}|+|\tilde{\vartheta}|+|u_s|+|\vartheta_s|+\fint|\tilde{\vartheta}|\Big]\\
&+\varepsilon P\Big(|\partial_t\nabla_x\tilde{u}|+|\partial_t\nabla_x\tilde{\vartheta}|+\Big|\partial_t\tilde{p}-\fint\partial_t\tilde{p}\Big|+|\tilde{\vartheta}|+\fint|\tilde{\vartheta}|+|\vartheta_s|+\fint|\vartheta_s|+|\partial_t\tilde{\vartheta}|\\
&+\fint|\partial_t\tilde{\vartheta}|+|\nabla_x\tilde{u}|+|\nabla_x\tilde{\vartheta}|+\Big|\tilde{p}-\fint\tilde{p}\Big|+|\nabla_x\tilde{u}_s|+|\nabla_x\tilde{\vartheta}_s|+\Big|p_s-\fint p_s\Big|\Big),\\
|\mathbf{P}\tilde{A}|\lesssim&\varepsilon P\Big(|\partial_t\nabla_x\tilde{u}|+|\partial_t\nabla_x\tilde{\vartheta}|+\Big|\partial_t\tilde{p}-\fint\partial_t\tilde{p}\Big|+|\tilde{\vartheta}|+\fint|\tilde{\vartheta}|+|\vartheta_s|+\fint|\vartheta_s|+|\partial_t\tilde{\vartheta}|+\fint|\partial_t\tilde{\vartheta}|\\
&+|\nabla_x\tilde{u}|+|\nabla_x\tilde{\vartheta}|+\Big|\tilde{p}-\fint\tilde{p}\Big|+|\nabla_x\tilde{u}_s|+|\nabla_x\tilde{\vartheta}_s|+\Big|p_s-\fint p_s\Big|\Big).\end{aligned}$$

By the standard Sobolev embedding and the trace theorem, we prove (3.6.4). The proof is completed.

Proof of Theorem 1.2 For the construction of the solution and the energy estimate, we consider $\tilde{R}^\ell(t,x,v)$ solving, for $\ell\in\mathbb{N}$,

$$\begin{aligned}
&\partial_t[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon^{-1}v\cdot\nabla_x[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon\Phi\cdot\nabla_v[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon^{-2}L[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]\\
=&\,\lambda[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon^{-1}L_1[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}]+\varepsilon^{-1}L_{\varepsilon^{1/2}R_s}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}]+\mathrm{e}^{-\lambda t}\varepsilon^{-1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell})\\
&+\varepsilon^{-1}L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)+\varepsilon\frac{\Phi\cdot v}{2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon^{-1/2}\mathrm{e}^{\lambda t}\tilde{A},\\
&\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}|_{\gamma_-}=P_\gamma\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}+\varepsilon\mathrm{e}^{\lambda t}\mathcal{Q}\tilde{R}^{\ell}+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\tilde{r},\quad \mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}|_{t=0}=\tilde{R}_0.
\end{aligned}\tag{3.6.6}$$

Here we set $\tilde{R}^0(t,x,v):=\mathrm{e}^{-ct}\tilde{R}_0(x,v)$.

Clearly $\tilde{R}^{\ell}_t:=\partial_t\tilde{R}^{\ell}$ solves

$$\begin{aligned}
&\partial_t[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon^{-1}v\cdot\nabla_x[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon\Phi\cdot\nabla_v[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon^{-2}L[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]\\
=&\,\lambda[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon^{-1}L_1[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t]+\varepsilon^{-1}L_{\varepsilon^{1/2}R_s}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t]+\varepsilon^{-1}L_{\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}]\\
&+\mathrm{e}^{-\lambda t}\varepsilon^{-1/2}[\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t,\mathrm{e}^{\lambda t}\tilde{R}^{\ell})+\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t)]\\
&+\varepsilon^{-1}L_{R_s}(\mathrm{e}^{\lambda t}\partial_t\tilde{f}_1+\mathrm{e}^{\lambda t}\varepsilon\partial_t\tilde{f}_2)+\varepsilon\frac{\Phi\cdot v}{2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon^{-1/2}\mathrm{e}^{\lambda t}\partial_t\tilde{A},\\
&\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t(t,x,v)|_{\gamma_-}=P_\gamma\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t+\varepsilon\mathrm{e}^{\lambda t}\mathcal{Q}\tilde{R}^{\ell}_t+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\partial_t\tilde{r},\quad \mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t|_{t=0}=\partial_t\tilde{R}_0.
\end{aligned}\tag{3.6.7}$$

As steady case, from (1.4.10) and $\int_{n\cdot v\gtrless 0}M^w|n\cdot v|\mathrm{d}v=1=\int_{n\cdot v\gtrless 0}\sqrt{2\pi}\mu|n\cdot v|\mathrm{d}v$,

$$\begin{aligned}
&\mathbf{P}\big(\varepsilon^{-1}L_1\tilde{R}+\varepsilon^{-1}L_{\varepsilon^{1/2}R_s}\tilde{R}+\varepsilon^{-1/2}\Gamma(\tilde{R},\tilde{R})+\varepsilon^{-1}L_{R_s}(\tilde{f}_1+\varepsilon\tilde{f}_2)\big)=0,\\
&\int_{\mathbb{R}^3}\tilde{A}(t,x,v)\sqrt{\mu}\mathrm{d}v=0=\int_{\mathbb{R}^3}\tilde{f}_2\sqrt{\mu}\mathrm{d}v,\quad \int_{n\cdot v<0}\mathcal{Q}\tilde{R}\{n\cdot v\}\mathrm{d}v=0=\int_{n\cdot v<0}\tilde{r}\mathrm{d}v.
\end{aligned}$$

Note that Proposition 3.2 guarantees the solvability of such linear problems (3.6.6) and (3.6.7).

Note that from the assumption (1.4.12), (.1), and (3.6.1),

$$P_{1,0}\lesssim\big[\|\partial_t\tilde{u}(0)\|_{L^2_x}+\|\partial_t\tilde{\vartheta}(0)\|_{L^2_x}\big]+\big[\|\tilde{u}(0)\|_{H^1_x}+\|\tilde{\vartheta}(0)\|_{H^1_x}\big]\lesssim||\tilde{u}(0)||_{H^2_x}+||\tilde{\vartheta}(0)||_{H^2_x}.\tag{3.6.8}$$

For $0<\eta_0\ll 1$ and $0<\eta_1<\infty$, we assume (induction hypothesis) that

$$\begin{aligned}
&\|\tilde{u}(0)\|^2_{H^2_x}+\|\tilde{\vartheta}(0)\|^2_{H^2_x}<\frac{c_0}{10}\eta_0,\quad \text{for } 0<c_0\ll 1,\\
&\|\mathbf{S}_1R_s\|^2_{L^3_x}+[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_{L^2_{x,v}}]^2+[\varepsilon^{-1}\|\mathbf{S}_2R_s\|_{L^2_{x,v}}]^2+\|f_{s,1}+\varepsilon f_{s,2}\|^2_{L^6_x}<\eta_0,
\end{aligned}\tag{3.6.9}$$

and

$$\begin{aligned}
&\sup_{0\le j\le\ell}\big\{\mathcal{E}^j(\infty)+\mathcal{D}^j(\infty)+\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^j\|^2_{L^3_xL^2_t}+\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^j_t\|^2_{L^3_xL^2_t}+\big[\varepsilon^{3/2}\|\mathrm{e}^{\lambda t}w\partial_t\tilde{R}^j\|_{L^\infty_{t,x,v}}\big]^2\big\}<\eta_0,\\
&\sup_{0\le j\le\ell}\big[\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^j\|_{L^\infty_{t,x,v}}\big]^2<\eta_1.
\end{aligned}\tag{3.6.10}$$

From (**??**), (3.6.1), and (3.6.9), we obtain

$$\|w\mathrm{e}^{\lambda t}[\tilde{f}_1+\varepsilon\tilde{f}_2]\|^2_{L^6_xL^\infty_tL^\infty_v}+\|w\mathrm{e}^{\lambda t}[\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2]\|^2_{L^6_xL^\infty_2L^\infty_v}<c_0\eta_0.$$

We also note that from (3.2.3), we have

$$\|\mathbf{S}_1\tilde{R}(0)\|_{L^3_x} \lesssim \Big\| \int_{\mathbb{R}^3} \tilde{R}(0,x,v)\langle v\rangle^2\sqrt{\mu}\mathrm{d}v\Big\|^2_{L^3_x} < \eta_0.$$

The condition for R_s and $f_{s,1}+\varepsilon f_{s,2}$ can be achieved by choosing further smaller $\|\vartheta^w\|_{H^{1+}(\partial\Omega)} + \|\Phi\|_{H^{\frac{3}{2}+}}(\Omega)$ in Theorem 1.2.

Throughout Step 2 to Step 4 we prove that (3.6.10) holds for all ℓ. For this, it suffices to show that (3.6.10) holds for $j=\ell+1$. Before proving the uniform estimate, we prove the crucial estimates involving operators Γ, L_{R_s}, $L_{f_1+\varepsilon f_2}$, $L_{\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}$.

Step 1 We apply (3.6.10) repeatedly. Applying (3.5.3) with $f=\mathrm{e}^{\lambda t}\tilde{R}^\ell=g$,

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell,\mathrm{e}^{\lambda t}\tilde{R}^\ell)\|_{L^2_{t,x,v}}\\
\lesssim{}& [\varepsilon^{3/2}\|\mathrm{e}^{\lambda t}w\tilde{R}^\ell\|_{L^\infty_{t,x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-\frac{1}{2}}\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell\|_{L^2_{t,x,v}}] + [\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell\|_{L^2_{t,x}}]\big\}\\
&+\big\{\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|^{1/2}_{L^3_xL^2_t}+\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell_t\|^{1/2}_{L^3_xL^2_t}+\|\mathbf{S}_1\tilde{R}^\ell(0)\|^{1/2}_{L^3_x}\big\}[\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^\ell\|_\infty]^{1/2}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|_{L^3_xL^2_t}\\
\lesssim{}& \varepsilon^{1/2}(\eta_1)^{1/2}(\eta_0)^{1/2} + (\eta_1)^{1/4}(\eta_0)^{3/4}.
\end{aligned}\tag{3.6.11}$$

Again applying (3.5.3) with $f=\mathrm{e}^{\lambda t}\tilde{R}^\ell, g=\mathrm{e}^{\lambda t}\tilde{R}^\ell_t$,

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell_t,\mathrm{e}^{\lambda t}\tilde{R}^\ell)\|_{L^2_{t,x,v}} + \|\nu^{-\frac{1}{2}}\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell,\mathrm{e}^{\lambda t}\tilde{R}^\ell_t)\|_{L^2_{t,x,v}}\\
\lesssim{}&[\varepsilon^{3/2}\|w\mathrm{e}^{\lambda t}\tilde{R}^\ell_t\|_{L^\infty_{t,x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-\frac{1}{2}}\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell\|_{L^2_{t,x,v}}] + [\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell\|_{L^2_{t,x}}]\big\}\\
&+[\varepsilon^{3/2}\|\mathrm{e}^{\lambda t}w\tilde{R}^\ell\|_{L^\infty_{t,x,v}}]\big\{[\varepsilon^{-1}\|\nu^{-\frac{1}{2}}\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell_t\|_{L^2_{t,x,v}}] + [\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell_t\|_{L^2_{t,x}}]\big\}\\
&+\big\{\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|^{1/2}_{L^3_xL^2_t}+\|\mathrm{e}^{\lambda t}\mathbf{S}_1\partial_t\tilde{R}^\ell\|^{1/2}_{L^3_xL^2_t}+\|\mathbf{S}_1f(0)\|^{1/2}_{L^3_x}\big\}[\varepsilon\|w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_{L^\infty_{t,x,v}}]^{1/2}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell_t\|_{L^3_xL^2_t}\\
\lesssim{}&\eta_0 + \varepsilon^{1/2}(\eta_1)^{1/2}(\eta_0)^{1/2} + (\eta_1)^{1/4}(\eta_0)^{3/4}.
\end{aligned}\tag{3.6.12}$$

Applying (3.5.4) with $f=R_s$ and $g=\mathrm{e}^{\lambda t}\tilde{R}^\ell$,

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}L_{\varepsilon^{1/2}R_s}\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_{L^2_{t,x,v}}\\
\lesssim{}& \|\nu^{-\frac{1}{2}}\varepsilon^{1/2}\Gamma(R_s,\mathrm{e}^{\lambda t}\tilde{R}^\ell)\|_{L^2_{t,x,v}} + \|\nu^{-\frac{1}{2}}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell,R_s)\|_{L^2_{t,x,v}}\\
\lesssim{}& \|\mathbf{S}_1R_s\|^{1/2}_{L^3_x}[\varepsilon\|wR_s\|_{L^\infty_x}]^{1/2}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|_{L^3_xL^2_t}\\
&+[\varepsilon^{3/2}\|wR_s\|_\infty]\big\{[\varepsilon^{-1}\|\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell\|_{L^2_{t,x,v}}] + \varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell\|_{L^2_{t,x}}\big\}\\
&+\varepsilon^{1/6}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|_{L^3_xL^2_t}\big\{[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_{L^2_{x,v}}]^{1/3}+[\varepsilon^{-1}\|\mathbf{S}_2R_s\|_{L^2_{x,v}}]^{1/3}\big\}[\varepsilon\|wR_s\|_\infty]^{2/3}\\
\lesssim{}& (\eta_1)^{1/4}(\eta_0)^{3/4} + \varepsilon^{1/2}\eta_0 + \varepsilon^{1/6}\eta_0.
\end{aligned}\tag{3.6.13}$$

Again applying (3.5.4) with $f=R_s$ and $g=\mathrm{e}^{\lambda t}\tilde{R}^\ell_t$,

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}L_{\varepsilon^{1/2}R_s}\mathrm{e}^{\lambda t}\tilde{R}^\ell_t\|_{L^2_{t,x,v}}\\
\lesssim{}& \|\nu^{-\frac{1}{2}}\varepsilon^{1/2}\Gamma(R_s,\mathrm{e}^{\lambda t}\tilde{R}^\ell_t)\|_{L^2_{t,x,v}} + \|\nu^{-\frac{1}{2}}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell_t,R_s)\|_{L^2_{t,x,v}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\mathbf{S}_1 R_s\|_{L^3_x}^{1/2}[\varepsilon\|wR_s\|_{L^\infty_x}]^{1/2}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell_t\|_{L^3_xL^2_t}\\
&\quad+[\varepsilon^{3/2}\|wR_s\|_\infty]\big\{[\varepsilon^{-1}\|\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell_t\|_{L^2_{t,x,v}}]+\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell_t\|_{L^2_{t,x}}\big\}\\
&\quad+\varepsilon^{1/6}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell_t\|_{L^3_xL^2_t}\big\{[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_{L^2_{x,v}}]^{1/3}+[\varepsilon^{-1}\|\mathbf{S}_2R_s\|_{L^2_{x,v}}]^{1/3}\big\}[\varepsilon\|wR_s\|_\infty]^{2/3}\\
&\lesssim \eta_0+\varepsilon^{1/2}\eta_0+\varepsilon^{1/6}\eta_0. \qquad (3.6.14)
\end{aligned}
$$

Applying the first estimate of (3.5.5) with $f=f_1+\varepsilon f_2$ and $g=\mathrm{e}^{\lambda t}\tilde{R}^\ell$

$$
\begin{aligned}
&\|\nu^{-\frac12}L_{f_1+\varepsilon f_2}\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_{L^2_{t,x,v}}\\
&\lesssim \|\nu^{-\frac12}\Gamma_\pm(f_1+\varepsilon f_2,\mathrm{e}^{\lambda t}\tilde{R}^\ell)\|_{L^2_{t,x,v}}+\|\nu^{-\frac12}\Gamma_\pm(\mathrm{e}^{\lambda t}\tilde{R}^\ell,f_1+\varepsilon f_2)\|_{L^2_{t,x,v}}\\
&\lesssim \{\|w[\tilde{f}_1+\varepsilon\tilde{f}_2]\|_{L^6_xL^\infty_tL^\infty_v}+\|w[f_{s,1}+\varepsilon f_{s,2}]\|_{L^6_xL^\infty_tL^\infty_v}\}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|_{L^3_xL^2_t}\\
&\quad+\{\varepsilon\|w[\tilde{f}_1+\varepsilon\tilde{f}_2]\|_{L^\infty_{t,x,v}}+\varepsilon\|w[f_{s,1}+\varepsilon f_{s,2}]\|_{L^\infty_{t,x,v}}\}\\
&\quad\times\big\{\varepsilon^{-1}\|\nu^{\frac12}\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell\|_{L^2_{t,x,v}}+\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell\|_{L^2_{t,x}}\big\}\\
&\lesssim c_0^{1/2}\eta_0^{1/2}+\varepsilon(P_1+\varepsilon P_2)\eta_0^{1/2}. \qquad (3.6.15)
\end{aligned}
$$

Again applying the first estimate of (3.5.5) with $f=f_1+\varepsilon f_2$ and $g=\mathrm{e}^{\lambda t}\tilde{R}^\ell_t$,

$$
\begin{aligned}
&\|\nu^{-\frac12}L_{f_1+\varepsilon f_2}\mathrm{e}^{\lambda t}\tilde{R}^\ell_t\|_{L^2_{t,x,v}}\\
&\lesssim \|f_1+\varepsilon f_2\|_{L^6_xL^\infty_tL^\infty_v}\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell_t\|_{L^3_xL^2_t}\\
&\quad+\varepsilon\|w[f_1+\varepsilon f_2]\|_{L^\infty_{t,x,v}}\big\{\varepsilon^{-1}\|\nu^{\frac12}\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell_t\|_{L^2_{t,x,v}}+\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell_t\|_{L^2_{t,x}}\big\}\\
&\lesssim c_0^{1/2}\eta_0^{1/2}+\varepsilon(P_1+\varepsilon P_2)\eta_0^{1/2}, \qquad (3.6.16)
\end{aligned}
$$

or applying (3.5.5) and (3.5.1) with $f=\partial_t f_1+\varepsilon\partial_t f_2$ and $g=\mathrm{e}^{\lambda t}\tilde{R}^\ell$,

$$
\begin{aligned}
&\|\nu^{-\frac12}L_{\partial_t f_1+\varepsilon\partial_t f_2}\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_{L^2_{t,x,v}}\\
&\lesssim \|\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2\|_{L^6_xL^2_{t,v}}\{\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell\|_{L^3_xL^2_t}+\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell_t\|_{L^3_xL^2_t}+\|\mathrm{e}^{\lambda t}\mathbf{S}_1\tilde{R}^\ell(0)\|_{L^3_x}\}\\
&\quad+\varepsilon\|w[\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2]\|_{L^\infty_{t,x,v}}\big\{\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\nu^{\frac12}(\mathbf{I}-\mathbf{P})\tilde{R}^\ell\|_{L^2_{t,x,v}}+\varepsilon^{-1}\|\mathrm{e}^{\lambda t}\mathbf{S}_2\tilde{R}^\ell\|_{L^2_{t,x}}\big\}\\
&\lesssim c_0^{1/2}(\eta_0)^{1/2}+\varepsilon(\eta_0)^{1/2}. \qquad (3.6.17)
\end{aligned}
$$

Again by the second estimate of (3.5.5) with $g=R_s$ and $f=\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2$,

$$
\begin{aligned}
&\|\nu^{-\frac12}L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)\|_{L^2_{t,x,v}}\\
&\lesssim \|\nu^{-\frac12}\Gamma_\pm(R_s,\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)\|_{L^2_{t,x,v}}+\|\nu^{-\frac12}\Gamma_\pm(\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2,R_s)\|_{L^2_{t,x,v}}\\
&\lesssim \|\mathbf{S}_1R_s\|_{L^3_xL^\infty_t}\|\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2\|_{L^6_xL^2_{t,v}}\\
&\quad+\varepsilon\|w[\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2]\|_{L^\infty_{x,v}L^2_t}\big\{\varepsilon^{-1}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})R_s\|_{L^2_{x,v}L^\infty_t}+\varepsilon^{-1}\|\mathbf{S}_2R_s\|_{L^2_xL^\infty_t}\big\}\\
&\lesssim c_0^{1/2}(\eta_0)^{1/2}+\varepsilon(\eta_0)^{1/2}, \qquad (3.6.18)
\end{aligned}
$$

and similarly applying the second estimate of (3.5.5) with $g = R_s$ and $f = \mathrm{e}^{\lambda t}\partial_t \tilde{f}_1 + \varepsilon \mathrm{e}^{\lambda t}\partial_t \tilde{f}_2$

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}} L_{R_s}(\mathrm{e}^{\lambda t}\partial_t \tilde{f}_1 + \varepsilon \mathrm{e}^{\lambda t}\partial_t \tilde{f}_2)\|_{L^2_{t,x,v}} \\
\lesssim\ & \|\mathbf{S}_1 R_s\|_{L^3_x L^\infty_t} \|\mathrm{e}^{\lambda t}\partial_t \tilde{f}_1 + \varepsilon \mathrm{e}^{\lambda t}\partial_t \tilde{f}_2\|_{L^6_x L^2_{t,v}} \\
&+\varepsilon \|w[\mathrm{e}^{\lambda t}\partial_t \tilde{f}_1 + \varepsilon \mathrm{e}^{\lambda t}\partial_t \tilde{f}_2]\|_{L^\infty_{x,v} L^2_t} \big\{\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})R_s\|_{L^2_{x,v}L^\infty_t} + \varepsilon^{-1}\|\mathbf{S}_2 R_s\|_{L^2_x L^\infty_t}\big\} \\
\lesssim\ & c_0^{1/2}(\eta_0)^{1/2} + \varepsilon(\eta_0)^{1/2}. \qquad (3.6.19)
\end{aligned}$$

Step 2 From (3.3.4), (1.4.10) and (3.6.4),

$$\begin{aligned}
&\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}(t)\|_2^2 + \frac{1}{\varepsilon}\int_0^t |\mathrm{e}^{\lambda s}\tilde{R}^{\ell+1}|_2^2 + \frac{1}{\varepsilon^2}\int_0^t \|\mathrm{e}^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde{R}^{\ell+1}\|_\nu^2 + \int_0^t \|\mathrm{e}^{\lambda s}\mathbf{P}\tilde{R}^{\ell+1}\|_2^2 \\
\lesssim & \|\tilde{R}^{\ell+1}(0)\|_2^2 + \varepsilon^{-1}\int_0^t |\mathrm{e}^{\lambda s}\varepsilon \mathcal{Q}\tilde{R}^\ell|_{2,-}^2 + \varepsilon^{-1}\int_0^t |\mathrm{e}^{\lambda s}\varepsilon^{1/2}\tilde{r}|_{2,-}^2 \\
&+\varepsilon \int_0^t ||\nu^{-\frac{1}{2}}\Gamma(\mathrm{e}^{\lambda s}\tilde{R}^\ell, \mathrm{e}^{\lambda s}\tilde{R}^\ell)||_2^2 + \int_0^t \|\nu^{-\frac{1}{2}}L_1 \mathrm{e}^{\lambda s}\tilde{R}^\ell\|_2^2 + \int_0^t \|\nu^{-\frac{1}{2}}L_{\varepsilon^{1/2}R_s}\mathrm{e}^{\lambda s}\tilde{R}^\ell\|_2^2 \\
&+\int_0^t \|\nu^{-\frac{1}{2}}\mathrm{e}^{\lambda s}\varepsilon^{1/2}(\mathbf{I}-\mathbf{P})\tilde{A}\|_2^2 + \varepsilon^{-2}\int_0^t \|\mathrm{e}^{\lambda s}\varepsilon^{1/2}\mathbf{P}\tilde{A}\|_2^2 + \int_0^t \|\nu^{-\frac{1}{2}}L_{R_s}(\mathrm{e}^{\lambda s}\tilde{f}_1 + \mathrm{e}^{\lambda s}\varepsilon \tilde{f}_2)\|_2^2 \\
\lesssim\ & \big[c_0^{1/2} + \varepsilon + \varepsilon^2 + \varepsilon\eta_1 + (\eta_1)^{1/2}(\eta_0)^{1/2} + (1+\varepsilon^{1/3}+\varepsilon)\eta_0\big]\eta_0 \\
&+\|\mathrm{e}^{\lambda' t}\tilde{\vartheta}\|_{L^2_t L^2_x}^2 + \varepsilon^2 P_{0,1} + \varepsilon[P_1 P_{0,1}]^2 + \varepsilon[P_1 P_{1,0}]^2 < \frac{\eta_0}{10},
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathrm{e}^{\lambda t}\tilde{R}_t^{\ell+1}(t)\|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon}\int_0^t |\mathrm{e}^{\lambda s}\tilde{R}_t^{\ell+1}|_2^2 + \frac{1}{\varepsilon^2}\int_0^t \|\mathrm{e}^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde{R}_t^{\ell+1}\|_\nu^2 + \int_0^t \|\mathrm{e}^{\lambda s}\mathbf{P}\tilde{R}_t^{\ell+1}\|_2^2 \\
\lesssim\ & \|\tilde{R}_t^{\ell+1}(0)\|_2^2 + \varepsilon^{-1}\int_0^t |\mathrm{e}^{\lambda s}\varepsilon \mathcal{Q}\tilde{R}_t^\ell|_{2,-}^2 + \varepsilon^{-1}\int_0^t |\mathrm{e}^{\lambda s}\varepsilon^{1/2}\tilde{r}_t|_{2,-}^2 \\
&+\varepsilon \int_0^t \big[||\nu^{-\frac{1}{2}}\Gamma(\mathrm{e}^{\lambda s}\tilde{R}_t^\ell, \mathrm{e}^{\lambda s}\tilde{R}^\ell)||_2^2 + ||\mathrm{e}^{-\lambda s}\nu^{-\frac{1}{2}}\Gamma(\mathrm{e}^{\lambda s}\tilde{R}^\ell, \mathrm{e}^{\lambda s}\tilde{R}_t^\ell)||_2^2\big] \\
&+\int_0^t \|\nu^{-\frac{1}{2}}L_1 \mathrm{e}^{\lambda s}\tilde{R}_t^\ell\|_2^2 + \int_0^t \|\nu^{-\frac{1}{2}}L_{\varepsilon^{1/2}R_s}\mathrm{e}^{\lambda s}\tilde{R}_t^\ell\|_2^2 + \int_0^t \|L_{\partial_t f_1 + \varepsilon\partial_t f_2}\mathrm{e}^{\lambda s}\tilde{R}\|_2^2 \\
&+\int_0^t \|\nu^{-\frac{1}{2}}\mathrm{e}^{\lambda s}\varepsilon^{1/2}(\mathbf{I}-\mathbf{P})\partial_t\tilde{A}\|_2^2 + \varepsilon^{-2}\int_0^t \|\mathrm{e}^{\lambda s}\varepsilon^{1/2}\mathbf{P}\partial_t\tilde{A}\|_2^2 \\
&+\int_0^t \|\nu^{-\frac{1}{2}}L_{R_s}(\mathrm{e}^{\lambda s}\partial_t\tilde{f}_1 + \mathrm{e}^{\lambda s}\varepsilon\partial_t\tilde{f}_2)\|_2^2 \\
\lesssim\ & \big[c_0^{1/2} + \varepsilon + \varepsilon^2 + \varepsilon\eta_1 + (\eta_1)^{1/2}(\eta_0)^{1/2} + (1+\varepsilon^{1/3}+\varepsilon)\eta_0\big]\eta_0 \\
&+\|\mathrm{e}^{\lambda' t}\partial_t\tilde{\vartheta}\|_{L^2_t L^2_x}^2 + \varepsilon^2 P_{1,1} + \varepsilon[P_2 P_{1,1}]^2 + \varepsilon[P_2 P_{2,0}]^2 < \frac{\eta_0}{10}.
\end{aligned}$$

Step 3 We apply Proposition 3.1 to (3.6.6): Set

$$\begin{aligned}
&f=\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1},\\
&g=-\varepsilon^{-1}L[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon\lambda[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+L_1[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}]+\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell})\\
&\qquad+L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)+\varepsilon^2\frac{\Phi\cdot v}{2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\tilde{A}.
\end{aligned}$$

Then

$$\begin{aligned}
&\|\mathbf{S}_1\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|_{L^3_xL^2_t}\\
\lesssim&\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|_{L^2_{t,x,v}}+\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|_{L^2_tL^2_\gamma}+\Big\|w^{-1}\Big[-\varepsilon^{-1}L[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon\lambda[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+L_1[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}]\\
&+\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell})+L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)+\varepsilon^2\frac{\Phi\cdot v}{2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}]+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\tilde{A}\Big]\Big\|_{L^2_{t,x,v}}\\
\lesssim&(1+\varepsilon\lambda+\varepsilon^2\|\Phi\|_\infty)\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|_{L^2_{t,x,v}}+\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|_{L^2_tL^2_\gamma}+\varepsilon^{-1}\|\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^{\ell+1}\|_{L^2_{t,x,v}}\\
&+\|\nu^{-1/2}L_1\mathrm{e}^{\lambda t}\tilde{R}^{\ell}\|_{L^2_{t,x,v}}+\varepsilon^{1/2}\|\nu^{-1/2}\mathrm{e}^{-\lambda t}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell})\|_{L^2_{t,x,v}}\\
&+\|\nu^{-1/2}L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1+\mathrm{e}^{\lambda t}\varepsilon\tilde{f}_2)\|_{L^2_{t,x,v}}+\varepsilon^{1/2}\|\mathrm{e}^{\lambda t}\tilde{A}\|_{L^2_{t,x,v}}.
\end{aligned}$$

Now we use Step 1 to bound

$$\|\mathbf{S}_1\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|^2_{L^3_xL^2_t}\lesssim\eta_0/10.$$

Similarly, we apply Proposition 3.1 to (3.6.7): Set

$$\begin{aligned}
&f=\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t,\\
&g=-\varepsilon^{-2}L[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\lambda[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon^{-1}L_1[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t]+\varepsilon^{-1}L_{\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell}]\\
&\qquad+\mathrm{e}^{-\lambda t}\varepsilon^{-1/2}[\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t,\mathrm{e}^{\lambda t}\tilde{R}^{\ell})+\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t)]\\
&\qquad+\varepsilon^{-1}L_{R_s}(\mathrm{e}^{\lambda t}\partial_t\tilde{f}_1+\mathrm{e}^{\lambda t}\varepsilon\partial_t\tilde{f}_2)+\varepsilon\frac{\Phi\cdot v}{2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t]+\varepsilon^{-1/2}\mathrm{e}^{\lambda t}\partial_t\tilde{A}.
\end{aligned}$$

Then

$$\begin{aligned}
&\|\mathbf{S}_1\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t\|_{L^3_xL^2_t}\\
\lesssim&\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t\|_{L^2_{t,x,v}}+\|\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}\|_{L^2_tL^2_\gamma}+\varepsilon^{-1}\|\mathrm{e}^{\lambda t}(\mathbf{I}-\mathbf{P})\tilde{R}^{\ell+1}_t\|_{L^2_{t,x,v}}\\
&+\|\nu^{-1/2}L_1\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t\|_{L^2_{t,x,v}}+\|\nu^{-1/2}L_{\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}\mathrm{e}^{\lambda t}\tilde{R}^{\ell}\|_{L^2_{t,x,v}}\\
&+\varepsilon^{1/2}\|\nu^{-1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t,\mathrm{e}^{\lambda t}\tilde{R}^{\ell})\|_{L^2_{t,x,v}}+\varepsilon^{1/2}\|\nu^{-1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^{\ell},\mathrm{e}^{\lambda t}\tilde{R}^{\ell}_t)\|_{L^2_{t,x,v}}\\
&+\|\nu^{-1/2}L_{R_s}(\mathrm{e}^{\lambda t}\partial_t\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\partial_t\tilde{f}_2)\|_{L^2_{t,x,v}}+\varepsilon^{1/2}\|\mathrm{e}^{\lambda t}\partial_t\tilde{A}\|_{L^2_{t,x,v}}.
\end{aligned}$$

Again by Step 1,

$$\|\mathbf{S}_1\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t\|^2_{L^3_xL^2_t}<\eta_0/10.$$

Step 4 We apply Proposition 3.2 to (3.6.6): Set $f=\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}$. Note

$$\varepsilon^{-1}L\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}=\varepsilon^{-1}\nu(v)\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}-\varepsilon^{-1}\int_{\mathbb{R}^3}\mathbf{k}(v,u)\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}(u)\mathrm{d}u$$

with $\nu(v)\sim\langle v\rangle$ and $|\mathbf{k}(v,u)|\lesssim\mathbf{k}_\beta(v,u)$. Moreover,

$$\varepsilon^{-1}\nu(v) - \varepsilon\lambda - \varepsilon\frac{\Phi\cdot v}{2} \gtrsim \varepsilon^{-1}C\langle v\rangle - \varepsilon\lambda - \varepsilon\|\Phi\|_\infty|v| \gtrsim \varepsilon^{-1}C_0\langle v\rangle.$$

Therefore (3.4.1), the condition of Proposition 3.4, is satisfied with the following setting

$$\begin{aligned}
&g = L_1[\mathrm{e}^{\lambda t}\tilde{R}^\ell] + \mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell, \mathrm{e}^{\lambda t}\tilde{R}^\ell) + L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1 + \varepsilon^2\mathrm{e}^{\lambda t}\tilde{f}_2) + \varepsilon^{1/2}\mathrm{e}^{\lambda t}\tilde{A},\\
&r = \varepsilon\mathcal{Q}\tilde{R}^\ell + \mathrm{e}^{\lambda t}\varepsilon^{1/2}\tilde{r}, \quad f_0 = \tilde{R}_0.
\end{aligned}$$

By Proposition 3.2, from (3.4.2),

$$\begin{aligned}
&\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^{\ell+1}\|_{L^\infty_{t,x,v}}\\
\lesssim&\varepsilon\|w\tilde{R}^{\ell+1}(0)\|_\infty + \varepsilon\max_{0\le j\le \ell}\sup_{0\le t\le\infty}\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^j\|_\infty + \varepsilon\sup_{0\le t\le\infty}\varepsilon^{1/2}\|\mathrm{e}^{\lambda t}w\tilde{r}\|_\infty\\
&+\varepsilon^2\sup_{0\le t\le\infty}\Big\|w\nu^{-1}\Big[L_1[\mathrm{e}^{\lambda t}\tilde{R}^\ell] + \varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell, \mathrm{e}^{\lambda t}\tilde{R}^\ell) + L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1 + \varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)\\
&+\varepsilon\frac{\Phi\cdot v}{2}[\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}] + \varepsilon^{1/2}\mathrm{e}^{\lambda t}\tilde{A}\Big]\Big\|_\infty\\
&+\|\mathrm{e}^{\lambda s}\mathbf{S}_1\tilde{R}^{\ell+1}(s)\|_{L^\infty_t L^3_x} + \frac{1}{\varepsilon^{1/2}}\|\mathrm{e}^{\lambda s}\mathbf{S}_2\tilde{R}^{\ell+1}(s)\|_{L^\infty_t L^2_x} + \frac{1}{\varepsilon^{1/2}}\|\mathrm{e}^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde{R}^{\ell+1}(s)\|_{L^\infty_t L^2_{x,v}}.
\end{aligned}$$

Using $|w\Gamma_\pm(w^{-1}, w^{-1})| \lesssim \langle v\rangle \lesssim \nu$,

$$\begin{aligned}
&\varepsilon^2\sup_{0\le t\le\infty}\big\|w\nu^{-1}\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell, \mathrm{e}^{\lambda t}\tilde{R}^\ell)\big\|_\infty\\
\lesssim&\, \varepsilon^{1/2}[\sup_{0\le t\le\infty}\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty]^2|\nu^{-1}w\Gamma(w^{-1}, w^{-1})| \lesssim \varepsilon^{1/2}[\sup_{0\le t\le\infty}\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty]^2,
\end{aligned}$$

$$\begin{aligned}
|\varepsilon^2 w\nu^{-1}L_1\mathrm{e}^{\lambda t}\tilde{R}^\ell| &\lesssim \varepsilon^{1/2}|w\nu^{-1}\Gamma_\pm(\varepsilon^{1/2}f_1 + \varepsilon^{3/2}f_2 + \varepsilon R_s, \mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell)|\\
&\lesssim \varepsilon^{1/2}\|w[\varepsilon^{1/2}f_1 + \varepsilon^{3/2}f_2 + \varepsilon R_s]\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell\|_\infty|\nu^{-1}w\Gamma_\pm(w^{-1}, w^{-1})|\\
&\lesssim \varepsilon\|w[f_1 + \varepsilon f_2]\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell\|_\infty + \varepsilon^{1/2}\|w\varepsilon R_s\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell\|_\infty\\
&\lesssim \varepsilon P_1\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell\|_\infty + \varepsilon^{1/2}\|w\varepsilon R_s\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell\|_\infty,
\end{aligned}$$

$$\begin{aligned}
|\varepsilon^2 w\nu^{-1}L_{R_s}(\mathrm{e}^{\lambda t}\tilde{f}_1 + \varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)| &\lesssim \varepsilon|w\nu^{-1}\Gamma_\pm(\varepsilon R_s, \mathrm{e}^{\lambda t}\tilde{f}_1 + \varepsilon\mathrm{e}^{\lambda t}\tilde{f}_2)|\\
&\lesssim \varepsilon\|w\varepsilon R_s\|_\infty\|w\mathrm{e}^{\lambda t}[\tilde{f}_1 + \varepsilon\tilde{f}_2]\|_\infty|\nu^{-1}w\Gamma_\pm(w^{-1}, w^{-1})|\\
&\lesssim \varepsilon P_1\|w\varepsilon R_s\|_\infty.
\end{aligned}$$

Altogether,

$$\big[\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^{\ell+1}\|_{L^\infty_{t,x,v}}\big]^2 \lesssim \big[c_0^{1/2} + \varepsilon + \varepsilon\eta_0 + \varepsilon^2(P_1)^2\big]\eta_0 + \varepsilon(P_1)^2 < \frac{\eta_0}{10}.$$

Now we consider $\partial_t\tilde{R}^{\ell+1}$. Apply Proposition 3.2 to (3.6.7): Set $f = \mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t$. Note

$$\varepsilon^{-1}L\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t = \varepsilon^{-1}\nu(v)\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t - \varepsilon^{-1}\int_{\mathbb{R}^3}\mathbf{k}(v,u)\mathrm{e}^{\lambda t}\tilde{R}^{\ell+1}_t(u)\mathrm{d}u$$

with $\nu(v) \sim \langle v\rangle$ and $|\mathbf{k}(v,u)| \lesssim \mathbf{k}_\beta(v,u)$. For Proposition 3.2 we set

$$g = L_1[\mathrm{e}^{\lambda t}\tilde{R}^\ell_t]+L_{\partial_t \tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}[\mathrm{e}^{\lambda t}\tilde{R}^\ell]+\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell_t,\mathrm{e}^{\lambda t}\tilde{R}^\ell)+\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell,\mathrm{e}^{\lambda t}\tilde{R}^\ell_t)$$
$$+L_{R_s}(\mathrm{e}^{\lambda t}\partial_t\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\partial_t\tilde{f}_2)+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\partial_t\tilde{A},$$
$$r=\varepsilon\mathrm{e}^{\lambda t}\partial_t\mathcal{Q}\tilde{R}^\ell_t+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\tilde{r},\quad f_0=\partial_t\tilde{R}_0.$$

From (3.4.3),

$$\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^{\ell+1}_t\|_{L^\infty_{t,x,v}}$$
$$\lesssim\varepsilon\|w\tilde{R}^{\ell+1}_t(0)\|_\infty+\varepsilon\max_{0\le j\le\ell}\|\varepsilon\mathrm{e}^{\lambda t}w\tilde{R}^\ell_t\|_{L^\infty_{t,x,v}}+\varepsilon\|\varepsilon^{1/2}w\mathrm{e}^{\lambda t}\tilde{r}_t\|_{L^\infty_{t,x,v}}+\frac{1}{\varepsilon^{1/2}}\|\mathrm{e}^{\lambda s}\tilde{R}^{\ell+1}_t(s)\|_{L^\infty_tL^2_{x,v}}$$
$$+\varepsilon^2\sup_{0\le t\le\infty}\big\|w\nu^{-1}\big[L_1[\mathrm{e}^{\lambda t}\tilde{R}^\ell_t]+\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell_t,\mathrm{e}^{\lambda t}\tilde{R}^\ell)+\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell,\mathrm{e}^{\lambda t}\tilde{R}^\ell_t)$$
$$+L_{\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}\mathrm{e}^{\lambda t}\tilde{R}^\ell+L_{R_s}(\mathrm{e}^{\lambda t}\partial_t\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\partial_t\tilde{f}_2)+\varepsilon^{1/2}\mathrm{e}^{\lambda t}\partial_t\tilde{A}\big]\big\|_\infty.$$

From $|w\Gamma_\pm(w^{-1},w^{-1})|\lesssim\langle v\rangle\lesssim\nu$,

$$\varepsilon^2\sup_{0\le t\le\infty}\big\|w\nu^{-1}\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell_t,\mathrm{e}^{\lambda t}\tilde{R}^\ell)\big\|_\infty+\varepsilon^2\sup_{0\le t\le\infty}\big\|w\nu^{-1}\mathrm{e}^{-\lambda t}\varepsilon^{1/2}\Gamma(\mathrm{e}^{\lambda t}\tilde{R}^\ell,\mathrm{e}^{\lambda t}\tilde{R}^\ell_t)\big\|_\infty$$
$$\lesssim\varepsilon^{1/2}[\sup_{0\le t\le\infty}\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty][\sup_{0\le t\le\infty}\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell_t\|_\infty]|\nu^{-1}w\Gamma(w^{-1},w^{-1})|$$
$$\lesssim\varepsilon^{1/2}[\sup_{0\le t\le\infty}\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty][\sup_{0\le t\le\infty}\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell_t\|_\infty],$$
$$|\varepsilon^2w\nu^{-1}L_{\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2}\mathrm{e}^{\lambda t}\tilde{R}^\ell|\lesssim\varepsilon\|w[\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2]\|_\infty\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty|\nu^{-1}w\Gamma_\pm(w^{-1},w^{-1})|$$
$$\lesssim\varepsilon P_2\|\varepsilon w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty,$$
$$|\varepsilon^2w\nu^{-1}L_1\mathrm{e}^{\lambda t}\tilde{R}^\ell_t|\lesssim\varepsilon\|w[f_1+\varepsilon f_2]\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell_t\|_\infty+\varepsilon^{1/2}\|w\varepsilon R_s\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell_t\|_\infty$$
$$\lesssim\varepsilon P_1\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell_t\|_\infty+\varepsilon^{1/2}\|w\varepsilon R_s\|_\infty\|w\mathrm{e}^{\lambda t}\varepsilon\tilde{R}^\ell_t\|_\infty,$$
$$|\varepsilon^2w\nu^{-1}L_{R_s}(\mathrm{e}^{\lambda t}\partial_t\tilde{f}_1+\varepsilon\mathrm{e}^{\lambda t}\partial_t\tilde{f}_2)|\lesssim\varepsilon\|w\varepsilon R_s\|_\infty\|w\mathrm{e}^{\lambda t}[\partial_t\tilde{f}_1+\varepsilon\partial_t\tilde{f}_2]\|_\infty|\nu^{-1}w\Gamma_\pm(w^{-1},w^{-1})|$$
$$\lesssim\varepsilon P_2\|w\varepsilon R_s\|_\infty.$$

Altogether

$$\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^{\ell+1}_t\|_{L^\infty_{t,x,v}}$$
$$\lesssim\varepsilon\|w\tilde{R}^{\ell+1}_t(0)\|_\infty+\varepsilon^{1/2}\big\{\varepsilon^{1/2}+\varepsilon^{1/2}P_1+\|\varepsilon wR_s\|_{L^\infty_{x,v}}+\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^\ell\|_\infty\big\}\times\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^\ell_t\|_{L^\infty_{t,x,v}}$$
$$+\varepsilon\{\varepsilon\|wR_s\|_{L^\infty_{x,v}}+\varepsilon\|w\mathrm{e}^{\lambda t}\tilde{R}^\ell\|_\infty+\varepsilon^{1/2}\}\times P_2+\frac{1}{\varepsilon^{1/2}}\sup_{0\le t\le\infty}\|\mathrm{e}^{\lambda s}\tilde{R}^{\ell+1}_t\|_{L^2_{x,v}},$$

and therefore

$$\big[\varepsilon\|\mathrm{e}^{\lambda t}w\tilde{R}^{\ell+1}_t\|_{L^\infty_{t,x,v}}\big]^2<\frac{\eta_0}{10}.$$

Step 5 We repeat Step 1∼Step 4 for $\tilde{R}^{\ell+1}-\tilde{R}^\ell$ to show that $\tilde{R}^\ell$ is a Cauchy sequence in $L^\infty\cap L^2$ for fixed ε. Now we pass a limit $\ell\to\infty$ in $L^\infty\cap L^2$ to conclude the existence. The proof of uniqueness is standard (See [20] for details). The proof is completed.

3.7 Positivity of Solutions

In this section, we prove the non-negativity of F_s in the main theorem. The proof is based on the asymptotical stability of F_s (Proposition 3.1) and the non-negativity of unsteady solution.

Proof of the non-negativity of $F(t,x,v)$ in Theorem 1.2 and $F_s(x,v)$ in Theorem 1.1 We use the positivity-preserving sequence as in [20, 32]. Set $F^0(t,x,v) = F_0(x,v) \geq 0$ and for $\ell \geq 0$

$$\partial_t F^{\ell+1} + \frac{1}{\varepsilon} v \cdot \nabla_x F^{\ell+1} + \varepsilon \Phi \cdot \nabla_v F^{\ell+1} + \frac{1}{\varepsilon^2} \nu(F^\ell) F^{\ell+1} = \frac{1}{\varepsilon^2} Q_+(F^\ell, F^\ell),$$

$$F^{\ell+1}(x,v)|_{\gamma_-} = M^w \int_{n(x)\cdot u>0} F^\ell(x,u)\{n(x)\cdot u\}\mathrm{d}u, \quad F^{\ell+1}(t,x,v)|_{t=0} = F_0(x,v).$$

Step 1 Assume $F^\ell(t,x,v) \geq 0$ for all $t \geq 0$. We claim $F^{\ell+1}(t,x,v) \geq 0$ for all $t \geq 0$. Along the trajectory, for $\tilde{t}_1(t,x,v) \leq s \leq t$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\Big\{F^{\ell+1}(s, Y_{\mathbf{cl}}(s;t,x,v), W_{\mathbf{cl}}(s;t,x,v)) \\
&\quad \times \exp\Big(-\int_s^t \frac{1}{\varepsilon^2}\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v), W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big)\Big\} \\
&= \frac{1}{\varepsilon^2} Q_+(F^\ell, F^\ell)(s, Y_{\mathbf{cl}}(s;t,x,v), W_{\mathbf{cl}}(s;t,x,v)) \\
&\quad \times \exp\Big(-\int_s^t \frac{1}{\varepsilon^2}\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v), W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big).
\end{aligned}$$

Then

$$\begin{aligned}
F^{\ell+1}(t,x,v) = {}& \mathbf{1}_{\tilde{t}_1<0} \exp\Big(-\int_0^t \frac{1}{\varepsilon^2}\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v), W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big) \\
& \times F_0(Y_{\mathbf{cl}}(0;t,x,v), W_{\mathbf{cl}}(0;t,x,v)) \\
& + \mathbf{1}_{\tilde{t}_1<0} \int_0^t \exp\Big(-\int_s^t \frac{1}{\varepsilon^2}\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v), W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big) \\
& \times \frac{1}{\varepsilon^2} Q_+(F^\ell, F^\ell)(s, Y_{\mathbf{cl}}(s;t,x,v), W_{\mathbf{cl}}(s;t,x,v))\mathrm{d}s \\
& + \mathbf{1}_{\tilde{t}_1>0} \int_{\tilde{t}_1}^t \exp\Big(-\int_s^t \frac{1}{\varepsilon^2}\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v), W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big) \\
& \times \frac{1}{\varepsilon^2} Q_+(F^\ell, F^\ell)(s, Y_{\mathbf{cl}}(s;t,x,v), W_{\mathbf{cl}}(s;t,x,v))\mathrm{d}s \\
& + \mathbf{1}_{\tilde{t}_1>0} \exp\Big(-\int_{\tilde{t}_1}^t \frac{1}{\varepsilon^2}\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v), W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big) \\
& \times M^w(\tilde{x}_1, \tilde{v}_1) \int_{n(\tilde{x}_1)\cdot u>0} F^\ell(\tilde{t}_1, \tilde{x}_1, u)\{n(\tilde{x}_1)\cdot u\}\mathrm{d}u.
\end{aligned}$$

From $Q_+(F^\ell, F^\ell)(s, Y_{\mathbf{cl}}(s;t,x,v), W_{\mathbf{cl}}(s;t,x,v)) \geq 0$ and $\nu(F^\ell)(\tau, Y_{\mathbf{cl}}(\tau;t,x,v),$

$W_(\tau;t,x,v)) \geq 0$, we prove our claim.

Step 2 We set $F^\ell = \mu + \varepsilon\{f_1 + \varepsilon f_2 + \varepsilon^{1/2}R^\ell\}\sqrt{\mu}$ where f_1 and f_2 are given by (1.2.5), (1.2.8) and let $F^0(t,x,v) := F_0(x,v)$. We claim that, there exist $0 < T = T(\|\varepsilon w R^\ell(t_0)\|_\infty) \ll 1$ and $C_1 = C_1(T) \gg 1$ for any $t_0 \geq 0$ such that

$$\sup_{t_0\leq t\leq t_0+\varepsilon^2 T} \|\varepsilon w R^\ell(t)\|_\infty \leq C_1\big\{\max_l \|\varepsilon w R^l(t_0)\|_\infty + O(\varepsilon^{5/2})\big\}. \tag{3.7.1}$$

It suffices to show (3.7.1) for $t_0 = 0$. Clearly (3.7.1) holds for $\ell = 0$. Now we assume (3.7.1) for $0 \leq l \leq \ell$.

Clearly, R^ℓ solves

$$\begin{aligned}
&\partial_t R^{\ell+1} + \varepsilon^{-1} v\cdot\nabla_x R^{\ell+1} + \varepsilon\Phi\cdot\nabla_v R^{\ell+1} + \varepsilon\frac{\Phi\cdot v}{2}R^{\ell+1} + \varepsilon^{-2}\nu R^{\ell+1} - \varepsilon^{-2}KR^\ell\\
=\,& \varepsilon^{-1}\nu([f_1+\varepsilon f_2]\sqrt{\mu})R^{\ell+1} + \varepsilon^{-1}\nu(R^\ell\sqrt{\mu})[f_1+\varepsilon f_2]\sqrt{\mu}\\
&-\varepsilon^{-1}[\Gamma_+(f_1+\varepsilon f_2, R^\ell) + \Gamma_+(R^\ell, f_1+\varepsilon f_2)]\\
&+\varepsilon^{-1/2}\Gamma_+(R^\ell,R^\ell) - \varepsilon^{-1/2}\nu(R^\ell\sqrt{\mu})R^{\ell+1} + \varepsilon^{-1/2}A(f_1,f_2),\\
&R^{\ell+1}|_{t=0} = R_0.
\end{aligned} \tag{3.7.2}$$

The boundary condition is given by

$$R^{\ell+1}|_{\gamma_-} = P_\gamma R^\ell + \varepsilon \mathcal{Q} R^\ell + \varepsilon^{1/2} r. \tag{3.7.3}$$

Define $h^\ell(t,x,v) := w(v)^{-1}R^\ell(t,x,v)$. Note that

$$\check{\nu}(v) := \nu(v) - \varepsilon^3\frac{\|\Phi\|_\infty|v|}{2} - \varepsilon\|f_1+\varepsilon f_2\|_{L^\infty_{t,x,v}} - \varepsilon^{1/2}\nu(\sqrt{\mu})\|\varepsilon w R^\ell\|_{L^\infty_{t,x,v}} \geq \frac{\nu(v)}{2} \gtrsim \langle v\rangle.$$

We define $\check{\mathbf{k}}$ such that where

$$\begin{aligned}
&\int_{\mathbb{R}^3}\check{\mathbf{k}}(v,u)\frac{w(v)}{w(u)}R^\ell(u)\mathrm{d}u\\
=\,& KR^\ell + \varepsilon\nu(R^\ell\sqrt{\mu})[f_1+\varepsilon f_2]\sqrt{\mu} - \varepsilon[\Gamma_+(f_1+\varepsilon f_2, R^\ell) + \Gamma_+(R^\ell, f_1+\varepsilon f_2)].
\end{aligned}$$

Then $\check{\mathbf{k}}(v,u) \lesssim \mathbf{k}_\beta(v,u)$.

Then, for $\tilde{t}_1 \leq s \leq t$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\Big[|\varepsilon h^{\ell+1}(s, Y_{\mathbf{cl}}(s;t,x,v), W_{\mathbf{cl}}(s;t,x,v))|\exp\Big(-\int_s^t \varepsilon^{-2}\check{\nu}(W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big)\Big]\\
\lesssim\,&\Big\{\varepsilon^{-2}\int_{\mathbb{R}^3}\mathbf{k}_\beta(W_{\mathbf{cl}}(s;t,x,v),u)|\varepsilon h^\ell(s, Y_{\mathbf{cl}}(s;t,x,v),u)|\mathrm{d}u\\
&+\varepsilon^{-3/2}\langle W_{\mathbf{cl}}(s;t,x,v)\rangle\|\varepsilon h^\ell(s)\|_\infty^2 + \varepsilon^{1/2}|A|\Big\}\exp\Big(-\int_s^t\varepsilon^{-2}\check{\nu}(W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big)\\
\lesssim\,&\big\{1+\varepsilon^{1/2}\langle W_{\mathbf{cl}}(s;t,x,v)\rangle\|\varepsilon h^\ell(s)\|_\infty\big\}\|\varepsilon h^\ell(s)\|_\infty\varepsilon^{-2}\exp\Big(-\int_s^t\varepsilon^{-2}\check{\nu}(W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big)\\
&+\|A\|_\infty\varepsilon^{5/2}\varepsilon^{-2}\exp\Big(-\int_s^t\varepsilon^{-2}\check{\nu}(W_{\mathbf{cl}}(\tau;t,x,v))\mathrm{d}\tau\Big),
\end{aligned}$$

where we used the fact $\int_{\mathbb{R}^3}\mathbf{k}_\beta(W_{\mathbf{cl}}(s;t,x,v),u)\mathrm{d}u\lesssim 1$ and $w\Gamma(\frac{\varepsilon h^\ell}{w},\frac{\varepsilon h^\ell}{w})(v)\lesssim\langle v\rangle\|\varepsilon h^\ell\|_\infty^2$.

Then, for $t\in[0,\varepsilon^2T]$,

$$
\begin{aligned}
&|\varepsilon h^{\ell+1}(t,x,v)|\\
\lesssim{}& \mathbf{1}_{\{\tilde t_1<0\}}\mathrm{e}^{-CT}\|\varepsilon h^{\ell+1}(0)\|_\infty+\int_{\max\{0,\tilde t_1(x,v)\}}^t \mathrm{d}s\,\frac{1}{\varepsilon^2}\exp\Big(-\int_s^t\frac{\check\nu(V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v))}{\varepsilon^2}\mathrm{d}\tau\Big)\\
&\times\Big\{\Big(1+\varepsilon^{1/2}\Big\langle V_{\mathbf{cl}}\Big(t-\frac{t-s}{\varepsilon};t,x,v\Big)\Big\rangle\Big)\|\varepsilon h^\ell(s)\|_\infty)\|\varepsilon h^\ell(s)\|_\infty+\varepsilon^{5/2}\|A\|_\infty\Big\}\\
&+\mathbf{1}_{\{\tilde t_1\ge 0\}}\exp\Big(-\int_{\tilde t_1}^t\frac{\tilde\nu(V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v))}{\varepsilon^2}\mathrm{d}\tau\Big)O(\varepsilon^{1/2})\mu(\tilde v_1)^{\frac12-}\\
&+\mathbf{1}_{\{\tilde t_1\ge 0\}}\frac{1}{\tilde w(v)}\exp\Big(-\int_{\tilde t_1}^t\frac{\tilde\nu(t-\frac{t-\tau}{\varepsilon};t,x,v)}{\varepsilon^2}\mathrm{d}\tau\Big)\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}H,
\end{aligned}
$$

where H is given by

$$
\begin{aligned}
&\sum_{l=1}^{k-1}\mathbf{1}_{\tilde t_{l+1}\le 0<\tilde t_l}\|\varepsilon h^{\ell-l}(0)\|_\infty\mathrm{d}\Sigma_l(0)+\sum_{l=1}^{k-1}\int_{\max\{0,\tilde t_{l+1}\}}^{\tilde t_l}\mathbf{1}_{\tilde t_l>0}\Big\{\Big(1+\varepsilon^{1/2}\Big\langle V_{\mathbf{cl}}\Big(t-\frac{t-\tau}{\varepsilon};t,x,v\Big)\Big\rangle\Big)\\
&\times\|\varepsilon h^{\ell-l}(\tau)\|_\infty\|\varepsilon h^{\ell-l}(\tau)\|_\infty+\varepsilon^{5/2}\|A\|_\infty\Big\}\mathrm{d}\Sigma_l(\tau)\mathrm{d}\tau\\
&+\sum_{l=1}^{k-1}\mathbf{1}_{\tilde t_l>0}O(\varepsilon^{1/2})\mu(v_l)^{\frac12-}\mathrm{d}\Sigma_l(\tilde t_{l+1})+\mathbf{1}_{\tilde t_k>0}\|\varepsilon h^{\ell+1-k}(\tilde t_k)\|_\infty\mathrm{d}\Sigma_{k-1}(\tilde t_k),
\end{aligned}
$$

and $\mathrm{d}\Sigma_{k-1}(\tilde t_k)$ is evaluated at $s=\tilde t_k$ of

$$
\begin{aligned}
\mathrm{d}\Sigma_l(s):=&\Big\{\prod_{j=l+1}^{k-1}\mathrm{d}\sigma_j\Big\}\Big\{\exp\Big(-\int_s^{\tilde t_l}\frac{\tilde\nu(V_{\mathbf{cl}}(\tilde t_l-\frac{\tilde t_l-\tau}{\varepsilon};\tilde t_l,x_l,v_l))}{\varepsilon^2}\mathrm{d}\tau\Big)\tilde w(v_l)\mathrm{d}\sigma_l\Big\}\\
&\prod_{j=1}^{l-1}\Big\{\exp\Big(-\int_{\tilde t_{j+1}}^{\tilde t_j}\frac{\tilde\nu(V_{\mathbf{cl}}(\tilde t_j-\frac{\tilde t_j-\tau}{\varepsilon};\tilde t_j,x_j,v_j))}{\varepsilon^2}\mathrm{d}\tau\Big)\mathrm{d}\sigma_j\Big\}.
\end{aligned}
$$

With the choice of $k=C_1T_0^{5/4}$ (clearly $0\le t\le\varepsilon^2T\ll\varepsilon T_0$), for $t\in[0,\varepsilon^2T]$,

$$
\begin{aligned}
&|\varepsilon h^{\ell+1}(t,x,v)|\\
\lesssim{}& C_1T_0^{5/4}\Big\{\mathrm{e}^{-CT}\max_{0\le l\le\ell+1}\|\varepsilon h^l(0)\|_\infty+T\max_{0\le l\le\ell+1}\sup_{0\le s\le t}\|\varepsilon h^l(s)\|_\infty+\varepsilon^{5/2}\|A\|_\infty+O(\varepsilon^{5/2})\\
&+\underbrace{\int_0^t\frac{\check\nu(V_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon};t,x,v))}{\varepsilon^2}\mathrm{e}^{-\int_s^t\frac{\check\nu(V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v))}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}s}_{\lesssim 1}\times\varepsilon^{1/2}\max_{0\le l\le\ell}\sup_{0\le s\le t}\|\varepsilon h^l(s)\|_\infty^2\Big\}\\
&+T_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}\max_{0\le l\le\ell}\sup_{0\le s\le t}\|\varepsilon h^l(s)\|_\infty.
\end{aligned}
$$

For $T_0\gg 1,\ 0<T\ll 1$, and $0<\varepsilon\ll 1$, using (3.7.1) for all $0\le l\le\ell$

$$
\begin{aligned}
&\sup_{0\le t\le \varepsilon^2 T}\|\varepsilon h^{\ell+1}(t)\|_\infty\\
\le& C_{T_0}\big\{\max_l\|\varepsilon h^l(0)\|_\infty+\varepsilon^{5/2}\|A\|_\infty+O(\varepsilon^{5/2})\big\}\\
&+[T+O(\varepsilon^{1/2})+o(1)]C_{T_0}C_1\big\{\max_l\|\varepsilon h^l(0)\|_\infty+\varepsilon^{5/2}\|A\|_\infty+O(\varepsilon^{5/2})\big\}\\
\le& C_1\big\{\max_l\|\varepsilon h^l(0)\|_\infty+\varepsilon^{5/2}\|A\|_\infty+O(\varepsilon^{5/2})\big\},
\end{aligned}
$$

for $C_1\ge 10C_{T_0}$.

Step 3 From Step 2, $wR^\ell\to wR$ weak-$*$ in $L^\infty([0,\varepsilon^2T]\times\Omega\times\mathbb{R}^3)$ up to subsequence. Clearly R satisfies the bound (3.7.1). On the other hand, following Step 2, wR^ℓ is a Cauchy sequence in $L^\infty([0,\varepsilon^2T]\times\Omega\times\mathbb{R}^3)$. It is standard to show that R solves (3.7.2) and (3.7.3) with $R^{\ell+1}=R=R^\ell$. Therefore $F=\mu+\varepsilon\{f_1+\varepsilon f_2+\varepsilon^{1/2}R\}\sqrt{\mu}$ solves the Boltzmann equation with diffuse BC. Since the unique solution R has a uniform-in-time bound from Theorem 1.2, we can continue the Step 2 for $[\varepsilon^2T,2\varepsilon^2T],[2\varepsilon^2T,3\varepsilon^2T],\cdots$, to conclude $wR^\ell\to wR$ in $L^\infty(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$. Therefore $F^\ell\to F\ge 0$ a.e.

Step 4 Let, for sufficiently large m,

$$F_0(x,v)=\mu+\sqrt{\mu}[\varepsilon(f_{1,s}+\tilde f_1(0))+\varepsilon^2(f_{2,s}+\tilde f_2(0)]+\sqrt{\mu}\varepsilon^{3/2}R_s+\mu^{3/4}\mathbf{1}_{|v|>m|\log\varepsilon|}.$$

Clearly, by the L^∞ estimate of R_s we have $F(0)\ge 0$. Moreover, F_0 satisfies the assumptions of Theorem 1.2. By Theorem 1.2, we have $\|F(t)-F_s\|_{L^2}\lesssim \mathrm{e}^{-\lambda t}$. Then, as $t\to\infty$, for any non negative test function $\psi(x,v)$,

$$
\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3}F_s(x,v)\psi(x,v)\mathrm{d}x\mathrm{d}v\\
=&\iint_{\Omega\times\mathbb{R}^3}F(t,x,v)\psi(x,v)\mathrm{d}x\mathrm{d}v+O(1)\iint_{\Omega\times\mathbb{R}^3}|F_s(x,v)-F(t,x,v)|\psi(x,v)\mathrm{d}x\mathrm{d}v\\
\ge& 0-O(1)\|F(t)-F_s\|_{L^2(\Omega\times\mathbb{R}^3)}\ge 0.
\end{aligned}
$$

This proves $F_s(x,v)\ge 0$ a.e. The proof is completed.

Appendix A Basic Estimates of the Fluid Equations

In this Appendix, to simplify the formulas, we set $\mathfrak{v}=1$ and $\kappa=1$, since they do not play any role in the estimates.

Lemma A.1 *Let (u_s,ϑ_s) be the H^2-solution to the steady INSF* (1.4.1)*. Assume* (1.4.4)*. Then*

$$
\begin{aligned}
&\|u_s\|_{W_x^{2,\frac32+}}+\|\vartheta_s\|_{H_x^{\frac32+}}+\|u_s\|_{L_x^6}+\|\vartheta_s\|_{L_x^6}+|\nabla_x u_s|_{L^2(\partial\Omega)}+|\nabla_x\vartheta_s|_{L^2(\partial\Omega)}\\
\lesssim& \|\Phi\|_{L_x^{\frac32+}}+|\vartheta^w|_{H^{1+}(\partial\Omega)}.
\end{aligned}
$$

If we further assume $\Phi\in H_x^r$ and $\theta_w\in H^{r+\frac32}(\partial\Omega)$ then $u_s,\vartheta_s\in H_x^{r+2}$.

Lemma A.2 *Let* (u, ϑ, p) *be* H^k *solution to* (1.2.7). *Set* $u = \tilde{u} + u_s$, $\vartheta = \tilde{\vartheta} + \vartheta_s$, $p = \tilde{p} + p_s$, *where* $(\tilde{u}, \tilde{\vartheta}, \tilde{p})$ *solves* (1.4.8). *Assume*

$$\|u_s\|_{H_x^1} + \|\tilde{u}_t(0)\|_{L_x^2} \ll 1, \quad \|\vartheta_s\|_{H_x^1} < \infty.$$

Then, for any $k \geq 0$ *and for* $0 < \lambda \ll 1$,

$$\sum_{0\leq i\leq k} \Big[\|\mathrm{e}^{\lambda t}\partial_t^i \tilde{u}(t)\|_{L_x^2}^2 + \|\mathrm{e}^{\lambda t}\partial_t^i \tilde{\vartheta}(t)\|_{L_x^2}^2 + \int_0^t \|\mathrm{e}^{\lambda t}\partial_t^i \tilde{u}\|_{H_x^1}^2 + \int_0^t \|\mathrm{e}^{\lambda t}\partial_t^i \tilde{\vartheta}\|_{H_x^1}^2\Big]$$
$$\lesssim P\bigg(\sum_{j=0}^{[\frac{k}{2}]} \big[\|\partial_t^j \tilde{u}(0)\|_{H_x^1} + \|\partial_t^j \tilde{\vartheta}(0)\|_{H_x^1}\big] + \sum_{j=[\frac{k}{2}]+1}^{k} \big[\|\partial_t^j \tilde{u}(0)\|_{L_x^2} + \|\partial_t^j \tilde{\vartheta}(0)\|_{L_x^2}\big]\bigg), \quad \text{(A.1)}$$

whenever the RHS is finite for some polynomial P.

Moreover, for some $0 < \lambda \ll 1$ *and polynomial* P_0 *with* $P_0(s) = O(s)$,

$$\|\partial_t^{k-(1+r)} \tilde{u}(t)\|_{H_x^{2r+2}} + \|\nabla_x \partial_t^{k-(1+r)} \tilde{p}(t)\|_{H_x^{2r}} \lesssim P_0\bigg(\sum_{j=0}^{k} \|\partial_t^j \tilde{u}(t)\|_{L_x^2}\bigg) \lesssim \mathrm{e}^{-\lambda t},$$

$$\|\partial_t^{k-(1+r)} \tilde{\vartheta}(t)\|_{H_x^{2r+2}} \lesssim P_0\bigg(\sum_{j=0}^{k} \|\partial_t^j \tilde{u}(t)\|_{L_x^2} + \sum_{j=0}^{k} \|\partial_t^j \tilde{\vartheta}(t)\|_{L_x^2}\bigg) \lesssim \mathrm{e}^{-\lambda t},$$

whenever the RHS of (A.1) *is finite and* $\vartheta_s, u_s \in H_x^{2r+1}$.

Furthermore,

$$\|\mathrm{e}^{\lambda t}\tilde{u}\|_{L_x^6 L_t^\infty} + \|\mathrm{e}^{\lambda t}\tilde{\vartheta}\|_{L_x^6 L_t^\infty} \lesssim P(\|\tilde{u}(0)\|_{H_x^1} + \|\tilde{\vartheta}(0)\|_{H_x^1} + \|\tilde{u}_t(0)\|_{L_x^2} + \|\tilde{\vartheta}_t(0)\|_{L_x^2}),$$
$$\|\mathrm{e}^{\lambda t}\tilde{u}_t\|_{L_x^6 L_t^\infty} + \|\mathrm{e}^{\lambda t}\tilde{\vartheta}_t\|_{L_x^6 L_t^\infty} \lesssim P\big(\|\partial_{tt}\tilde{u}(0)\|_{L_x^2} + \|\partial_t \tilde{u}(0)\|_{H_x^1} + \|\tilde{u}(0)\|_{H_x^1}$$
$$+ \|\partial_{tt}\tilde{\vartheta}(0)\|_{L_x^2} + \|\tilde{\vartheta}_t(0)\|_{H_x^1} + \|\tilde{\vartheta}(0)\|_{H_x^1}\big).$$

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