Ann. of Appl. Math.
36:3(2020), 221-234

TRAVELING WAVE SOLUTIONS FOR A PREDATOR-PREY MODEL WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE*†

Zhuanling Gao, Xiangkui Zhao‡, Zhihong Zhao

(*School of Mathematics and Physics, University of Science & Technology Beijing, Beijing 100083, PR China*)

Abstract

In this paper, we study a class of predator-prey models with Beddington-DeAngelis functional response. And the predator equation has singularity in zero prey population, where a smoothing auxiliary function is introduced to overcome it. Our aim is to see if the predator and prey can eventually survive when an alien predator enters the habitat of an existing prey by employing traveling wave solutions, based on the upper and lower solutions and Schauder's fixed point theorem. In addition, the non-existence of traveling wave solutions is discussed by the comparison principle. At the same time, some simulations are carried out to further verify the results.

Keywords predator-prey model; Beddington-DeAngelis; traveling wave solution; existence

2000 Mathematics Subject Classification 35K10; 35K57

1 Introduction

In recent years, we have found that some species are endangered by their predators or other reasons. This will cause ecologically bankrupt. Therefore we pay more attention to it. The predator-prey model is an important tool to study the relationship of several species. And it is a topic attracting more and more attention from mathematicians and ecologists [3, 5–7, 12, 16]. So it has very broad application prospects. Whether the predator and prey can survive eventually is equivalent to the existence of traveling wave solution for specific model. So the traveling wave solution has studied by many scholars, see [1, 9, 10, 15, 19, 20] and their references.

*Supported by the NNSF of PR China (11071014,11071205,11001032), the Fundamental Research Funds for the Central Universities (06108114).

†Manuscript received March 13, 2020

‡Corresponding author. E-mail: xiangkuizh@ustb.edu.cn

In 2016, Chen, Yao, and Guo [4] studied the diffusion predator-prey model of Lotka-Volterra type functional response. The model is as follows:

$$\begin{cases} u_t = u_{xx} + ru(1-u) - rkuv, \\ v_t = dv_{xx} + sv\Big(1 - \dfrac{v}{u}\Big), \end{cases} \tag{1.1}$$

where u, v represent the population densities of the prey and predator species at position x and time t, respectively, d, r, s, k are constant numbers, d is the diffusion coefficient, and r, s are the intrinsic growth rates of species u, v, respectively. The functional response of the predator to the prey is given by the Lotka-Volterra type rku.

Then, Zhao [20] studied the model as $k = 1$, namely

$$\begin{cases} u_t = u_{xx} + ru(1-u) - ruv, \\ v_t = dv_{xx} + sv\Big(1 - \dfrac{v}{u}\Big). \end{cases} \tag{1.2}$$

In 2017, Ai, Du, Peng [1] studied the traveling wave solution of the Holling-Tanner predator-prey model:

$$\begin{cases} u_t = d_1 u_{xx} + u(1-u) - \dfrac{\alpha u^m}{1+\beta u^m} v, \\ v_t = d_2 v_{xx} + rv\Big(1 - \dfrac{v}{u}\Big), \end{cases} \tag{1.3}$$

where u, v represent the population densities of the prey and the predator at position x and time t, respectively, the parameters α, m and r are positive and β is non-negative. Here, the predation rate in the prey equation is controlled by a so-called Holling type functional response. The predator equation is also singular at zero prey population. When $m = 1, \beta = 0$, model (1.3) becomes model (1.2).

The functional response function of (1.3) is a class of Holling-II type functional response function only depending on the prey. However, in reality, it is not independent of predator either. In fact, the B-D functional response function $\Phi(u, v) = \frac{cu}{m_0 + m_1 u + m_2 v}$ maintains all the advantages of the ratio-dependent response function and avoids the controversy caused by the low-density problem, so it can better reflect the real relationship of the two species [8, 11, 17, 18]. When $m_1 = 0$ and $m_2 \neq 0$, the Beddington-DeAngelis type functional response is also a class of Holling-II type [13, 14].

We will consider the following model:

$$\begin{cases} U_t = U_{xx} + U(1-U) - \dfrac{\alpha U}{1 + \beta_1 U + \beta_2 V} V, \\ V_t = dV_{xx} + rV\Big(1 - \dfrac{V}{U}\Big), \end{cases} \tag{1.4}$$

where U, V represent the population densities of the prey and the predator at position x and time t, respectively, and the constant d is the diffusion coefficient corresponding to U, V. The parameters r, c, d, α are all normal and β_1, β_2 are non-negative. r is the intrinsic growth rate of the prey.

It is easy to conclude that (1.4) has two equilibrium points $E_0 = (1, 0)$ and $E^* = (U^*, V^*)$, where

$$U^* = V^* = \frac{\beta_1 + \beta_2 - \alpha - 1 + \sqrt{(\alpha - \beta_1 - \beta_2 + 1)^2 + 4(\beta_1 + \beta_2)}}{2(\beta_1 + \beta_2)}.$$

In this paper, we consider the case that the habitat is the entire space R. What we are interested in is if an exotic predator is introduced into the habitat of an existing prey, whether the predator and prey can eventually survive. In fact, this problem is equivalent to whether the solution of (1.4) tends to be the only positive constant steady state as time approaches infinity. Therefore, we study the traveling wave solution as below.

If there are positive functions u and v defined on R such that $U(x,t) = u(x+ct)$, $V(x,t) = v(x+ct)$, then the solution of (1.4) is called the traveling wave of speed c. Here u, v is the waveforms, set $z = x + ct$ and bring $(U, V)(x,t) = (u, v)(z)$ into (1.4). Then the traveling wave satisfies the following system of equations:

$$\begin{cases} u''(z) - cu'(z) + u(z)(1 - u(z)) - \dfrac{\alpha u(z)}{1 + \beta_1 u(z) + \beta_2 v(z)} v(z) = 0, \\ dv''(z) - cv'(z) + rv(z)\Big(1 - \dfrac{v(z)}{u(z)}\Big) = 0. \end{cases} \tag{1.5}$$

As above, this paper mainly considers the traveling wave solutions of connections (1,0) to (U^*, V^*). It means that (u, v) satisfies the following condition

$$\lim_{z \to -\infty} (u, v) = (1, 0), \quad \lim_{z \to +\infty} (u, v) = (U^*, V^*). \tag{1.6}$$

In order to overcome the singularity of the predator equation, an auxiliary system of (1.5) similar to [1] was introduced.

The rest of the paper is organized as follows: In the second section, an auxiliary system is introduced. We construct a pair of upper and lower solutions of the auxiliary system and study the existence of the weak traveling wave solutions of the auxiliary system by Schauder's fixed point theorem. Here, the weak traveling wave solution means that the solution is connected (1,0) at $z \to -\infty$, but at $z \to +\infty$ is not necessarily connected (U^*, V^*). By proving that $u(z)$ has a lower bound, we know the weak traveling wave of the auxiliary system is the weak traveling wave of the original model. In the third section, we use a squeeze method to prove that the weak traveling wave is actually a traveling wave. That is, $\lim_{z \to +\infty} (u, v) = (U^*, V^*)$. And the traveling wave solution of the original model is obtained. In the forth section, some simulations are given for the theoretical results by Matlab. Nonexistence of traveling waves solutions is proved in Section 5. Finally, we give the details of the verifications of all constructed upper and lower solutions in Supplementary.

For convenience, we will use x instead of variable z below.

2 The Solutions of Auxiliary System

To overcome the singularity of predator equation, we introduce the following auxiliary system [1]:

$$\begin{cases} u''-cu'+u(1-u)-\dfrac{\alpha u}{1+\beta_1 u+\beta_2 v}v=0,\\ dv''-cv'+rv\Big(1-\dfrac{v}{\sigma_\varepsilon(u)}\Big)=0. \end{cases} \tag{2.1}$$

Here we replace the reaction function $rv(1-\frac{v}{u})$ with a smooth funtion $rv(1-\frac{v}{\sigma_\varepsilon(u)})$, where

$$\sigma_\varepsilon(u)=\begin{cases} u, & u\ge\varepsilon\\ u+\varepsilon \mathrm{e}^{\frac{1}{u-\varepsilon}}, & 0\le u<\varepsilon \end{cases} \tag{2.2}$$

with $\varepsilon>0$ sufficiently small.

2.1 Upper and lower solutions

First, we give the definition of upper and lower solutions of system (2.1) as follows.

Definition 2.1 The functions $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ are called a pair of upper and lower solutions of (2.1), if $\overline{u}'$, $\underline{u}'$, $\overline{v}'$, $\underline{v}'$, $\overline{u}''$, $\underline{u}''$, $\overline{v}''$, $\underline{v}''$ are bounded and the inequalities

$$\begin{cases} \overline{u}''(x)-c\overline{u}'(x)+\overline{u}(x)(1-\overline{u}(x))-\dfrac{\alpha\overline{u}(x)}{1+\beta_1\overline{u}(x)+\beta_2\underline{v}(x)}\underline{v}(x)\le 0,\\ \underline{u}''(x)-c\underline{u}'(x)+\underline{u}(x)(1-\underline{u}(x))-\dfrac{\alpha\underline{u}(x)}{1+\beta_1\underline{u}(x)+\beta_2\overline{v}(x)}\overline{v}(x)\ge 0,\\ d\overline{v}''(x)-c\overline{v}'(x)+r\overline{v}(x)\Big(1-\dfrac{\overline{v}(x)}{\sigma_\varepsilon(\overline{u}(x))}\Big)\le 0,\\ d\underline{v}''(x)-c\underline{v}'(x)+\underline{v}(x)\Big(1-\dfrac{\underline{v}(x)}{\sigma_\varepsilon(\underline{u}(x))}\Big)\ge 0 \end{cases} \tag{2.3}$$

hold for $x\in\mathbb{R}\setminus D$ with some finite set $D=\{x_1,\cdots,x_m\}$.

Throughout the paper, for $c^*=2\sqrt{dr}$, we denote

$$\lambda_1=\frac{c-\sqrt{c^2-4dr}}{2d},\quad \lambda_2=\frac{c+\sqrt{c^2-4dr}}{2d},\quad \lambda_3=\frac{c+\sqrt{c^2+4r}}{2}.$$

In fact, we have

$$d\lambda_i^2 - c\lambda_i + r = 0,\ i = 1,2 \quad \text{and} \quad \lambda_3^2 - c\lambda_3 - r = 0.$$

Next, to consider the existence of upper and lower solutions of (2.1), we divide it into two cases: $c > c^*$ and $c = c^*$.

2.1.1 The case $c > c^*$

First, for given constants $A > 1,\ \eta > 0$, we consider a function

$$f(x) = \mathrm{e}^{\lambda_1 x} - A\mathrm{e}^{(\lambda_1+\eta)x}.$$

Then it is easy to check that the function has a unique zero point at $x_0 = -\frac{\ln A}{\eta}$ and a unique maximum point at $x_M = -\frac{\ln(A(1+\frac{\eta}{\lambda_1}))}{\eta} < x_0$. And f is continuous on $\mathbb{R}$ and positive on $(-\infty,z_0)$.

Next, we choose the constants $\eta, \gamma, \beta, \alpha, \beta_1, \beta_2$ and A satisfying the following conditions

(A1) $\eta \in (0, \min\{\lambda_1, \lambda_2 - \lambda_1\})$, $\gamma > 0$ is small enough such that $\gamma < \lambda_1$ and $\gamma^2 - c\gamma < 0$;

(A2) $\beta > -\frac{1}{\gamma^2 - c\gamma}$;

(A3) $A > \max\left\{1, \frac{r}{-\varepsilon \mathrm{e}^{-\frac{1}{\varepsilon}}(d(\lambda+\eta)^2 - c(\lambda+\eta) - r)}\right\}$;

(A4) $0 < \alpha - \beta_2 < 1,\ \beta_1 > \beta_2$.

Now we introduce functions $\overline{u}(x),\ \overline{v}(x),\ \underline{u}(x),\ \underline{v}(x)$ as follows:

$$\overline{u}(x) = 1, \quad x \in \mathbb{R}, \tag{2.4}$$

$$\underline{u}(x) = \begin{cases} 1 - \beta \mathrm{e}^{\gamma x}, & x \le a_1, \\ 0, & x > a_1, \end{cases} \tag{2.5}$$

$$\overline{v}(x) = \begin{cases} \mathrm{e}^{\lambda_1 x}, & x \le 0, \\ 1, & x \ge 0, \end{cases} \tag{2.6}$$

$$\underline{v}(x) = \begin{cases} \mathrm{e}^{\lambda_1 x}(1 - A\mathrm{e}^{\eta x}), & x \le a_0, \\ 0, & x \ge a_0, \end{cases} \tag{2.7}$$

where $a_0 = -\frac{1}{\eta}\ln A$, $a_1 = -\frac{1}{\gamma}\ln \beta$.

Lemma 2.1 *Assume that $c > c^*$, then the functions $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ defined by* (2.4)-(2.7) *are a pair of upper and lower solutions of* (2.1).

For the specific details about the proof of Lemma 2.1, we refer the readers to Supplementary of this paper.

2.1.2 The case $c = c^*$

In this subsection ,we consider the existence of the upper and lower solutions of (2.1), when $c = c^* = 2\sqrt{dr}$. In this case, we have $\lambda_1 = \lambda_2 = \frac{c}{2d}$. For given positive constants $M_1 = \frac{\lambda_1 \mathrm{e}^2}{2}$ and $N = M_1\sqrt{\frac{2}{\lambda_1}}$. Introduce a function in [4]

$$g(x) = [-M_1 x - N(-x)^{\frac{1}{2}}]\mathrm{e}^{\lambda_1 x}, \quad z \le 0,$$

then $x_0 = -\left(\frac{N}{M_1}\right)^2$ is the unique zero of g in $(-\infty, 0)$. Moreover, $g > 0$ in $(-\infty, x_0)$ has a unique maximum point $\tilde{x}$ in $(-\infty, x_0)$.

Next, we choose the constants $\beta, \gamma, N, \alpha, \beta_1$ and β_2 satisfying the following conditions:

(B1) $0 < \gamma \ll 1$ satisfies $\gamma^2 - c\gamma < 0,\ \lambda_1 > 2\gamma$;

(B2) $\beta > \max\left\{\mathrm{e}, -\frac{M_1}{\gamma \mathrm{e}(\gamma^2 - c\gamma)}\right\}$;

(B3) $N > \max\left\{M_1\sqrt{\frac{2}{\lambda_1}}, \frac{4rM_1^2}{d\varepsilon \mathrm{e}^{-\frac{1}{\varepsilon}}}\left(\frac{7}{2\mathrm{e}\lambda}\right)^{\frac{7}{2}}\right\}$;

(B4) $0 < \alpha - \beta_2 < 1,\ \beta_1 > \beta_2$.

Now we introduce functions $\overline{u}(x),\ \overline{v}(x),\ \underline{u}(x),\ \underline{v}(x)$ as follows:

$$\overline{u}(x) = 1, \quad x \in \mathbb{R}, \tag{2.8}$$

$$\underline{u}(x) = \begin{cases} 1 - \beta \mathrm{e}^{\gamma x}, & x \le a_1, \\ 0, & x > a_1, \end{cases} \tag{2.9}$$

$$\overline{v}(x) = \begin{cases} -M_1 x \mathrm{e}^{\lambda_1 x}, & x \le 0, \\ 1, & x > 0, \end{cases} \tag{2.10}$$

$$\underline{v}(x) = \begin{cases} [-M_1 x - N(-x)^{\frac{1}{2}}]\mathrm{e}^{\lambda_1 x}, & x \le a_0, \\ 0, & x > a_0, \end{cases} \tag{2.11}$$

where $a_0 = -\frac{1}{\eta}\ln A$, $a_1 = -\frac{1}{\gamma}\ln \beta$.

Lemma 2.2 *Assume that $c = c^*$, then the functions $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ defined by* (2.8)-(2.11) *are a pair of upper and lower solutions of* (2.1).

For the specific details about the proof of Lemma 2.1, we refer the readers to Supplementary of this paper.

2.2 Existence of weak traveling wave solutions

According to the form of the upper and lower solutions constructed in the previous text, we will use the Schauder's fixed point theorem to prove the existence of solution of system (2.1) in this section.

First, we introduce the following function spaces

$$X = \{\Phi = (u, v) : \Phi \text{ is a bounded and uniformly continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\},$$

then X is a Banach space equipped with the standard supremum norm. In the paper, we use the standard partial ordering and order intervals in $\mathbb{R}$ or $\mathbb{R}^2$, and apply $\|\cdot\|$ to denote the norm in $\mathbb{R}^2$. Further define

$$X_k = \{(u, v) \in X : 0 \le u(x) \le 1 \text{ and } 0 \le v(x) \le 1 \text{ for all } x \in \mathbb{R}\},$$

Next, define functions

$$\begin{cases} F_1(y_1,y_2)=\beta y_1+y_1\Big(1-y_1-\dfrac{\alpha}{1+\beta_1 y_1+\beta_2 y_2}y_2\Big),\\ F_2(y_1,y_2)=\beta y_2+ry_2\Big(1-\dfrac{y_2}{\sigma_\varepsilon(y_1)}\Big),\end{cases}$$

for some constant β. By $\beta>\max\{1+\frac{\alpha^2}{1+\beta_1+\beta_2},\frac{2r}{\varepsilon e^{-\frac{1}{\varepsilon}}}\}$, we get that F_1 is nondecreasing in y_1 and is decreasing in y_2 for $\delta_0\le y_1\le 1$ and $0\le y_2\le 1$. At the same time, F_2 is nondecreasing with respect to y_1 and y_2 for $\delta_0\le y_1\le 1$ and $0\le y_2\le 1$.

Now we define

$$\lambda_1^{\pm}=\frac{1}{2}(c\pm\sqrt{c^2+4\Lambda}),\quad \lambda_2^{\pm}=\frac{1}{2d}(c\pm\sqrt{c^2+4d\Lambda}).$$

Obviously, $\lambda_1^-<0<\lambda_1^+$, $\lambda_2^-<0<\lambda_2^+$ and

$$\begin{cases}(\lambda_1^{\pm})^2-c(\lambda_1^{\pm})-\beta=0,\\ d(\lambda_2^{\pm})^2-c(\lambda_2^{\pm})-\beta=0.\end{cases}$$

For $\Phi=(\phi_1,\phi_2)\in X_k$, define an operator $P=(P_1,P_2):X_k\to X$ as follows:

$$\begin{cases} P_1(u,v)(x)=\dfrac{1}{\lambda_1^+-\lambda_1^-}\Big(\displaystyle\int_{-\infty}^{x}e^{\lambda_1^-(x-y)}+\int_x^{+\infty}e^{\lambda_1^+(x-y)}\Big)F_1(u,v)\mathrm{d}y,\\ P_2(u,v)(x)=\dfrac{1}{\lambda_2^+-\lambda_2^-}\Big(\displaystyle\int_{-\infty}^{x}e^{\lambda_2^-(x-y)}+\int_x^{+\infty}e^{\lambda_2^+(x-y)}\Big)F_2(u,v)\mathrm{d}y.\end{cases}$$

It is easy to see the operator $P=(P_1,P_2):X_k\to X$ is a solution of (2.1).

Lemma 2.3 *Let $c\ge c^*$. Assume that $(\overline{u},\overline{v}),(\underline{u},\underline{v})$ in X_k is a pair of upper and lower solutions of* (2.1) *satisfying*

(1) $\overline{u}(x)\ge\underline{u}(x),\ \overline{v}(x)\ge\underline{v}(x),\ x\in\mathbb{R}$,

(2) $\overline{u}'(x-)\ge\overline{u}'(x+),\ \overline{v}'(x-)\ge\overline{v}'(x+),\ \underline{u}'(x-)\le\underline{u}'(x+),\ \underline{u}'(x-)\le\underline{u}'(x+)$.

Then (2.1) *has a positive solution (u,v) such that $\overline{u}(x)\ge u\ge\underline{u}(x),\ \overline{v}(x)\ge v\ge\underline{v}(x),\ x\in\mathbb{R}$.*

The proof can be shown by a similar argument as in [4]. We will not repeat it here.

Theorem 2.1 *Assume that $c\ge c^*$, there is a set of positive solutions (u,v) in* (2.1) *such that $\lim\limits_{z\to-\infty}(u,v)(x)=(1,0)$ and $\underline{u}(x)<u(x)<\overline{u}(x)$, $\underline{v}(x)<v(x)<\overline{v}(x)$, $x\in\mathbb{R}$.*

Proof First we prove that the case $c>c^*$. By Lemma 2.1, we know that (2.4)-(2.7) are a pair of upper and lower solutions of (2.1). Now we show that conditions (1) and (2) in Lemma 2.3 hold for the case $c>c^*$. When $x\ge a_1$, we have

$$\overline{u}-\underline{u}=1>0.$$

When $x<a_1$, we have

$$\overline{u}-\underline{u}=\beta e^{\gamma x}>0.$$

Similarly we can show that $\overline{v} \geq \underline{v}$. Therefore, condition (1) of Lemma 2.3 holds. Then, for condition (2), we have

$$\begin{aligned}
&\underline{u}'(a_1-) = -\gamma < 0 = \underline{u}'(a_1+),\\
&\underline{v}'(a_0-) = -\eta e^{\lambda a_0} < 0 = \underline{v}'(a_0+),\\
&\underline{v}'(a_2-) = \lambda > 0 = \underline{v}'(a_2+).
\end{aligned}$$

Thus there exists a pair of functions with $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$.

That $\lim\limits_{z\to-\infty}(u,v)(x) = (1,0)$ is trivial. The case $c = c^*$ can be proved similarly. The proof is completed.

Based on the above theorem, we show that the prey component in this weak traveling wave solution has a positive lower bound, that is:

Theorem 2.2 *$u(x) \geq \varepsilon$ for $x \in \mathbb{R}$.*

Proof

$$\begin{cases}
M_1 = \sup\Big\{\dfrac{\alpha v}{1+\beta_1 u+\beta_2 v} - u + 1,\ 0 \leq u \leq 1, 0 \leq u \leq 1\Big\},\\[2ex]
T_1^+ = \dfrac{c+\sqrt{c^2+4M_1}}{2},\\[2ex]
M_2 = \dfrac{r}{d} + c\Big(\Big|\dfrac{1}{d}-1\Big|+1\Big)T_1^+ + 2M_1.
\end{cases}$$

Define a δ_0, where δ_0 is a positive solution of

$$(1-u)(1+\beta_1 u) - \frac{dM_2}{r}u[\alpha - \beta_2(1-u)] = 0,$$

here we let $\varepsilon < \delta_0$. Assume $(u(x), v(x))$ to be a weak traveling wave solution of (2.12). Here we know $0 < u(x) \leq 1$ and $0 < v(x) \leq 1$ for all $x \in \mathbb{R}$. It is obviously that $\delta_0 > 0$, $M_1 > 0$ and

$$-\frac{1}{1+\beta_1 u+\beta_2 v}[(1-u)(1+\beta_1 u) - v(\alpha - \beta_2(1-u))] < M_1.$$

We complete the proof in four steps.

Step 1 Show that $\frac{|u'(x)|}{u(x)} \leq T_1^+$ for all $x \in \mathbb{R}$.

Let $T_1 = \frac{u'}{u}$. Using the u-equation we can get

$$\begin{aligned}
T_1' = \frac{u''}{u} - T_1^2 &= cT_1 - \frac{1}{1+\beta_1 u+\beta_2 v}[(1-u)(1+\beta_1 u) - v(\alpha - \beta_2(1-u))] - T_1^2\\
&\leq cT_1 + M_1 - T_1^2.
\end{aligned}$$

Since $T_1(-\infty) = 0$ and T_1^+ is a positive constant solution of $T' = cT + M_1 - T^2$, according the comparison theorem, we have $T_1(x) < T_1^+$ for all $x \in \mathbb{R}$. Similarly, if

$T_1 < -T_1^+$ for some x_0, by letting $T(x)$ be the solution of $T' = cT + M_1 - T^2$ with $T(x_0) = T_1(x_0)$, and applying the comparison theorem, we have $T_1(x) \leq T(x)$ for $x \geq x_0$. Note that

$$cT(x_0) + M_1 - T^2(x_0) < c(-T_1^+) + M_1 - (-T_1^+)^2 < 0$$

means $T(x) \to -\infty$ as $x \to x_1$ for some finite value $x_1 > x_0$. Therefore $T_1(x) \to \infty$ as $x \to x_2$ for some $x_2 \in [x_0, x_1]$, a contradiction. It derives that $T_1(x)$ is defined for all $x \in \mathbb{R}$. To sum up, we know that $\frac{|u'(x)|}{u(x)} = |T_1| \leq T_1^+$ on $\mathbb{R}$.

Steps 2 Show that $\frac{v'(x)}{v(x)} \leq \frac{c-\sqrt{c^2-4dr}}{2d}$ for $x \in \mathbb{R}$.

Steps 3 Show that $\frac{v(x)}{\sigma(u(x))} \leq \frac{dM_2}{r}$ for $x \in \mathbb{R}$.

The proofs of Steps 2 and 3 are similar to Lemma 2.3 in [10]. We omit them here.

Step 4 Show that $u(x) > \varepsilon$ for $x \in \mathbb{R}$.

For contradiction, we suppose that $u(x) \leq \varepsilon$ for some $x \in \mathbb{R}$. Then since $u(-\infty) = 1$ there is a smallest x_0 such that $u(x_0) = \varepsilon$ and $u'(x_0) \leq 0$. According the u-equation of (2.1), since

$$v(x_0) \leq \frac{dM_2}{r}\sigma(u(x_0)) = \frac{dM_2}{r}u(x_0)$$

from Step 3, then

$$v(x_0) \leq \frac{dM_2}{r}u(x_0) = \frac{dM_2}{r}\varepsilon < \frac{dM_2}{r}\delta_0 = \frac{(1-\delta_0)(1+\beta_1\delta_0)}{\alpha - \beta_2 + \beta_2\delta_0}.$$

Thus

$$(1-\delta_0)(1+\beta_1\delta_0) - (\alpha - \beta_2 + \beta_2\delta_0)v(x_0) > 0,$$

we deduce

$$u''(x_0) = cu'(x_0) - \varepsilon(1-\varepsilon) + \frac{\alpha\varepsilon}{1 + \beta_1\varepsilon + \beta_2 v(x_0)}v(x_0) < 0$$

from the choices of ε and δ_0, from which we conclude that $u'(x) \leq 0$ and is not identical to 0 for all $x > x_0$. This contradicts $u(x) \geq 0$. This shows $u(x) > \varepsilon$ for $x \in \mathbb{R}$. The proof is complete.

Consequently $\sigma_\varepsilon(u) \equiv u$, thus the weak traveling wave solution of auxiliary system is the solution of the original model (1.5).

3 The Traveling Wave Solutions

In this section, we show that the solutions (u, v) converge to the coexistence equilibrium (u^*, v^*) as $x \to \infty$ under additional conditions.

Theorem 3.1 *For any $c \geq c^* = 2\sqrt{dr}$, and the parameters satisfying $0 < \alpha - \beta_2 < 1$, $\beta_1 > \beta_2$, there exists a traveling solution (u, v) for (1.5) such that*

$$\lim_{x\to-\infty}(u,v)(x)=(1,0), \tag{3.1}$$
$$\lim_{x\to\infty}(u,v)(x)=(u^*,v^*). \tag{3.2}$$

Here

$$U^*=V^*=\frac{\beta_1+\beta_2-\alpha-1+\sqrt{(\alpha-\beta_1-\beta_2+1)^2+4(\beta_1+\beta_2)}}{2(\beta_1+\beta_2)}.$$

Proof By Theorem 2.2, we know that the solution (u,v) of (2.1) satisfies $u(x)>\varepsilon$ for $x\in\mathbb{R}$. Therefore (1.5) has a positive solution satisfying (3.1) if $c\ge c^*$. Now we consider the tail behavior of the traveling wave for (1.5) at ∞.

Rewrite (1.5) to the following form

$$\begin{cases} u''-cu'+\dfrac{u}{1+\beta_1u+\beta_2v}(\alpha-\beta_2(1-u))(h(u)-v)=0,\\ dv''-cv'+\dfrac{rv}{u}(u-v)=0. \end{cases} \tag{3.3}$$

Here $h(u)=\frac{(1-u)(1+\beta_1u)}{\alpha-\beta_2(1-u)}$. It is easy to verify that there is a unique fixed point $U^*\in(0,1)$ such that $h(U^*)=U^*$.

Moreover, let the second iteration

$$h(h(u))=\frac{[1-h(u)][1+\beta_1h(u)]}{\alpha-\beta_2(1-h(u))}=u,$$

that is $q(u)=h(h(u))-u=0$. Simplify this equation, we have

$$\begin{aligned}&-\beta_1[(1-u)(1+\beta_1u)]^2+(\beta_1-1-\beta_2u)(1-u)(1+\beta_1u)(\alpha-\beta_2(1-u))\\&+(\alpha-\beta_2(1-u))^2(1-(\alpha-\beta_2)u)=0.\end{aligned}$$

Here, we can see that $q(u)$ is a polynomial of degree 4 with $q(\pm)=-\infty$, and

$$\begin{aligned}&q(1)=\alpha^2(1-(\alpha-\beta_2))>0,\\&q(0)=-\beta_1+(\alpha-\beta_2)(\beta_1-1+\alpha-\beta_2)<-\beta_1+(\alpha-\beta_2)\beta_1<0,\\&q\Big(-\frac{1}{\beta_1}\Big)=\Big(\alpha-\beta_2\Big(1+\frac{1}{\beta_1}\Big)\Big)^2\Big(1+\Big(\alpha-\beta_2\Big)\frac{1}{\beta_1}\Big)>0.\end{aligned}$$

Four-term coefficient $-\beta_1^3u^4+\beta_2\beta_1\beta_2u^4=(-\beta_1^3u^4+\beta_1\beta_2^2)u^4<0$.

So $q(u)=0$ does not have any fixed point in the interval $(0,1)$ other than $u=U^*$. See Figure 1.

According to [1], we also can define a similar sequence $\{\zeta_n\}_{n=-1}^{\infty}$ as follows:

$$\zeta_{-1}=0,\quad \zeta_n=h(\zeta_{n+1}),\quad \text{for any } n=-1,0,1,\cdots.$$

Note that h is strictly monotone decreasing on [0,1].

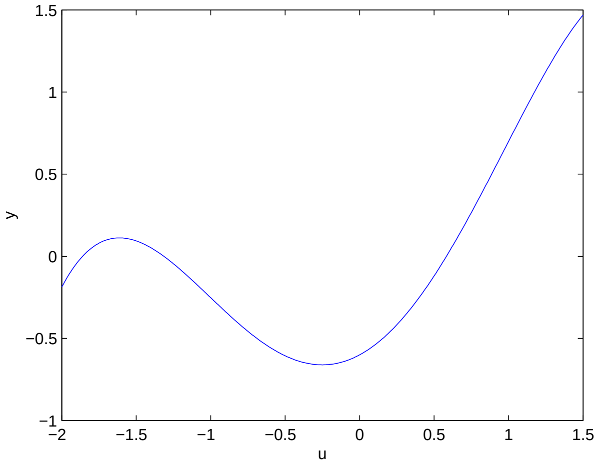

Figure 1: $y = q(u)$: parameters $\alpha = 1.0$, $\beta_1 = 0.7$, $\beta_2 = 0.5$.

Next we apply the method of mathematical induction (see [1]) to get the fact that there exists an increasing sequence $\{x_n\}_{n=0}^{\infty}$ with $x_n \to \infty$ as $n \to \infty$ such that, for $x \geq x_n$ $(n = 0, 1, 2, \cdots)$

$$\zeta_{2n+1} < u(x) < \zeta_{2n}, \quad \zeta_{2n+1} < v(x) < \zeta_{2n}. \tag{3.4}$$

Since $\lim_{n\to\infty} \zeta_n = U^*$, it derives from the fact that $\lim_{x\to\infty} (u(x), v(x)) = (U^*, U^*)$. Consequently, (u, v) is a traveling wave solution of (1.5), and the proof of Theorem 3.1 is complete.

4 Numerical Simulation

Below we use matlab to further find the existence of the heteroclinic orbit between the two equilibrium points, that is, the traveling wave solution corresponding to system (1.5). Further we verify the theoretical results we have obtained.

Figure 2 shows the traveling wave solution of the prey population $u(x, t)$ tends to the positive equilibrium point (U^*, V^*) with parameters $\alpha = 1.0$, $\beta_1 = 0.7$, $\beta_2 = 0.5$; Figure 3 is the traveling wave solution of the prey population in different positions. Figure 4 shows that the traveling wave solution of the predator population $v(x, t)$ tends to the positive equilibrium point (U^*, V^*) with parameters $\alpha = 1.0$, $\beta_1 = 0.7$, $\beta_2 = 0.5$; Figure 5 is the traveling wave solution of the predator population in different positions.

5 Nonexistence of Traveling Waves

In this section, we will show the nonexistence of solutions for (1.5) by use some conclusions of [2, 4].

Theorem 5.1 *For any speed $c < c^*$, there do not exist positive solutions of system* (1.5).

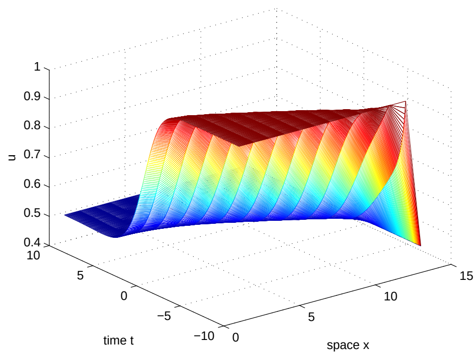

Figure 2: The relationship between u,x and t.

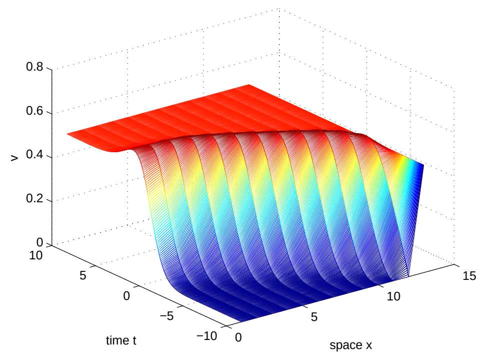

Figure 3: The relationship between v,x and t.

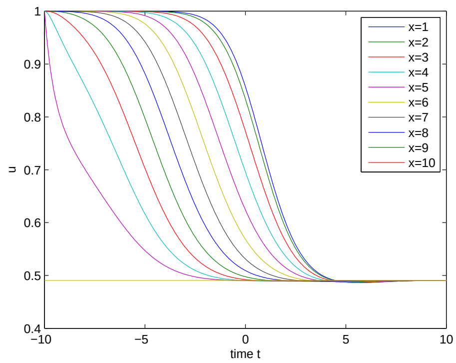

Figure 4: The relationship between t and u in different positions.

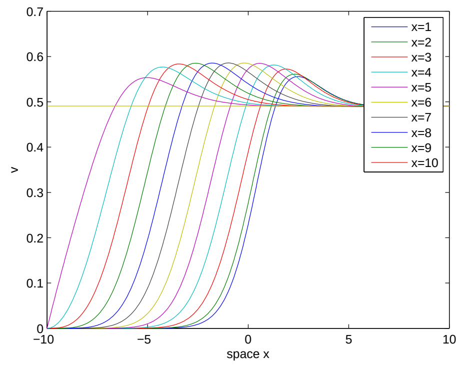

Figure 5: The relationship between t and v in different positions.

Proof Assume the statement does not hold, then there exists some $c_1 < c^*$, such that (1.5) has a strictly positive solution $(u(x), v(x))$, Since $c_1 < c^* = 2\sqrt{dr}$, there exists a $\varrho \in (0, 1)$ with $0 < 1 - \varrho \ll 1$ such that $2\sqrt{dr\varrho} > c_1$. According to the V-equation in the model, we can get that

$$V_t = dV_{xx} + rV\left(1 - \frac{V}{U}\right) \geq dV_{xx} + rV(1 - V).$$

By (1.6) and the positivity of v, there exists a positive constant κ such that $\alpha(x,t) := v(x + c_1 t)$ satisfies

$$\begin{cases} \alpha_t(x,t) \geq d\alpha_{xx}(x,t) + r\alpha(x,t)[1 - \kappa\alpha(x,t)], \\ \alpha(x,0) = v(x). \end{cases}$$

Now we consider that $y(t) = -(2\sqrt{dr\varrho} + c_1)\frac{t}{2}$. Note that $|y(t)| < 2\sqrt{dr\varrho}|t|$. Then, according to Theorem 4.4 in [2], we obtain

$$\lim_{t\to\infty} \inf \alpha(y(t), t) \geq \frac{1}{\kappa} > 0.$$

On the other hand, $y(t)+c_1t=(c_1-2\sqrt{dr\varrho})\frac{t}{2}\to -\infty$ as $t\to +\infty$. Hence we obtain that

$$\lim_{t\to\infty}\sup\alpha(y(t),t)=\lim_{t\to\infty}\sup v(y(t)+c_1t)=\lim_{x\to\infty}v(x)=0,$$

a contradiction. Therefore, the proof of this theorem is completed.

References

[1] S. Ai, Y. Du, R. Peng, Traveling waves for a generalized Holling-Tanner predator-prey model, *Journal of Differential Equations*, **263**:11(2017),7782-7814.

[2] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, *Lecture Notes in Mathematics*, **446**(1975),5-49.

[3] L. Chen, F. Chen, L. Chen, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge, *Nonlinear Analysis: Real World Application*, **11**:1(2010),246-252.

[4] Y. Chen, J. Guo, C. Yao, Traveling wave solutions for a continuous and discrete diffusive predator-prey model, *Journal of Mathematical Analysis and Applications*, **445**:1(2016),212-239.

[5] J. Cushing, Periodic time-dependent predator-prey systems, *SIAM Journal on Applied Mathematics*, **32**:1(1977),82-95.

[6] Y. Du, Y. Lou, Qualitative behaviour of positive solutions of a predator-prey model: effects of saturation, *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, **131**:2(2001),321-349.

[7] Y. Du, Y. Lou, Some uniqueness and exact multiplicity results for a predator-prey model, *Transactions of the American Mathematical Society*, **349**:6(1997),2443-2476.

[8] M. Fan, Y. Kuang, Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional, *Mathematical Computer Modelling*, **48**:11(2008), 1755-1764.

[9] C. Hsu, C. Yang, T. Yang, Existence of traveling wave solutions for diffusive predator-prey type systems, *Journal of Differential Equations*, **252**:4(2012),3040-3075.

[10] J. Huang, G. Lu, S. Ruan, Existence of traveling wave solutions in a diffusive predator-prey model, *Journal of Mathematical Biology*, **46**:2(2003),132-152.

[11] T.W. Hwang, Global analysis of the predator-prey system with Beddington-DeAngelis functional response, *Journal of Mathematical Analysis Applications*, **281**:1(2003),395-401.

[12] R. Jones, K.R.T. Gardnerc, Stability of travelling wave solutions of diffusive predator-prey systems, *Transactions of the American Mathematical Society*, **327**:2(1991),465-524.

[13] Z. Jun, K. Changyun, Positive solutions for a Lotka-Volterra prey-predator model with cross-diffusion and Holling type-II functional response, *Science China Mathematics*, **57**:5(2014),991-1010.

[14] Y. Li, H. Gao, Existence, uniqueness and global asymptotic stability of positive solutions of a predator-prey system with Holling II functional response with random perturbation, *Nonlinear Analysis*, **68**:6(2008),1694-1705.

[15] D. Liang, P. Weng, J. Wu, Travelling wave solutions in a delayed predator-prey diffusion PDE system point-to-periodic and point-to-point waves, *IMA Journal of Applied Mathematics*, **77**:4(2012),516-545.

[16] Z. Ling, L. Zhang, M. Zhu, M. Banerjee, Dynamical behaviour of a generalist predator-prey model with free boundary, *Boundary Value Problems*, **2017**:1(2017),1-20.

[17] S. Liu, E. Beretta, A stage-structured predator-prey model of Beddington-DeAngelis type, *SIAM Journal on Applied Mathematics*, **66**:4(2006),1101-1129.

[18] W. Wang, L. Zhang, Y. Xue, Z. Jin, Spatiotemporal pattern formation of Beddington-DeAngelis-type predator-prey model, arXiv.org e-Print Archive 2008.

[19] G. Zhang, W. Li, G. Lin, Traveling waves in delayed predator-prey systems with nonlocal diffusion and stage structure, *Mathematical and Computer Modelling*, **49**:5(2009),1021-1029.

[20] X. Zhao, A note on traveling wave solutions of a diffusive predator-prey model, *Communications in Nonlinear Science and Numerical Simulation*, **62**:62(2018),174-183.

(*edited by Liangwei Huang*)