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BICYCLIC GRAPHS WITH UNICYCLIC OR BICYCLIC INVERSES*†

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Abstract

A graph G is nonsingular if its adjacency matrix $A(G)$ is nonsingular. A nonsingular graph G is said to have an inverse G^+ if $A(G)^{-1}$ is signature similar to a nonnegative matrix. Let $\mathcal{H}$ be the class of connected bipartite graphs with unique perfect matchings. We present a characterization of bicyclic graphs in $\mathcal{H}$ which possess unicyclic or bicyclic inverses.

Keywords inverse graph; unicyclic graph; bicyclic graph; perfect matching

2000 Mathematics Subject Classification 05C90

1 Introduction

Let G be a simple, undirected graph on n vertices. We denote its vertex set by $V(G)$ and its edge set by $E(G)$. We use P_n to denote the path on n vertices. And we use $[i,j]$ to denote an edge between the vertices i and j. The adjacency matrix $A(G)$ of G is a square symmetric matrix of size n whose (i,j)th entry a_{ij} is 1 if $[i,j] \in E(G)$ and 0 otherwise.

A graph G is nonsingular if its adjacency matrix $A(G)$ is nonsingular. Let G be an unweighted graph and G_W be the positive weighted graph obtained from G by giving weights to its edges using the positive weight function $W\colon E(G) \to (0,\infty)$.

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The unweighted graph G may be viewed as a weighted graph where each edge has weight 1. A perfect matching in a graph G is a collection of vertex disjoint edges that covers every vertex. If a graph G has a unique perfect matching, then we denote it by $\mathcal{M}$. In addition, when u is a vertex, we shall always use u' to denote the matching mate for u, where the edge $[u, u'] \in \mathcal{M}$. If G is a bipartite graph with a unique perfect matching then it is nonsingular (see [2]).

A unicyclic graph G is a connected simple graph which satisfies $|E(G)| = |V(G)|$. A bicyclic graph G is a connected simple graph which satisfies $|E(G)| = |V(G)| + 1$. There are two type of basic bicyclic graphs: ∞-graphs and θ-graphs. More concisely, an ∞-graph, denoted by $\infty(p, q, l)$, is obtained from two vertex-disjoint cycles C_p and C_q by connected one vertex of C_p and one of C_q with a path P_l of length $l - 1$ (in the case of $l = 1$, identifying the above two vertices); and a θ-graph, denoted by $\theta(p, q, l)$, is a union of three internally disjoint paths P_{p+1}, P_{q+1}, P_{l+1} of length p, q, l respectively with common end vertices, where $p, q, l \geq 1$ and at most one of them is 1. Observe that any bicyclic graph G is obtained from an ∞-graph or a θ-graph by attaching trees to some of its vertices (see [15]).

One motivation for considering a connected bipartite graph with a unique perfect matching is that in some cases in quantum chemistry, an Hückel graph can be considered as a connected bipartite graph with a unique perfect matching.

We say λ is an eigenvalue of G if λ is an eigenvalue of $A(G)$. We use $\sigma(G)$ to denote the spectrum of G.

In 1976, the notion of an inverse graph was introduced by Harary and Minc (see [5]). A nonsingular graph G is invertible if $A(G)^{-1}$ is a matrix with entries from $\{0, 1\}$, and the graph H with adjacency matrix $A(G)^{-1}$ is called the inverse graph of G. However, in the same article, when only one connected graph is invertible, the author proved that a connected graph G is invertible if and only if $G = P_2$. In 1985, another notion of an inverse graph was supplied by Godsil (see [2]). This concept generalizes the definition given by Harary and Minc.

Let $\mathcal{H}$ denote the class of connected bipartite graphs with unique perfect matchings. Let $G \in \mathcal{H}$, then $A(G)^{-1}$ is signature similar to a nonnegative matrix, that is, there exists a diagonal matrix S with diagonal entries from $\{1, -1\}$ such that $SA(G)^{-1}S \geq 0$. The weighted graph associated to the matrix $SA(G)^{-1}S \geq 0$ is called the inverse of G and is denoted by G^+. A invertible graph G is said to be a self-inverse graph if G is isomorphic to its inverse graph. Let $\mathcal{H}_g$ denote the class of connected bipartite graph with unique perfect matching $\mathcal{M}$ such that $G/\mathcal{M}$ is bipartite.

Definition 1[7] Let $G \in \mathcal{H}$, then G has at least two pendant (degree one) vertices. An edge of a graph is said to be pendant if one of its vertices is a pendant vertex.

A corona graph $G \circ K_1$ is a graph which is obtained from a graph G by adding a new pendant vertex to every vertex of G.

Some other notions about inverse graphs are introduced in the following literatures. In 1978, Cvetković, Gutman and Simić introduced the pseudo-inverse of a graph. Let G be a graph. The pseudo-inverse graph $PI(G)$ of G is the graph, defined on the same vertex set as G, in which the vertices x and y are adjacent if and only if $G - x - y$ has a perfect matching (see [3]). In 1988, Buckley, Doty and Harary introduced the signed inverse of a graph (see [11]). A signed graph is a graph in which each edge has a positive or negative sign (see [4]). An adjacency matrix of a signed graph is symmetric and entries from $\{0, 1, -1\}$. A nonsingular graph G has a signed inverse if $A(G)^{-1}$ is the adjacency matrix of some signed graph H. In 1990, Pavlíková and Jediný introduced another notion of inverse graphs. The inverse of a nonsingular graph with the spectrum $\lambda_1, \lambda_2, \cdots, \lambda_n$ is a graph with the spectrum $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$ (see [14]). This type of inverse of a graph should not be unique. In this article, we follow the notion of inverse graph given by Godsil.

In [2], Godsil introduced the notion of a graph inverse and supplied a class $\mathcal{H}_g$ of $\mathcal{H}$ which posses inverses. He posed the problem of characterizing the graphs in $\mathcal{H}$ which possess inverses. In [9], utilizing constructions derived from the graph itself, Tifenbach and Kirkland supplied necessary and sufficient conditions for graphs in $\mathcal{H}$ to possess inverses. In [1], Akbari and Kirkland provided a complete characterization of a unicyclic graph in $\mathcal{H}$ which possesses inverses. In [6], Panda and Pati supplied a large class of graph in $\mathcal{H}$ for which G^+ exists. The characterization of finding a graph with an inverse in $\mathcal{H}$ is still open.

Consider the problem of characterizing a graph which is isomorphic to its inverses in $\mathcal{H}$. In [8], Simion and Cao showed that for any $G \in \mathcal{H}_g$, we have $G \cong G^+$ if and only if G is a corona graph. In [10], Tifenbach supplied necessary and sufficient conditions for graph satisfying $G \cong G^+$ via a particular isomorphism in $\mathcal{H}$. In [9], Tifenbach and Kirkland obtained necessary and sufficient conditions for an invertible unicyclic graph in $\mathcal{H}$ to be self-inverse.

In [9], Tifenbach and Kirkland supplied necessary and sufficient conditions for an invertible unicyclic graph in $\mathcal{H}$ satisfying that G^+ is unicyclic. In [13], Panda and Delhi supplied necessary and sufficient conditions for an invertible unicyclic graph in $\mathcal{H}$ satisfying that G^+ is bicyclic. Considering this result, we can naturally ask the following question. Can we characterize the bicyclic graphs G in $\mathcal{H}$ such that G^+ is unicyclic or bicyclic? In this article, we supply such characterizations.

2 Preliminaries

In order to study inverse graphs better, the concepts of odd type and even type nonmatching edges were proposed by Panda and Pati (see [6]).

Definition 2[6] Let $G \in \mathcal{H}$, a path $P = [u_1, u_2, \cdots, u_{2k}]$ is called an alternating path if and only if the edges $[u_i, u_{i+1}] \in \mathcal{M}$ for each $i = 1, 3, \cdots, 2k-1$, and the other edges are nonmatching edges. We say P is an mm-alternating path if $[u_1, u_2], [u_{2k-1}, u_{2k}] \in \mathcal{M}$. We say P is an nn-alternating path if $[u_1, u_2], [u_{2k-1}, u_{2k}] \in E(G) \setminus \mathcal{M}$.

Definition 3[6,12] (a) Let $G \in \mathcal{H}$ and $[u, v] \in E(G) \setminus \mathcal{M}$. An extension at $[u, v]$ is called even type (resp. odd type) if the number of nonmatching edges on that extension is even (resp. odd).

(b) Let $[u, v] \in E(G) \setminus \mathcal{M}$. We say $[u, v]$ is an even type edge, if each extension at $[u, v]$ is even type. We say $[u, v]$ is an odd type edge, if either each extension at $[u, v]$ is odd type or there are no extensions at $[u, v]$. We say $[u, v]$ is a mixed type edge, if it has an even type and an odd type extensions.

Example 1 Consider the graph G shown in Figure 1. Let $G \in \mathcal{H}$. $[i_3', i_4]$ is an even type edge, because $[i_3', u_1, u_1', u_2, u_2', u_3, u_3', i_4]$ and $[i_3', u_1, u_1', u_2, u_2', v_1, v_1', v_2, v_2', u_3, u_3', i_4]$ are two even type extensions at $[i_3', i_4]$. Every other nonmatching edge is an odd type edge.

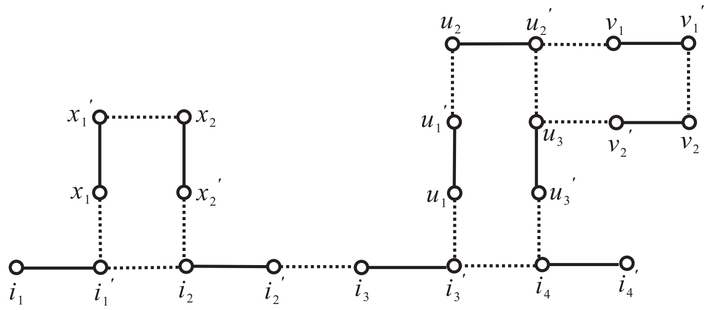

Figure 1: The solid edges are matching edges

3 Bicyclic Graphs with Possess Unicyclic or Bicyclic Inverses

The inverse graph G^+ of an unweighted graph G may be weighted. In this section, we characterize the bicyclic graph G in $\mathcal{H}$ such that G^+ is unicyclic or bicyclic, that is, G^+ is an unweighted connected unicyclic or bicyclic graph.

3.1 Bicyclic graphs in $\mathcal{H}_g$

Remark 1[6] Let $G \in \mathcal{H}_g$, then the following points are true.

(1) Each nonmatching edge is odd in G.

(2) Let $[i, j] \in E(G^+)$, the inverse graph G^+ of G is an unweighted if and only if the number of mm-alternating $i - j$-paths is at most one.

Next we present the inverse of the adjacency matrix of a graph in $\mathcal{H}$ given in [1]. We follow the convention that sum over an empty set is zero.

Lemma 1[1] *Let $G \in \mathcal{H}$, and $B = [b_{ij}]$,*

$$b_{ij} = \sum_{P \in \mathcal{P}(i,j)} (-1)^{\frac{\|P\|-1}{2}},$$

where $\mathcal{P}(i,j)$ is the set of mm-alternating $i-j$-paths in G and $\|P\|$ is the number of edges in P. Then $B = A(G)^{-1}$.

Theorem 1[12] *Let $G \in \mathcal{H}_g$. Then the following are equivalent.*

(1) $G \cong G^+$.

(2) $|\mathcal{P}_G| = |E(G^+)|$, *where $\mathcal{P}_G$ is the set of mm-alternating paths in G.*

(3) $G = G_1 \circ K_1$, *for some connected bipartite graph G_1.*

The following is a description of a unicyclic or bicyclic graph as the inverse of a bicyclic graph.

Theorem 2 *Let $G \in \mathcal{H}_g$ be bicyclic. There is no G^+ which is unicyclic.*

Proof Since $G \in \mathcal{H}_g$, G^+ exists. First we consider $G \in \mathcal{H}_g$ and G is a bicyclic graph on n vertices, so the number of matching edges is $\frac{n}{2}$ and the number of nonmatching edges is $\frac{n}{2}+1$. Since G^+ is unweighted, Remark 1 yields that there is at most one mm-alternating path in G from one vertex to another vertex. By Lemma 1, we have $A(G^+)_{u',v'} \neq 0$ for each nonmatching edge $[u,v]$ in G. $A(G^+)_{u,u'} \neq 0$ for each matching edge $[u,u']$ in G. Hence, G^+ has at least $n+1$ edges, so there is no G^+ which is unicyclic. The proof is completed.

Theorem 3 *Let $G \in \mathcal{H}_g$ be bicyclic. Then G^+ is bicyclic if and only if $G \cong G^+$.*

Proof Since $G \in \mathcal{H}_g$, G^+ exists. First we assume that G^+ is a bicyclic graph on n vertices, Remark 1 yields that there is at most one mm-alternating path in G from one vertex to another vertex. G has no mm-alternating path of length 5, otherwise by Lemma 1 and Theorem 1 G^+ has at least $n+2$ edges, which is not possible. Then the length of each mm-alternating path in G is either 1 or 3. Using this fact one can easily show that each matching edge in G is a pendant edge. Hence G is a corona graph, by Theorem 1, $G \cong G^+$.

Conversely, if $G \cong G^+$, it is clear that G is bicyclic, then G^+ is also bicyclic. The proof is completed.

3.2 Bicyclic graphs in $\mathcal{H}\backslash\mathcal{H}_g$

In this subsection, we characterize the bicyclic graphs $G \in \mathcal{H}\backslash\mathcal{H}_g$ such that G^+ is unicyclic or bicyclic.

Corollary 1[1] *Let $G \in \mathcal{H}$, and $A(G)$ be its adjacency matrix. Then $A(G)^{-1}$ is diagonally similar to a nonnegative matrix if and only if the product of the edge weights on any cycle in G^+ is 1.*

In [6], a class is supplied for which G^+ exists. It is the class in $\mathcal{H}$ in which each nonmatching edge is either even type or odd type and such that the extensions at two distinct even type edge never have an odd type edge in common. We shall denote this class by $\mathcal{F}$. Let $G \in \mathcal{H}$ and ε be the set of all even type edges in G. The author proved that for $G \in \mathcal{F}$, G^+ exists if and only if $(G-\varepsilon)/\mathcal{M}$ is bipartite.

Let $G \in \mathcal{H}\backslash\mathcal{H}_g$ be a bicyclic graph. It is clear that G has either one even type extension, or two even type extensions.

Case 1 Let G be ∞-graph.

(a) If G has exactly one even type extension, it is clear that $G \in \mathcal{F}$ such that $(G-\varepsilon)/\mathcal{M}$ is bipartite, then G^+ exists.

(b) If G has exactly two even type extensions, it is clear that $G \in \mathcal{F}$ such that $(G-\varepsilon)/\mathcal{M}$ is bipartite, then G^+ exists.

Case 2 Let G be θ-graph.

(a) If G has exactly one even type extension.

(1) There is no mixed type edge in G. We are not sure whether the inverse of this graph exists. But we can give two examples of such graphs as shown in Figure 2, where the inverse of Figure 2(a) exists, the inverse of Figure 2(b) does not exist.

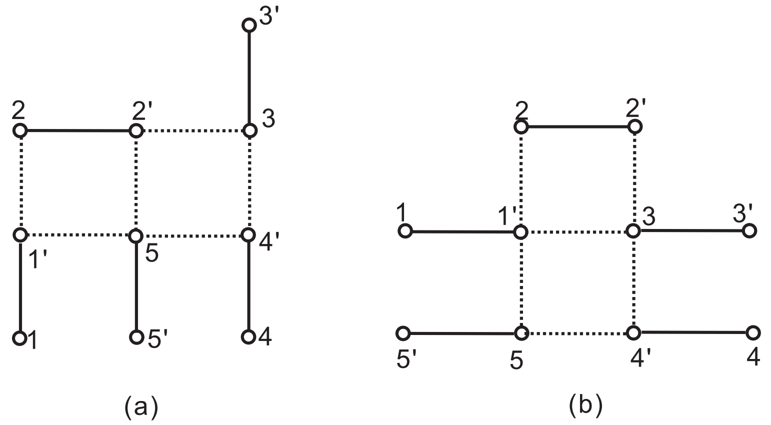

Figure 2: The solid edges are matching edges

Example 2 Consider the graph shown in Figure 2(a). The graph $G \in \mathcal{F}$ and $(G-\varepsilon)/\mathcal{M}$ is bipartite, then G^+ exists. The graph G has an inverse where $S = \text{diag}[1,-1,1,-1,1,1,-1,1,-1,1]$. In particular, this graph is a θ-graph with bicyclic inverse.

Consider the graph shown in Fig 2(b). This graph G^+ does not exist. To see this put $B = A(G)^{-1}$. If G^+ exists, by Corollary 1, for cycle $[1,2',2,3',4,5',1]$, we must have $B(1,2')B(2',2)B(2,3')B(3',4)B(4,5')B(5',1) = 1$, which is not possible, because $B(1,2') = B(2,3') = B(3',4) = B(4,5') = B(5',1) = -1$ and $B(2',2) = 1$.

(2) There is a mixed type edge, which has an even type extension and an odd type extension. We are not sure whether the inverse of this graph exists. But we can give an example of such a graph as shown in Figure 3, whose inverse does not exist.

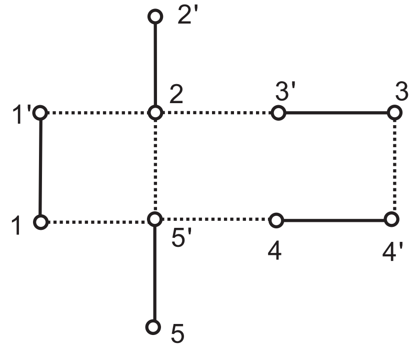

Figure 3: The solid edges are matching edges

Example 3 Consider the graph shown in Figure 3. This graph G^+ does not exist. To see this put $B = A(G)^{-1}$. Note that $B(1, 1') = 1$ and $B(1', 5) = B(5, 2') = B(2', 1) = -1$. By Corollary 1, we know that if G^+ exists, for cycle $[1, 1', 5, 2', 1]$, we have $B(1, 1')B(1', 5)B(5, 2')B(2', 1) = 1$, which is not the case here.

(b) If G has exactly two even type extensions.

(1) G has exactly one even type edge with two even type extensions, it is clear that $G \in \mathcal{F}$ such that $(G - \varepsilon)/\mathcal{M}$ is bipartite, then G^+ exists.

(2) G has exactly two even type extensions on two distinct even type edges. We are not sure if the inverse of this graph exists. But we can give two examples of such graphs as shown in Figure 4, where the inverse of Figure 4(a) does not exist, the inverse of Figure 4(b) exists.

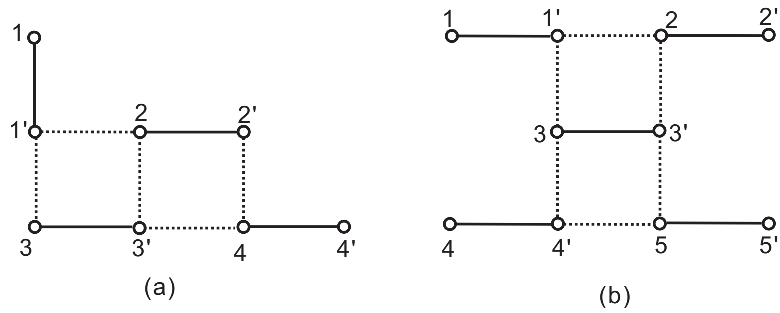

Figure 4: The solid edges are matching edges

Example 4 Consider the graph shown in Figure 4(a). The graph G^+ does not exist. To see this put $B = A(G)^{-1}$. If G^+ exists, by Corollary 1, we must have $B(3,2')B(2',2)B(2,4')B(4',3) = 1$ for the cycle $[3,2',2,4',3]$, which is not possible, because $B(3,2') = B(2,4') = B(4',3) = -1$ and $B(2,2') = 1$.

In the graph shown in Figure 4(b). The graph $G \in \mathcal{F}$ and $(G-\varepsilon)/\mathcal{M}$ is bipartite, then G^+ exists. The graph G has an inverse where $S = \mathrm{diag}[1,1,-1,1,1,1,1,-1,1,1]$. In particular, this graph is a θ-graph with bicyclic inverse.

(3) G has exactly two even type extensions on two distinct nonmatching edges which have a mixed type edge in G. We are not sure if the inverse of this graph exists. But we can give an example of such a graph as shown in Figure 5, whose inverse exists.

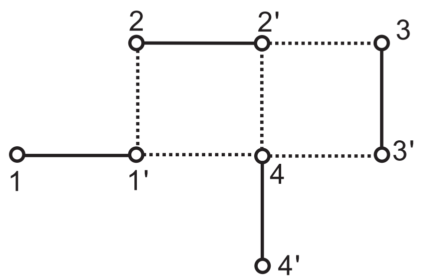

Figure 5: The solid edges are matching edges

Example 5 Consider the graph shown in Figure 5. The edge $[1',4]$ is a mixed type edge, so $G \notin \mathcal{F}$. But the graph G has an inverse where $S = \mathrm{diag}[1,-1,1,-1,1,-1,1,-1]$. In particular, this graph is a θ-graph with bicyclic inverse.

Definition 4 Let $G \in \mathcal{H}$. A minimal path in G from a vertex u to a vertex v is a mm-alternating $u-v$-path which does not contain an even type extension (at some nonmatching edge in G). A simple minimal path is a minimal path which does not contain an even type edge.

Example 6 In the graph G shown in Figure 1, $[i_1,i_1',i_2,i_2',i_3,i_3',i_4,i_4']$ is an mm-alternating path from i_1 to i_4' and $[i_1,i_1',i_2,i_2',i_3,i_3',u_1,u_1',u_2,u_2',u_3,u_3',i_4,i_4']$ is an mm-alternating i_1-i_4'-path which is not a minimal path.

Consider the path $P = [i_1,i_1',x_1,x_1',x_2,x_2',i_2,i_2']$ which is a minimal path from i_1 to i_2'. Each nonmatching edge on P is odd type. Hence P is a simple minimal path.

Lemma 2 *Let $G \in \mathcal{H}\backslash\mathcal{H}_g$ be ∞-graph. Assume that ε is the set of all even type edges in G. Then $|E(G-\varepsilon)| \le |E(G^+)|$.*

Proof Let $[u,v] \notin \varepsilon$ be any nonmatching edge in G. By Lemma 1, there are mm-alternating $u'-v'$-paths in G such that $A(G^+)_{u',v'} \ne 0$. For each matching edge $[u,u']$, by Lemma 1, we have $A(G^+)_{u,u'} \ne 0$. Hence, for each edge $[u,v] \notin \varepsilon$ in G, there is an edge in G^+. Thus $|E(G-\varepsilon)| \le |E(G^+)|$. The proof is completed.

Lemma 3[13] *Let $G \in \mathcal{H}\backslash\mathcal{H}_g$ be an invertible bicyclic graph. Assume that ε is the set of all even type edges in G. Then G^+ has a nonmatching $[u,v] \notin \varepsilon$, such that $[u',v']$ is not in G if and only if G has a simple minimal $u-v$-path of length at least 5.*

Lemma 4 *Let $G \in \mathcal{B}$ be θ-graph, where $\mathcal{B}$ contains* Case 2(b)(1)*, Then G^+ exists. Assume that G has exactly one even type edge with two even type extensions, and assume that $[x,y]$ is an even type edge. Then $|E(G)| \le |E(G^+)|$.*

Proof Let $[u,v] \neq [x,y]$ be any nonmatching edge in G. By Lemma 1, there is exactly one mm-alternating $u'-v'$-path in G such that $A(G^+)_{u',v'} \neq 0$. For each matching edge $[u,u']$, by Lemma 1, we have $A(G^+)_{u,u'} \neq 0$. Furthermore, by Lemma 1, we have $A(G^+)_{x,y} \neq 0$ for the even type edge $[x,y]$. Hence, for each edge $[u,v]$ in G, there is an edge in G^+. Thus $|E(G)| \le |E(G^+)|$. The proof is completed.

Theorem 4 *Let $G \in \mathcal{H}\backslash\mathcal{H}_g$ be ∞-graph. Then G^+ is unicyclic if and only if:*

(1) *G has no simple minimal path of length 5, when G has exactly one even type extension;*

(2) *G has exactly one simple minimal path of length 5, when G has exactly two even type extensions.*

Proof Since $G \in \mathcal{H}\backslash\mathcal{H}_g$ is ∞-graph, it contains two circles, at least one of which is composed of even type extension and even type edge, and G^+ exists.

(1) First we assume that G^+ is unicyclic. We now show that when G has exactly one even type extension, G has no simple minimal path of length 5. Assume that G has one simple minimal path of length 5. Then by virtue of Lemmas 2 and 3, G^+ has $n+1$ edges, which contradicts to the fact that G^+ is unicyclic. Then G has no simple minimal path of length 5, when G has exactly one even type extension.

Conversely, when G has exactly one even type extension, G has no simple minimal path of length 5. By Lemmas 2 and 3, G^+ has exactly n edges. Hence, G^+ is unicyclic.

(2) Now we show that when G has exactly two even type extensions, G has exactly one simple minimal path of length 5. Assume that G has no simple minimal path of length 5. By Lemmas 2 and 3, G^+ has $n-1$ edges, which contradicts to the fact that G^+ is unicyclic. Suppose that G has two simple minimal paths of length 5, say P_1 and P_2, the set of end vertices of P_1 is not equal to the set of end vertices of P_2. Using Lemmas 2 and 3, G^+ has at least $n+1$ edges, which is not possible. Then G has exactly one simple minimal path of length 5, when G has exactly two even type extensions.

We now show the converse, when G has exactly two even type extensions, G has exactly one simple minimal path of length 5. By Lemmas 2 and 3, G^+ has exactly n edges. Hence, G^+ is unicyclic. The proof is completed.

Theorem 5 *Let $G \in \mathcal{H}\backslash\mathcal{H}_g$ be ∞-graph. Then G^+ is bicyclic if and only if:*

(1) *G has exactly one simple minimal path of length* 5*, when G has exactly one even type extension;*

(2) *G has exactly two simple minimal paths of length* 5*, when G has exactly two even type extensions.*

Proof Since $G \in \mathcal{H}\backslash\mathcal{H}_g$ is ∞-graph, it contains two circles, at least one of which is composed of even type extension and even type edge, and G^+ exists.

(1) First we assume that G^+ is bicyclic. We now show that G has exactly one simple minimal path of length 5, when G has exactly one even type extension. Assume that G has no simple minimal path of length 5. By Lemmas 2 and 3, G^+ has n edges, which contradicts to the fact that G^+ is bicyclic. Hence G has at least one simple minimal path of length 5. Assume that G has two simple minimal path of length 5 say P_1 and P_2, the set of end vertices of P_1 is not equal to the set of end vertices of P_2. By Lemmas 2 and 3, G^+ has at least $n+2$ edges, which is not possible. Then G has exactly one simple minimal path of length 5, when G has exactly one even type extension.

Conversely, when G has exactly one even type extension, G has exactly one simple minimal path of length 5. By Lemmas 2 and 3, G^+ has $n+1$ edges. Hence, G^+ is bicyclic.

(2) We now show that G has exactly two simple minimal paths of length 5, when G has exactly two even type extensions. Assume that G has no simple minimal path of length 5. Then by virtue of Lemmas 2 and 3, G^+ has $n-1$ edges, which contradicts to the fact that G^+ is bicyclic. Suppose that G has three simple minimal paths of length 5, say P_i for $i = 1, 2, 3$, the set of end vertices of P_i is not equal to the set of end vertices of P_j for $i, j = 1, 2$ and $i \neq j$. Using Lemmas 2 and 3, we get G^+ has $n+2$ edges, which is not possible. If G has exactly one simple minimal path of length 5, by Lemmas 2 and 3, G^+ has n edges, which is not possible. Hence G has exactly two simple minimal paths of length 5, when G has exactly two even type extensions.

Conversely, when G has exactly two even type extensions, G has exactly two simple minimal paths of length 5. By Lemmas 2 and 3, G^+ has $n+1$ edges. Hence, G^+ is bicyclic. The proof is completed.

Theorem 6 *Let $G \in \mathcal{B}$ be θ-graph, where $\mathcal{B}$ contains Case* 2(b)(1)*. There is no G^+ which is unicyclic.*

Proof Since $G \in \mathcal{B}$, then G^+ exists. G has exactly two even type extensions. That is, G has exactly one even type edge with two even type extensions. By Lemma 4, G^+ has at least $n+1$ edges. Hence there is no G^+ which is unicyclic. The proof is completed.

Theorem 7 *Let $G \in \mathcal{B}$ be θ-graph, where $\mathcal{B}$ contains Case* 2(b)(1)*. Then G^+ is bicyclic if and only if G has no simple minimal path of length* 5.

Proof Since $G \in \mathcal{B}$, then G^+ exists. First we assume that G^+ is bicyclic. We now show that G has no simple minimal paths of length 5, when G has exactly two even type extensions, that is, G has exactly one even type edge with two even type extensions. Assume that G has one simple minimal path of length 5. Then by virtue of Lemmas 3 and 4, G^+ has at least $n+2$ edges, which is not possible. Hence G has no simple minimal paths of length 5.

Conversely, if $G \in \mathcal{B}$ and G has no simple minimal path of length 5, by Lemmas 3 and 4, G^+ has exactly $n+1$ edges. Hence G^+ is bicyclic. The proof is completed.

4 Conclusion

In [2], Godsil introduced the notion of a graph inverse and supplied a class of graphs in $\mathcal{H}$ which possess inverses. In [1], the authors provided a complete characterization of unicyclic graphs in $\mathcal{H}$ which possess inverses. In [9], the authors presented a characterization of unicyclic in $\mathcal{H}$ which possesses a unicyclic inverse. In [6], the authors supplied a lager class of invertible graph in $\mathcal{H}$ which properly contains those in Godsil [2]. Here we presented a characterization of bicyclic graphs in $\mathcal{H}$ which possess unicyclic or bicyclic inverses. We divided the bicyclic graphs in $\mathcal{H}$ into two subclasses which are $\mathcal{H}_g$ and $\mathcal{H}\backslash\mathcal{H}_g$. We characterized bicyclic graphs with unicyclic or bicyclic inverses in $\mathcal{H}_g$, and supplied a necessary and sufficient conditions for ∞-graph in $\mathcal{H}\backslash\mathcal{H}_g$ to have unicyclic or bicyclic inverse. However, for θ-graph in $\mathcal{H}\backslash\mathcal{H}_g$, we only give partial characterization.

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