POSITIVE SOLUTIONS TO A BVP WITH TWO INTEGRAL BOUNDARY CONDITIONS[*][†]

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Abstract

Based on the Guo-Krasnoselskii's fixed-point theorem, the existence and multiplicity of positive solutions to a boundary value problem (BVP) with two integral boundary conditions

$$\begin{cases} v^{(4)} = f(s, v(s), v'(s), v''(s)), \quad s \in [0,1], \\ v'(1) = v'''(1) = 0, \\ v(0) = \int_0^1 g_1(\tau) v(\tau) d\tau, \quad v''(0) = \int_0^1 g_2(\tau) v''(\tau) d\tau \end{cases}$$

are obtained, where f, g_1, g_2 are all continuous. It generalizes the results of one positive solution to multiplicity and improves some results for integral BVPs. Moreover, some examples are also included to demonstrate our results as applications.

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1   Introduction

Boundary value problems (BVPs) for ordinary differential equations have wide applications in many scientific areas such as physics, mechanics of materials, ecology and so on. For example, deformations of elastic beams can be represented for some fourth-order BVPs, and there are some appealing results [1–3] can be referred.

Especially, much attention has been drawn to BVPs with integral boundary conditions [4–9] recent years because of their applications in thermodynamics and chemical engineering. In 2011, by global bifurcation theory and the Krein-Rutman theorem, Ma [4] investigated positive solutions to a class of BVP with integral boundary conditions. Hereafter, based on the famous Krasnoselskii's fixed-point theorem,

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Lv [5] considered the monotone and concave positive solutions to the following integral BVP

$$\begin{cases} v^{(4)}(s) = f(s, v(s), v'(s), v''(s)), & s \in [0, 1], \\ v(0) = v'(1) = v'''(1) = 0, \\ v''(0) = \int_0^1 g(\tau)v''(\tau)\mathrm{d}\tau, \end{cases} \tag{1.1}$$

where $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$, $g \in C([0, 1], [0, +\infty))$. For more this kind of results, one can refer to [7–9].

Nevertheless, there are few results on multiplicity and the properties of positive solutions to BVPs with integral boundary conditions. Recently, Yang [6] investigated the multiplicity of positive solutions to a fourth-order integral BVP. Motivated by Yang's ideas in [6] and based on the work of Lv [5], this paper deals with the following BVP with two integral boundary conditions

$$\begin{cases} v^{(4)}(s) = f(s, v(s), v'(s), v''(s)), & s \in [0, 1], \\ v'(1) = v'''(1) = 0, \\ v(0) = \int_0^1 g_1(\tau)v(\tau)\mathrm{d}\tau, \quad v''(0) = \int_0^1 g_2(\tau)v''(\tau)\mathrm{d}\tau, \end{cases} \tag{1.2}$$

and the existence and multiplicity of positive solutions are thus established. Notice that (1.2) is reduced to (1.1) when $g_1 \equiv 0$ in (1.2), and therefore we generalize the result of one positive solution in [5] to the case of multiple positive solutions. Moreover, some results of positive solutions to integral BVPs we mentioned in [4,5,9] are also improved.

2　Preliminaries

We first state several notations and lemmas in this paper.

We always assume that, throughout this paper, $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \times (-\infty, 0] \to [0, +\infty)$ and $g_1, g_2 : [0, 1] \to [0, +\infty)$ are all continuous. Furthermore,

$$\zeta_1 := \int_0^1 g_1(\tau)\mathrm{d}\tau < \frac{1}{2}, \quad \zeta_2 := \int_0^1 g_2(\tau)\mathrm{d}\tau < 1 \quad \text{and} \quad g_1 \le \frac{1 - 2\zeta_1}{2(1 - \zeta_2)}g_2.$$

Denote

$$\overline{f_0} = \limsup_{v_0+v_1-v_2\to 0+} \max_{s\in[0,1]} \frac{f(s, v_0, v_1, v_2)}{v_0 + v_1 - v_2}, \quad \overline{f_\infty} = \limsup_{v_0+v_1-v_2\to +\infty} \max_{s\in[0,1]} \frac{f(s, v_0, v_1, v_2)}{v_0 + v_1 - v_2},$$

$$\underline{f_0} = \liminf_{v_0+v_1-v_2\to 0+} \min_{s\in[0,1]} \frac{f(s, v_0, v_1, v_2)}{v_0 + v_1 - v_2}, \quad \underline{f_\infty} = \liminf_{v_0+v_1-v_2\to +\infty} \min_{s\in[0,1]} \frac{f(s, v_0, v_1, v_2)}{v_0 + v_1 - v_2},$$

$$M_1 = \frac{3}{2(1 - \zeta_2)}, \quad M_2 = \frac{1}{4}\int_0^1 \tau^2\Big(1 - \frac{1}{2}\tau\Big)\Big[\frac{1}{2} + \frac{1}{1 - \zeta_2}\int_0^1 sg_2(s)\mathrm{d}s\Big]\mathrm{d}\tau.$$

Lemma 2.1[10] (Guo-Krasnoselskii's fixed-point theorem)   *Let E be a Banach space, $J \subset E$ is a cone. Suppose that I_1 and I_2 are bounded open subsets of E with $\theta \in I_1$ and $\overline{I}_1 \subset I_2$, and let $L : J \cap (\overline{I}_2 \backslash I_1) \longrightarrow J$ be a completely continuous operator such that either*

(i) $\|Lv\| \geq \|v\|$, $v \in J \cap \partial I_1$ *and* $\|Lv\| \leq \|v\|$, $v \in J \cap \partial I_2$; *or*

(ii) $\|Lv\| \leq \|v\|$, $v \in J \cap \partial I_1$ *and* $\|Lv\| \geq \|v\|$, $v \in J \cap \partial I_2$

holds, then L has a fixed point in $J \cap (\overline{I}_2 \backslash I_1)$.

Let $C^2[0,1]$ be bestowed on the maximum norm

$$\|v\|_2 = \max \Big\{ \max_{s \in [0,1]} |v(s)|, \ \max_{s \in [0,1]} |v'(s)|, \ \max_{s \in [0,1]} |v''(s)| \Big\}.$$

Similar to Lemma 2.1 in [5], we obtain the following result by a direct computation.

Lemma 2.2   *If $y \in C[0,1]$, then the following BVP*

$$\begin{cases} v^{(4)}(s) = y(s), & s \in [0,1], \\ v'(1) = v'''(1) = 0, \\ v(0) = \displaystyle\int_0^1 g_1(\tau)v(\tau)\mathrm{d}\tau, & v''(0) = \displaystyle\int_0^1 g_2(\tau)v''(\tau)\mathrm{d}\tau \end{cases} \tag{2.1}$$

has a unique solution

$$v(s) = \int_0^1 \big[G_1(s,\tau) + H(s,\tau) \big] y(\tau)\mathrm{d}\tau, \tag{2.2}$$

where

$$H(s,\tau) = \frac{1}{1-\zeta_1} \int_0^1 G_1(\eta,\tau)g_1(\eta)\mathrm{d}\eta + \frac{2\beta + s(2-s)(1-\zeta_1)}{2(1-\zeta_1)(1-\zeta_2)} \int_0^1 G_2(\eta,\tau)g_2(\eta)\mathrm{d}\eta,$$

$$G_1(s,\tau) = \begin{cases} s\tau - \dfrac{1}{2}s\tau^2 - \dfrac{1}{6}s^3, & 0 \leq s \leq \tau \leq 1, \\ \tau s - \dfrac{1}{2}\tau s^2 - \dfrac{1}{6}\tau^3, & 0 \leq \tau \leq s \leq 1, \end{cases} \qquad G_2(s,\tau) = \begin{cases} s, & 0 \leq s \leq \tau \leq 1, \\ \tau, & 0 \leq \tau \leq s \leq 1, \end{cases}$$

and

$$\beta = \int_0^1 \frac{\tau(2-\tau)}{2} g_1(\tau)\mathrm{d}\tau.$$

Lemma 2.3[5]   *Let $G_1(s,\tau)$ be as in Lemma 2.2. For $s,\tau \in [0,1]$,*

$$\frac{1}{2}s\tau - \frac{1}{4}s^2\tau \leq G_1(s,\tau).$$

Lemma 2.4   *If $y \in C\big([0,1],[0,+\infty)\big)$, then the unique solution $v = v(s)$ to the BVP (2.1) satisfies the following assertions:*

(i) $v(s) \geq 0$, $v'(s) \geq 0$, $v''(s) \leq 0$ *for $s \in [0,1]$;*

(ii) $\|v\| \leq \|v''\|$ *and* $\|v'\| \leq \|v''\|$ *for $v(s) \in C^2[0,1]$;*

(iii) $v(s) \geq \frac{1}{2}(s - \frac{1}{2}s^2)\|v\|_2$ *for $s \in [0,1]$.*

Proof The proof of (i) is analogous to Lemma 2.3 in [5] and so we omit it here.

(ii) By the facts that $v'(1) = 0$ and $v'(s) \geq 0$ for $s \in [0,1]$, it follows that

$$0 \leq v'(s) \leq \int_s^1 |v''(\tau)| d\tau \leq \|v''\|, \quad s \in [0,1].$$

Therefore, $\|v'\| \leq \|v''\|$.

In turn, according to the facts that $g_1(s) \leq \frac{1-2\zeta_1}{2(1-\zeta_2)} g_2(s)$ for any $s \in [0,1]$,

$$\beta = \int_0^1 \frac{\tau(2-\tau)}{2} g_1(\tau) d\tau \leq \frac{1}{2} \int_0^1 g_1(\tau) d\tau = \frac{1}{2}\zeta_1 \tag{2.3}$$

and

$$G_1(s,\tau) \leq s\tau \leq G_2(s,\tau), \quad \text{for any } (s,\tau) \in [0,1] \times [0,1], \tag{2.4}$$

we get

$$H(s,\tau) = \frac{1}{1-\zeta_1} \int_0^1 G_1(\eta,\tau) g_1(\eta) d\eta + \frac{2\beta + s(2-s)(1-\zeta_1)}{2(1-\zeta_1)(1-\zeta_2)} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta$$

$$\leq \frac{1}{1-\zeta_1} \int_0^1 G_2(\eta,\tau) g_1(\eta) d\eta + \frac{1}{2(1-\zeta_1)(1-\zeta_2)} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta$$

$$\leq \frac{1-2\zeta_1}{2(1-\zeta_1)(1-\zeta_2)} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta + \frac{1}{2(1-\zeta_1)(1-\zeta_2)} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta$$

$$= \frac{1}{1-\zeta_2} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta, \quad (s,\tau) \in [0,1] \times [0,1]. \tag{2.5}$$

Recalling that (2.2) and $\frac{\partial^2}{\partial s^2} G_1(s,\tau) = -G_2(s,\tau)$, we have

$$v'(s) = \int_0^1 \left[\frac{\partial}{\partial s} G_1(s,\tau) + \frac{1-s}{1-\zeta_2} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta \right] y(\tau) d\tau, \quad s \in [0,1], \tag{2.6}$$

and

$$v''(s) = -\int_0^1 \left[G_2(s,\tau) + \frac{1}{1-\zeta_2} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta \right] y(\tau) d\tau, \quad s \in [0,1]. \tag{2.7}$$

Therefore, by (2.2), (2.4)-(2.7), one has

$$v(s) = \int_0^1 \left[G_1(s,\tau) + H(s,\tau) \right] y(\tau) d\tau$$

$$\leq \int_0^1 \left[G_2(s,\tau) + \frac{1}{1-\zeta_2} \int_0^1 G_2(\eta,\tau) g_2(\eta) d\eta \right] y(\tau) d\tau$$

$$= -v''(s), \quad s \in [0,1], \tag{2.8}$$

which means that $\|v\| \leq \|v''\|$. Combining the fact that $\|v'\| \leq \|v''\|$, it is natural that

$$\|v\|_2 = \max \Big\{ \max_{s\in[0,1]} |v(s)|, \ \max_{s\in[0,1]} |v'(s)|, \ \max_{s\in[0,1]} |v''(s)| \Big\} = \|v''\|.$$

(iii) By (2.7), we have

$$\|v\|_2 \leq \int_0^1 \Big[\tau + \frac{1}{1-\zeta_2} \int_0^1 G_2(\eta,\tau) g_2(\eta) \mathrm{d}\eta\Big] y(\tau) \mathrm{d}\tau. \tag{2.9}$$

Moreover, by Lemma 2.3 and (2.2), we obtain

$$
\begin{aligned}
v(s) &= \int_0^1 \big[G_1(s,\tau) + H(s,\tau)\big] y(\tau) \mathrm{d}\tau \\
&\geq \int_0^1 \Big[G_1(s,\tau) + \frac{s(2-s)}{2(1-\zeta_2)} \int_0^1 G_2(\eta,\tau) g_2(\eta) \mathrm{d}\eta\Big] y(\tau) \mathrm{d}\tau \\
&\geq \int_0^1 \Big[\frac{1}{2}s\tau - \frac{1}{4}s^2\tau + \frac{s(2-s)}{2(1-\zeta_2)} \int_0^1 G_2(\eta,\tau) g_2(\eta) \mathrm{d}\eta\Big] y(\tau) \mathrm{d}\tau \\
&\geq \frac{1}{2}\Big(s - \frac{1}{2}s^2\Big) \int_0^1 \Big[\tau + \frac{1}{1-\zeta_2} \int_0^1 G_2(\eta,\tau) g_2(\eta) \mathrm{d}\eta\Big] y(\tau) \mathrm{d}\tau, \quad s\in[0,1]. \tag{2.10}
\end{aligned}
$$

Therefore, $v(s) \geq \frac{1}{2}(s - \frac{1}{2}s^2)\|v\|_2$ for $s\in[0,1]$ by (2.9)-(2.10). The proof is completed.

Define a cone J in $C^2[0,1]$ by

$$J = \Big\{ v \in C^2[0,1] : v(s) \geq \frac{1}{2}\Big(s - \frac{1}{2}s^2\Big)\|v\|_2, \ v'(s) \geq 0, \ v''(s) \leq 0, \ s\in[0,1]\Big\},$$

and define an operator $L : J \to C^2[0,1]$ by

$$Lv(s) = \int_0^1 [G_1(s,\tau) + H(s,\tau)] f(\tau, v(\tau), v'(\tau), v''(\tau)) \mathrm{d}\tau.$$

Clearly, $v(\cdot)$ is a positive solution with monotonicity and concavity of the BVP (1.2) if and only if $v(\cdot)$ is a fixed point of L. In addition, in view of the fact $L(J) \subset J$ and by Lemma 2.4, it is trivial to establish the following result:

Lemma 2.5 $L : J \to J$ *is completely continuous.*

3   Existence of One Positive Solution

In this section, our purpose is to investigate the existence of one positive solution to the BVP (1.2).

Theorem 3.1 *There is at least one monotone and concave positive solution to the BVP* (1.2) *if either*

(i) $\overline{f_0} < \frac{1}{M_1}$, $\underline{f_\infty} > \frac{1}{M_2}$; *or*

(ii) $\underline{f_0} > \frac{1}{M_2}$, $\overline{f_\infty} < \frac{1}{M_1}$

holds.

Proof (i) Since $\overline{f_0} < \frac{1}{M_1}$, there exists a number $\delta_1 > 0$ such that

$$M_1(\overline{f_0} + \delta_1) < 1. \tag{3.1}$$

From the continuity of f and the definition of $\overline{f_0}$, there is a number $\lambda > 0$ such that, for $s \in [0,1]$ and $v_0 + v_1 - v_2 \in [0, \lambda]$,

$$f(s, v_0, v_1, v_2) < (\overline{f_0} + \delta_1)(v_0 + v_1 - v_2). \tag{3.2}$$

Let $I_1 = \{v \in C^2[0,1] : \|v\|_2 < \frac{\lambda}{3}\}$ for all $v \in J \cap \partial I_1$, by (3.1) and (3.2), one can obtain

$$|(Lv)''(s)| = \int_0^1 \left[G_2(s,\tau) + \frac{1}{1-\zeta_2}\int_0^1 G_2(\eta,\tau)g_2(\eta)\mathrm{d}\eta\right]f(\tau, v(\tau), v'(\tau), v''(\tau))\mathrm{d}\tau$$

$$\leq \int_0^1 \left[\tau + \frac{1}{1-\zeta_2}\int_0^1 \tau g_2(\eta)\mathrm{d}\eta\right](\overline{f_0} + \delta_1)(v(\tau) + v'(\tau) - v''(\tau))\mathrm{d}\tau$$

$$\leq M_1(\overline{f_0} + \delta_1)\|v\|_2 < \|v\|_2, \quad s \in [0,1].$$

Therefore

$$\|Lv\|_2 = \|(Lv)''\| < \|v\|_2, \quad \text{for any } v \in J \cap \partial I_1. \tag{3.3}$$

In addition, since $\underline{f_\infty} > \frac{1}{M_2}$, there exists a number $\delta_2 > 0$ such that

$$M_2(\underline{f_\infty} - \delta_2) > 1. \tag{3.4}$$

Similar to the case of $\overline{f_0} < \frac{1}{M_1}$, when $\underline{f_\infty} > \frac{1}{M_2}$, there is a number $\Lambda \gg \lambda$ such that, for $s \in [0,1]$ and $v_0 + v_1 - v_2 \in [\Lambda, +\infty)$,

$$f(s, v_0, v_1, v_2) > (\underline{f_\infty} - \delta_2)(v_0 + v_1 - v_2). \tag{3.5}$$

Let $I_2 = \{v \in C^2[0,1] : \|v\|_2 < \Lambda\}$ for all $v \in J \cap \partial I_2$, by Lemma 2.3, (3.4) and (3.5), we have

$$Lv(1) = \int_0^1 \left[G_1(1,\tau) + \frac{1}{1-\zeta_1}\int_0^1 G_1(\eta,\tau)g_1(\eta)\mathrm{d}\eta\right]f(\tau, v(\tau), v'(\tau), v''(\tau))\mathrm{d}\tau$$

$$+\frac{2\beta + (1-\zeta_1)}{2(1-\zeta_1)(1-\zeta_2)}\int_0^1\int_0^1 G_2(\eta,\tau)g_2(\eta)f(\tau, v(\tau), v'(\tau), v''(\tau))\mathrm{d}\eta\mathrm{d}\tau$$

$$\geq \int_0^1 \left[\frac{1}{4}\tau + \frac{1}{2(1-\zeta_2)}\int_0^1 \eta\tau g_2(\eta)\mathrm{d}\eta\right](\underline{f_\infty} - \delta_2)(v(\tau) + v'(\tau) - v''(\tau))\mathrm{d}\tau$$

$$\geq \frac{1}{2}\int_0^1 \left[\frac{1}{2}\tau + \frac{\tau}{1-\zeta_2}\int_0^1 \eta g_2(\eta)\mathrm{d}\eta\right](\underline{f_\infty} - \delta_2)\frac{1}{2}\left(\tau - \frac{1}{2}\tau^2\right)\|v\|_2\mathrm{d}\tau$$

$$= M_2(\underline{f_\infty} - \delta_2)\|v\|_2 > \|v\|_2, \quad s \in [0,1],$$

which indicates

$$\|Lv\|_2 \geq \|Lv\| \geq Lv(1) > \|v\|_2, \quad \text{for any } v \in J \cap \partial I_2. \tag{3.6}$$

Therefore, the operator L has at least one fixed point $v \in J \cap (\overline{I_2} \setminus I_1)$ by (3.3), (3.6) and Lemma 2.1, that is, there is at least one monotone and concave positive solution to the BVP (1.2).

The proof of (ii) is much similarly, and thus we omit it here. The proof is complete.

Corollary 3.1 *There is at least one monotone and concave positive solution to the BVP* (1.2) *if either*

(i) $\overline{f_0} = 0$, $\underline{f_\infty} = +\infty$; *or*

(ii) $\underline{f_0} = +\infty$, $\overline{f_\infty} = 0$

holds.

Example 3.1 Consider the following BVP

$$
\begin{cases}
v^{(4)}(s) = f(s, v, v', v''), & s \in [0, 1], \\
v'(1) = v'''(1) = 0, \\
v(0) = \displaystyle\int_0^1 \frac{3}{11}\tau^3 v(\tau)\mathrm{d}\tau, & v''(0) = \displaystyle\int_0^1 3\tau^3 v''(\tau)\mathrm{d}\tau,
\end{cases}
\tag{3.7}
$$

where

$$
f(s, v, v', v'') = \frac{1}{1+s}\left[\frac{20(v + v' - v'')}{1 + \ln(1 + v + v' - v'')} + \frac{(v + v' - v'')^2}{7(1 + v + v' - v'')}\right],
$$
$$
g_1(\tau) = \frac{3}{11}\tau^3, \quad g_2(\tau) = 3\tau^3.
$$

It is not difficult to examine that f, g_1 and g_2 satisfy the assumptions of Section 2 in this paper. By some calculations, we also obtain that $\overline{f_\infty} = \frac{1}{7}$, $\underline{f_0} = 10$, $M_1 = 6$ and $M_2 = \frac{29}{192}$, which means (ii) of Theorem 3.1 holds. Therefore, there exists at least one monotone and concave positive solution to the BVP (3.7) by Theorem 3.1.

4   Existence of Multiple Positive Solutions

In this section, our aim is to establish the existence of multiple positive solutions to the BVP (1.2).

Theorem 4.1 *There are at least two monotone and concave positive solutions to the BVP* (1.2) *if one of the following conditions holds:*

(i) $\overline{f_0}, \overline{f_\infty} < \frac{1}{M_1}$ *and there is a number $R_0 > 0$ satisfying $\lambda \ll R_0 \ll \Lambda$, such that $N_1(R_0) \geq \frac{64}{3}R_0$, where $N_1(\Lambda) = \min\left\{f(s, v_0, v_1, v_2) : \frac{7}{64}\Lambda \leq v(s) \leq \Lambda, \ |v''(s)| \leq \Lambda, \ s \in [\frac{1}{4}, \frac{3}{4}]\right\}$, $\lambda > 0$ is small enough and $\Lambda > 0$ is large enough;*

(ii) $\underline{f_0}, \underline{f_\infty} > \frac{1}{M_2}$ *and there is a number $\widetilde{R_0} > 0$ satisfying $\widetilde{\lambda} \ll \widetilde{R_0} \ll \widetilde{\Lambda}$, such that $N_2(\widetilde{R_0}) \leq 2(1 - \zeta_2)\widetilde{R_0}$, where $N_2(\widetilde{\Lambda}) = \max\left\{f(s, v_0, v_1, v_2) : |v''(s)| \leq \widetilde{\Lambda}, \ s \in [0, 1]\right\}$, $\widetilde{\lambda} > 0$ is small enough and $\widetilde{\Lambda} > 0$ is large enough.*

Proof (i) Let $E = C^2[0,1]$, $I_1 = \{v \in E : \|v\|_2 < \frac{\lambda}{3}\}$, $I_2 = \{v \in E : \|v\|_2 < \Lambda\}$. By Theorem 3.1, it follows that $\|Lv\|_2 \leq \|v\|_2$ when $v \in J \cap \partial I_1$ or $v \in J \cap \partial I_2$. Choose $I_3 = \{v \in E : \|v\|_2 < R_0\}$ such that $\overline{I_1} \subset I_3$, $\overline{I_3} \subset I_2$. For all $v \in J \cap \partial I_3$, we get

$$Lv\left(\frac{1}{2}\right) \geq \int_{\frac{1}{4}}^{\frac{3}{4}} G_1\left(\frac{1}{2}, \tau\right) f(s, v(\tau), v'(\tau), v''(\tau)) \mathrm{d}\tau$$

$$\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G_1\left(\frac{1}{2}, \tau\right) N_1(R_0) \mathrm{d}\tau \geq \frac{3}{64} N_1(R_0) \geq R_0 = \|v\|_2.$$

Therefore, $\|Lv\|_2 \geq \|Lv\| \geq Lv(\frac{1}{2}) \geq \|v\|_2$, for any $v \in J \cap \partial I_3$. By Lemma 2.1, there exist two positive solutions $v_1 \in J \cap (\overline{I_3} \setminus I_1)$ and $v_2 \in J \cap (\overline{I_2} \setminus I_3)$ to the BVP (1.2). Furthermore, they are two distinct monotone and concave positive solutions to the BVP (1.2) by Theorem 3.1.

The proof of (ii) is much similar, so we omit it here. The proof is complete.

Corollary 4.1 *There are at least two monotone and concave positive solutions to the BVP* (1.2) *if either*

(i) $\overline{f_0} = \overline{f_\infty} = 0$, $N_1(1) \geq \frac{64}{3}$; *or*

(ii) $\underline{f_0} = \underline{f_\infty} = +\infty$, $N_2(1) \leq 2(1 - \zeta_2)$

holds.

Example 4.1 Consider the following BVP

$$\begin{cases} v^{(4)}(s) = f(s, v, v', v''), & s \in [0,1], \\ v'(1) = v'''(1) = 0, \\ v(0) = \int_0^1 \frac{1}{5}\tau v(\tau)\mathrm{d}\tau, & v''(0) = \int_0^1 \tau v''(\tau)\mathrm{d}\tau, \end{cases} \tag{4.1}$$

where

$$f(s, v, v', v'') = \frac{1}{1+s}\left[\frac{(v + v' - v'')^{\frac{1}{2}}}{10} + \frac{(v + v' - v'')^2}{15}\right], \quad g_1(\tau) = \frac{1}{5}\tau, \quad g_2(\tau) = \tau.$$

It is not difficult to check that f, g_1 and g_2 satisfy the assumptions of Section 2 in this paper. By some calculations, we also obtain $\underline{f_0} = \underline{f_\infty} = +\infty$ and $N_2(1) = \max\{f : |v''(s)| \leq 1, s \in [0,1]\} = \frac{\sqrt{3}}{10} + \frac{3}{5} \approx 0.773 \leq 2(1 - \zeta_2) = 2(1 - \frac{1}{2}) = 1$, which implies condition (ii) of Corollary 4.1 holds. Therefore, there are at least two monotone and concave positive solutions to the BVP (4.1) by Corollary 4.1.

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