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NEW OSCILLATION CRITERIA FOR THIRD-ORDER HALF-LINEAR ADVANCED DIFFERENTIAL EQUATIONS*†

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Abstract

The theme of this article is to provide some sufficient conditions for the asymptotic property and oscillation of all solutions of third-order half-linear differential equations with advanced argument of the form

$$\left(r_2(t)((r_1(t)(y'(t))^\alpha)')^\beta\right)' + q(t)y^\gamma(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $\int^\infty r_1^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty$ and $\int^\infty r_2^{-\frac{1}{\beta}}(s)\mathrm{d}s < \infty$. The criteria in this paper improve and complement some existing ones. The results are illustrated by two Euler-type differential equations.

Keywords third-order differential equation; advanced argument; oscillation; asymptotic behavior; noncanonical operators

2000 Mathematics Subject Classification 34C10; 34K11

1 Introduction

In 2019, Chatzarakis ([1]) offered sufficient conditions for the oscillation and asymptotic behavior of second-order half-linear differential equations with advanced argument of the form

$$\left(r(y')^\alpha\right)'(t) + q(t)y^\alpha(\sigma(t)) = 0,$$

where $\int^\infty r^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty$.

In 2018, Džurina ([2]) presented new oscillation criteria for third-order delay differential equations with noncanonical operators of the form

$$\left(r_2\left(r_1y'\right)'\right)'(t) + q(t)y(\tau(t)) = 0, \quad t \geq t_0 > 0.$$

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In this paper, we consider the oscillatory and asymptotic behavior of solutions to the third-order half-linear advanced differential equations of the form

$$\left(r_2(t)((r_1(t)(y'(t))^\alpha)')^\beta\right)' + q(t)y^\gamma(\sigma(t)) = 0, \quad t \geq t_0 > 0. \tag{1.1}$$

Throughout the whole paper, we assume that

(H_1) α, β and γ are quotients of odd positive integers;

(H_2) the functions $r_1, r_2 \in C([t_0,\infty),(0,\infty))$ are of noncanonical type (see Trench [2]), that is,

$$\pi_1(t_0) := \int_{t_0}^\infty r_1^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty, \quad \pi_2(t_0) := \int_{t_0}^\infty r_2^{-\frac{1}{\beta}}(s)\mathrm{d}s < \infty;$$

(H_3) $q \in C([t_0,\infty),[0,\infty))$ does not vanish eventually;

(H_4) $\sigma \in C^1([t_0,\infty),(0,\infty))$, $\sigma(t) \geq t$, $\sigma'(t) \geq 0$ for all $t \geq t_0$.

By a solution of equation (1.1), we mean a nontrivial real valued function $y \in C([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, which has the property that y, $r_1(y')^\alpha$, $r_2\big((r_1(y')^\alpha)'\big)^\beta$ are continuous and differentiable for all $t \in [T_x,\infty)$, and satisfy (1.1) on $[T_x,\infty)$. We only need to consider those solutions of (1.1) which exist on some half-line $[T_x,\infty)$ and satisfy the condition

$$\sup\{|y(t)| : T \leq t < \infty\} > 0$$

for any $T \geq T_x$. In the sequel, we assume that (1.1) possesses such solutions.

As is customary, a solution $y(t)$ of (1.1) is called oscillatory if it has arbitrary large zeros on $[T_x,\infty)$. Otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

Following classical results of Kiguradze and Kondrat'ev [3], we say that (1.1) has property A if any solution y of (1.1) is either oscillatory or satisfies $\lim\limits_{t\to\infty} y(t) = 0$, which is also called that equation (1.1) is almost oscillatory.

For brevity, we define operators

$$L_0y = y, \quad L_1y = r_1(y')^\alpha, \quad L_2y = r_2\big((r_1(y')^\alpha)'\big)^\beta, \quad L_3y = \big(r_2((r_1(y')^\alpha)')^\beta\big)'.$$

Also, we use the symbols $\uparrow$ and $\downarrow$ to indicate whether the function is nondecreasing and nonincreasing, respectively.

2 Main Results

As usual, all functional inequalities considered in this paper are supposed to hold eventually, that is, they are satisfied for all t large enough.

Without loss of generality, we need only to consider eventually positive solutions of (1.1), since if y satisfies (1.1), so does $-y$.

The following lemma on the structure of possible nonoscillatory solutions of (1.1) plays a crucial role in the proofs of the main results.

Lemma 2.1 *Assume* (H_1)-(H_4)*, and that* y *is an eventually positive solution of equation* (1.1)*. Then there exists a* $t_1 \in [t_0, \infty)$ *such that* y *eventually belongs to one of the following classes:*

$$\begin{aligned} S_1 &= \{y : y > 0,\ L_1 y < 0,\ L_2 y < 0,\ L_3 y < 0\}; \\ S_2 &= \{y : y > 0,\ L_1 y < 0,\ L_2 y > 0,\ L_3 y < 0\}; \\ S_3 &= \{y : y > 0,\ L_1 y > 0,\ L_2 y > 0,\ L_3 y < 0\}; \\ S_4 &= \{y : y > 0,\ L_1 y > 0,\ L_2 y < 0,\ L_3 y < 0\}, \end{aligned}$$

for $t \geq t_1$*.*

The proof is straightforward and hence is omitted.

Now, we will establish one-condition criteria of property A of (1.1).

Theorem 2.1 *Assume* (H_1)-(H_4)*. If*

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha}}(v) \left(\int_{t_0}^{v} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u \right)^{\frac{1}{\alpha}} \mathrm{d}v = \infty, \tag{2.1}$$

then (1.1) *has property A.*

Proof First of all, it is important to note that if (H_2) and (2.1) hold, then

$$\int_{t_0}^{\infty} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u = \left(\int_{t_0}^{\infty} q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} = \infty, \tag{2.2}$$

that is,

$$\int_{t_0}^{\infty} q(s)\mathrm{d}s = \infty. \tag{2.3}$$

Now, suppose on the contrary that y is a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1$. Using Lemma 2.1, we know that y eventually belongs to one of the four classes in Lemma 2.1. We will consider each of them separately.

Assume $y \in S_1$. Then from $L_1 y < 0$, that is, $r_1(y')^\alpha < 0$, we see that $y' < 0$ and y is decreasing. On the other words, there exists a finite constant $\ell \geq 0$ such that $\lim\limits_{t\to\infty} y(t) = \ell$. Obviously, $\lim\limits_{t\to\infty} y(\sigma(t)) = \ell$, too.

We claim that $\ell = 0$. Assume on the contrary that $\ell > 0$. Then there exists a $t_2 \geq t_1$ such that $y(t) \geq y(\sigma(t)) \geq \ell$ for $t \geq t_2$. Thus,

$$-L_3 y(t) = q(t) y^\gamma(\sigma(t)) \geq \ell^\gamma \cdot q(t), \tag{2.4}$$

for $t \geq t_2$. Integrating (2.4) from t_2 to t, we have

$$-L_2y(t) \geq -L_2y(t_2) + \ell^{\gamma}\int_{t_2}^{t} q(s)\mathrm{d}s \geq \ell^{\gamma}\int_{t_2}^{t} q(s)\mathrm{d}s.$$

Therefore,

$$-\left(L_1y\right)'(t) \geq \ell^{\frac{\gamma}{\beta}} r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_2}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}. \tag{2.5}$$

Integrating (2.5) again from t_2 to t, we have

$$\begin{aligned}-L_1y(t) &\geq -L_1y(t_2) + \ell^{\frac{\gamma}{\beta}}\int_{t_2}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\\ &\geq \ell^{\frac{\gamma}{\beta}}\int_{t_2}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u,\end{aligned}$$

that is,

$$-y'(t) \geq \ell^{\frac{\gamma}{\alpha\beta}} r_1^{-\frac{1}{\alpha}}(t)\left(\int_{t_2}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{\frac{1}{\alpha}}. \tag{2.6}$$

Integrating (2.6) from t_2 to t, and taking account of (2.1), we have

$$y(t) \leq y(t_2) - \ell^{\frac{\gamma}{\alpha\beta}}\int_{t_2}^{t} r_1^{-\frac{1}{\alpha}}(v)\left(\int_{t_2}^{v} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}v \to -\infty,$$

as $t \to \infty$, which contradicts the positivity of y. Thus, $\lim\limits_{t\to\infty} y(t) = 0$.

Assume $y \in S_2$. Proceeding the same steps as above, we arrive at (2.4). Integrating (2.4) from t_2 to t, we have

$$L_2y(t) \leq L_2y(t_2) - \ell^{\gamma}\int_{t_2}^{t} q(s)\mathrm{d}s \to -\infty, \quad t \to \infty, \tag{2.7}$$

where we used (2.3). This contradicts the positivity of L_2y and thus $\lim\limits_{t\to\infty} y(t) = 0$.

Assume $y \in S_3$. We define a function

$$w(t) := \frac{L_2y(t)}{y^{\gamma}(\sigma(t))}, \quad t \geq t_2.$$

Obviously, $w(t)$ is positive for $t \geq t_2$. Using (1.1), we obtain

$$\begin{aligned}w'(t) &= \frac{(L_2y)'(t)}{y^{\gamma}(\sigma(t))} - \frac{L_2y(t)\cdot\gamma\cdot y^{\gamma-1}(\sigma(t))\cdot y'(\sigma(t))\cdot\sigma'(t)}{y^{2\gamma}(\sigma(t))}\\ &= \frac{L_3y(t)}{y^{\gamma}(\sigma(t))} - \gamma\frac{L_2y(t)\cdot y'(\sigma(t))\cdot\sigma'(t)}{y^{\gamma+1}(\sigma(t))}\\ &\leq \frac{L_3y(t)}{y^{\gamma}(\sigma(t))} = -q(t).\end{aligned}$$

Integrating the above inequality from t_2 to t, and taking (2.3) into account, we have

$$w(t) \le w(t_2) - \int_{t_2}^{t} q(s)\mathrm{d}s \to -\infty, \quad t \to \infty.$$

This contradicts the positivity of w. Hence, $S_3 = \emptyset$.

Assume $y \in S_4$. Considering that y is increasing, and integrating (1.1) from t_2 to t, we obtain

$$-L_2 y(t) = -L_2 y(t_2) + \int_{t_2}^{t} q(s) y^{\gamma}(\sigma(s))\mathrm{d}s \ge y^{\gamma}(\sigma(t_2)) \int_{t_2}^{t} q(s)\mathrm{d}s,$$

that is,

$$-(L_1 y)'(t) \ge k^{\frac{1}{\beta}} r_2^{-\frac{1}{\beta}}(t) \left(\int_{t_2}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}, \tag{2.8}$$

where $k := y^{\gamma}(\sigma(t_2))$. Integrating (2.8) from t_2 to t and using (2.2), we have

$$L_1 y(t) \le L_1 y(t_2) - k^{\frac{1}{\beta}} \int_{t_2}^{t} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \to -\infty, \quad t \to \infty. \tag{2.9}$$

This contradicts the positivity of $L_1 y$. Thus, $S_4 = \emptyset$. The proof is complete.

Remark 2.1 It is clear that any nonoscillatory solution in Theorem 2.1 eventually belongs to either S_1 or S_2 in Lemma 2.1, that is, $S_3 = S_4 = \emptyset$.

Next, we formulate some additional information about the monotonicity of solutions in S_2 or S_1.

Lemma 2.2 *Assume* (H_1)-(H_4). *Let* $y \in S_2$ *in Lemma* 2.1 *on* $[t_1, \infty)$ *for some* $t_1 \ge t_0$, *and define a function*

$$\pi(t) := \int_{t}^{\infty} r_1^{-\frac{1}{\alpha}}(s) \pi_2^{\frac{1}{\alpha}}(s)\mathrm{d}s. \tag{2.10}$$

If

$$\int_{t_0}^{\infty} q(s) \pi^{\gamma}(\sigma(s))\mathrm{d}s = \infty, \tag{2.11}$$

then there exists a $t_2 \ge t_1$ *such that*

$$\frac{y(t)}{\pi(t)} \downarrow 0, \tag{2.12}$$

for $t \ge t_2$.

Proof Let $y \in S_2$ in Lemma 2.1 on $[t_1, \infty)$ for some $t_1 \ge t_0$. First, we prove that (2.11) implies

$$\lim_{t \to \infty} \frac{y(t)}{\pi(t)} = 0. \tag{2.13}$$

Using I'Hospital rule, we obtain

$$\lim_{t\to\infty}\frac{y(t)}{\pi(t)}=-\left(\lim_{t\to\infty}\frac{L_1y(t)}{\pi_2(t)}\right)^{\frac{1}{\alpha}}=\left(\lim_{t\to\infty}L_2y(t)\right)^{\frac{1}{\alpha\beta}}.$$

Taking the decrease of $L_2y(t)$ into account, there exists a finite constant $\ell\ge 0$ such that $\lim\limits_{t\to\infty}L_2y(t)=\ell$. We claim that $\ell=0$. If not, then $L_2y(t)\ge\ell>0$, and $y(t)\ge\ell^{\frac{1}{\alpha\beta}}\pi(t)$ eventually, for $t\ge t_2$ and $t_2\in[t_1,\infty)$. Using this relation in (1.1), we obtain

$$-L_3y(t)\ge\ell^{\frac{\gamma}{\alpha\beta}}q(t)\pi^{\gamma}(\sigma(t)),\quad t\ge t_2.$$

Integrating the above inequality from t_2 to t, we have

$$L_2y(t)\le L_2y(t_2)-\ell^{\frac{\gamma}{\alpha\beta}}\int_{t_2}^{t}q(s)\pi^{\gamma}(\sigma(s))\mathrm{d}s\to-\infty,\quad t\to\infty,$$

which is a contradiction. Thus (2.13) holds and consequently, also

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}L_1y(t)=0,\tag{2.14}$$

due to the decreasing properties of $\pi(t)$ and $\pi_2(t)$, respectively. Considering the monotonicity of L_2y together with (2.14) yields

$$-L_1y(t)=L_1y(\infty)-L_1y(t)=\int_t^{\infty}r_2^{-\frac{1}{\beta}}(s)(L_2y(s))^{\frac{1}{\beta}}\mathrm{d}s\le\pi_2(t)(L_2y(t))^{\frac{1}{\beta}},$$

hence, there exists a $t_3\ge t_2$ such that

$$\left(\frac{L_1y}{\pi_2}\right)'(t)=\frac{(L_2y(t))^{\frac{1}{\beta}}\cdot\pi_2(t)+L_1y(t)}{r_2^{\frac{1}{\beta}}(t)\cdot\pi_2^2(t)}\ge 0,\quad t\ge t_3.$$

Then $\frac{L_1y}{\pi_2}$ is increasing on $[t_3,\infty)$. Using it together with (2.14) leads to

$$y(t)=y(t)-y(\infty)=-\int_t^{\infty}\frac{\pi_2^{\frac{1}{\alpha}}(s)(L_1y(s))^{\frac{1}{\alpha}}}{r_1^{\frac{1}{\alpha}}(s)\pi_2^{\frac{1}{\alpha}}(s)}\mathrm{d}s\le-\left(\frac{L_1y(t)}{\pi_2(t)}\right)^{\frac{1}{\alpha}}\pi(t).$$

Therefore, there exists a $t_4\ge t_3$ such that

$$\left(\frac{y}{\pi}\right)'(t)=\frac{(L_1y(t))^{\frac{1}{\alpha}}\pi(t)+y(t)\pi_2^{\frac{1}{\alpha}}(t)}{r_1^{\frac{1}{\alpha}}(t)\pi^2(t)}\le 0,\quad t\ge t_4,$$

and we conclude that y/π is decreasing on $[t_4,\infty)$. Hence, (2.12) holds. The proof is complete.

Corollary 2.1 *Assume* (H$_1$)-(H$_4$). *Let* $y\in S_2$ *in Lemma* 2.1 *on* $[t_1,\infty)$ *for some* $t_1\ge t_0$, *and a function* $\pi(t)$ *be defined by* (2.10). *If* (2.11) *holds, then there exists a* $t_2\ge t_1$ *such that*

$$y(t)\le k\pi(t),\tag{2.15}$$

for every constant $k>0$ *and* $t\ge t_2$.

Lemma 2.3 *Assume* (H_1)-(H_4). *Let* $y \in S_1$ *in Lemma* 2.1 *on* $[t_1, \infty)$ *for some* $t_1 \geq t_0$. *If* (2.11) *holds, then there exists a* $t_2 \geq t_1$ *such that*

$$\frac{y(t)}{\pi_1(t)} \uparrow, \tag{2.16}$$

for $t \geq t_2$.

Proof Let $y \in S_1$ in Lemma 2.1 on $[t_1, \infty)$ for some $t_1 \geq t_0$. It follows from the monotonicity of $L_1 y$ that, for $\ell \geq t$,

$$y(t) \geq -\int_t^\ell r_1^{-\frac{1}{\alpha}}(s)(L_1 y(s))^{\frac{1}{\alpha}} \mathrm{d}s \geq -(L_1 y(t))^{\frac{1}{\alpha}} \int_t^\ell r_1^{-\frac{1}{\alpha}}(s) \mathrm{d}s.$$

Letting ℓ to ∞, we have

$$y(t) \geq -(L_1 y(t))^{\frac{1}{\alpha}} \cdot \pi_1(t). \tag{2.17}$$

From (2.17), we conclude that y/π_1 is nondecreasing, since

$$\left(\frac{y}{\pi_1}\right)'(t) = \frac{(L_1 y(t))^{\frac{1}{\alpha}} \pi_1(t) + y(t)}{r_1^{\frac{1}{\alpha}}(t) \pi_1^2(t)} \geq 0. \tag{2.18}$$

The proof is complete.

Theorem 2.2 *Assume* (H_1)-(H_4). *If*

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha}}(v) \left(\int_{t_0}^{v} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} \pi^{\gamma}(\sigma(s)) q(s) \mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u \right)^{\frac{1}{\alpha}} \mathrm{d}v = \infty, \tag{2.19}$$

then (1.1) *has property A*.

Proof Suppose on the contrary and assume that y is a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $y(t) > 0$ and $y(\sigma(t)) > 0$ for $t \in [t_1, \infty) \subseteq [t_0, \infty)$. Then we obtain that y eventually belongs to one of the four classes in Lemma 2.1. We will consider each of them separately.

Assume $y \in S_1$. Note that (2.3) and (2.11) are necessary for (2.19) to be valid. In fact, since the function $\int_{t_0}^t \pi^\gamma(\sigma(s)) q(s) \mathrm{d}s$ is unbounded due to (H_2) and $\pi' < 0$, (2.3) and (2.11) must hold. Furthermore, by (2.19), we see that (2.1) holds, and we also obtain

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha}}(v) \left(\int_{t_0}^{v} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} \pi_1^{\gamma}(\sigma(s)) q(s) \mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u \right)^{\frac{1}{\alpha}} \mathrm{d}v = \infty. \tag{2.20}$$

Then using Lemma 2.3, it follows from (2.16) that there exist $c > 0$ and $t_2 \geq t_1$ such that $y(t) \geq c\pi_1(t)$ for $t \geq t_2$. Substituting this inequality into (1.1), we obtain

$$-(L_2 y)'(t) = q(t) y^\gamma(\sigma(t)) \geq c^\gamma q(t) \pi_1^\gamma(\sigma(t)), \quad t \geq t_2. \tag{2.21}$$

Integrating (2.21) from t_2 to t, we have

$$-L_2y(t) \geq -L_2y(t_2) + c^\gamma \int_{t_2}^t q(s)\pi_1^\gamma(\sigma(s))\mathrm{d}s \geq c^\gamma \int_{t_2}^t \pi_1^\gamma(\sigma(s))q(s)\mathrm{d}s,$$

that is,

$$-(L_1y)'(t) \geq c^{\frac{\gamma}{\beta}} r_2^{-\frac{1}{\beta}}(t) \left(\int_{t_2}^t \pi_1^\gamma(\sigma(s))q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}.$$

Integrating the above inequality from t_2 to t, we have

$$\begin{aligned}-L_1y(t) &\geq -L_1y(t_2) + c^{\frac{\gamma}{\beta}} \int_{t_2}^t r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u \pi_1^\gamma(\sigma(s))q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \\ &\geq c^{\frac{\gamma}{\beta}} \int_{t_2}^t r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u \pi_1^\gamma(\sigma(s))q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u,\end{aligned}$$

that is,

$$-y'(t) \geq c^{\frac{\gamma}{\alpha\beta}} r_1^{-\frac{1}{\alpha}}(t) \left(\int_{t_2}^t r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u \pi_1^\gamma(\sigma(s))q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u\right)^{\frac{1}{\alpha}}. \tag{2.22}$$

Integrating (2.22) from t_2 to t, and taking (2.20) into account, we have

$$\begin{aligned}y(t) \leq y(t_2) - c^{\frac{\gamma}{\alpha\beta}} \int_{t_2}^t r_1^{-\frac{1}{\alpha}}(v) \left(\int_{t_2}^v r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u \pi_1^\gamma(\sigma(s))q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u\right)^{\frac{1}{\alpha}} \mathrm{d}v \\ \to -\infty, \quad t \to \infty,\end{aligned}$$

which contradicts the positivity of y. Thus, $S_1 = \emptyset$.

Assume $y \in S_2$. Noting (2.1) is necessary for the validity of (2.20), we have $\lim\limits_{t\to\infty} y(t) = 0$.

Finally, noting (2.3) and (2.2) are necessary for the validity of (2.19), it follows immediately from Remark 2.1 that $S_3 = S_4 = \emptyset$. The proof is complete.

Theorem 2.3 *Assume* (H_1)-(H_4). *If*

$$\limsup_{t\to\infty} \pi_1^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_1}^t r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u > 1, \tag{2.23}$$

for any $t_1 \geq t_0$, *and* $\gamma = \alpha\beta$, *then* (1.1) *has property A.*

Proof On the contrary, suppose that y is a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $y(t) > 0$ and $y(\sigma(t)) > 0$ for $t \in [t_1, \infty) \subseteq [t_0, \infty)$. Then we obtain that y eventually belongs to one of the four classes in Lemma 2.1. We will consider each of them separately.

First, note that (2.23) along with (H_2) implies (2.3) and (2.2). Then, using Theorem 2.1, we get $S_3 = S_4 = \emptyset$. Moreover, if $y \in S_2$, then $\lim\limits_{t\to\infty} y(t) = 0$.

Next, we consider the class S_1. Assume $y \in S_1$. Integrating (1.1) from t_1 to t and using the decrease of y, we have

$$\begin{aligned} -L_2y(t) &= -L_2y(t_1) + \int_{t_1}^{t} q(s)y^{\gamma}(\sigma(s))\mathrm{d}s \geq \int_{t_1}^{t} q(s)y^{\gamma}(\sigma(s))\mathrm{d}s \\ &\geq y^{\gamma}(\sigma(t)) \int_{t_1}^{t} q(s)\mathrm{d}s, \end{aligned} \tag{2.24}$$

that is,

$$-(L_1y)'(t) \geq y^{\frac{\gamma}{\beta}}(\sigma(t))r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}. \tag{2.25}$$

Integrating the above inequality from t_1 to t, we have

$$\begin{aligned} -L_1y(t) &\geq -L_1y(t_1) + \int_{t_1}^{t} y^{\frac{\gamma}{\beta}}(\sigma(u))r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u \\ &\geq y^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u. \end{aligned} \tag{2.26}$$

Similar to the proof of Lemma 2.3, we obtain (2.17), which along with (2.26) leads to

$$\begin{aligned} -L_1y(t) &\geq -(L_1y)^{\frac{\gamma}{\alpha\beta}}(\sigma(t))\pi_1^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u \\ &\geq -(L_1y)^{\frac{\gamma}{\alpha\beta}}(t)\pi_1^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u. \end{aligned}$$

Taking $\gamma = \alpha\beta$ into account, the above inequality becomes

$$-L_1y(t) \geq -L_1y(t)\pi_1^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u,$$

which results in a contradiction

$$\limsup_{t\to\infty} \pi_1^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u \leq 1.$$

Thus, $S_1 = \emptyset$. The proof is complete.

Theorem 2.4 *Assume* (H_1)-(H_4) *and suppose that* (2.1) *holds. If*

$$\limsup_{t\to\infty} \pi_1^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_0}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u > 1, \tag{2.27}$$

and $\gamma = \alpha\beta$, then (1.1) *has property A.*

Proof Using Theorem 2.1, we have $S_3 = S_4 = \emptyset$, and if $y \in S_2$, then $\lim\limits_{t\to\infty} y(t) = 0$.

Now, we only need to consider the class S_1. Assume $y \in S_1$. Similar to the proof of Theorem 2.3, we arrive at

$$\begin{aligned}-L_2y(t) &\geq -L_2y(t_1) + y^{\gamma}(\sigma(t)) \int_{t_1}^{t} q(s)\mathrm{d}s \\ &\geq -L_2y(t_1) - y^{\gamma}(\sigma(t)) \int_{t_0}^{t_1} q(s)\mathrm{d}s + y^{\gamma}(\sigma(t)) \int_{t_0}^{t} q(s)\mathrm{d}s.\end{aligned}$$

Since $\lim\limits_{t\to\infty} y(t) = 0$, there exists a $t_2 > t_1$ such that

$$-L_2y(t_1) - y^{\gamma}(\sigma(t)) \int_{t_0}^{t_1} q(s)\mathrm{d}s > 0,$$

for $t \geq t_2$. Thus, for $t \geq t_2$, we have

$$-L_2y(t) \geq y^{\gamma}(\sigma(t)) \int_{t_0}^{t} q(s)\mathrm{d}s.$$

Integrating the above inequality from t_2 to t, we have

$$\begin{aligned}-L_1y(t) \geq &- L_1y(t_2) - y^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_0}^{t_2} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \\ &+ y^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_0}^{t} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u.\end{aligned}$$

There also exists a $t_3 > t_2$ such that

$$-L_1y(t_2) - y^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_0}^{t_2} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u > 0,$$

for $t \geq t_3$. Thus, for $t \geq t_3$, we obtain

$$-L_1y(t) \geq y^{\frac{\gamma}{\beta}}(\sigma(t)) \int_{t_0}^{t} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u.$$

The rest of proof is similar and hence we omit it. Finally, we obtain $S_1 = \emptyset$. The proof is complete.

Next, we will establish various oscillation criteria for (1.1).

Theorem 2.5 *Assume* (H_1)-(H_4). *If*

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} r_1^{-\frac{1}{\alpha}}(v) \left(\int_{t_0}^{v} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u\right)^{\frac{1}{\alpha}} \mathrm{d}v > \frac{1}{e} \tag{2.28}$$

and

$$\liminf_{t\to\infty} \int_{t}^{\sigma(\sigma(t))} r_1^{-\frac{1}{\alpha}}(v) \left(\int_{v}^{\sigma(t)} r_2^{-\frac{1}{\beta}}(u) \left(\int_{u}^{\sigma(t)} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u\right)^{\frac{1}{\alpha}} \mathrm{d}v > \frac{1}{e} \tag{2.29}$$

hold, and moreover, $\alpha\beta = \gamma$, *then* (1.1) *is oscillatory.*

Proof Suppose that y is a nonoscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that $t_1 \ge t_0$ such that $y(t)>0$ and $y(\sigma(t))>0$ for $t\ge t_1$. Then we obtain that y eventually belongs to one of the four classes in Lemma 2.1. In following, we consider each of these classes separately.

Assume $y\in S_1$. Similar to the proof of Theorem 2.3, we arrive at (2.26), that is

$$y' + \left(r_1^{-\frac{1}{\alpha}}(t) \left(\int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^{u} q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u \right)^{\frac{1}{\alpha}} \right) y^{\frac{\gamma}{\alpha\beta}}(\sigma(t)) \le 0.$$

Using $\alpha\beta=\gamma$, the above inequality becomes

$$y' + \left(r_1^{-\frac{1}{\alpha}}(t) \left(\int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^{u} q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u \right)^{\frac{1}{\alpha}} \right) y(\sigma(t)) \le 0. \tag{2.30}$$

However, it is well-known (see, e.g., [5, Theorem 2.4.1]) that condition (2.28) implies the oscillation of (2.30). Thus, it contradicts our initial assumption. Then $S_1=\emptyset$.

Assume $y\in S_2$. Integrating (1.1) from t to u $(t<u)$, and using the monotonicity of y, we obtain

$$L_2y(t) \ge L_2y(t) - L_2y(u) = \int_t^u q(s)y^{\gamma}(\sigma(s))\mathrm{d}s \ge y^{\gamma}(\sigma(u)) \int_t^u q(s)\mathrm{d}s,$$

that is,

$$(L_1y)'(t) \ge y^{\frac{\gamma}{\beta}}(\sigma(u)) r_2^{-\frac{1}{\beta}}(t) \left(\int_t^u q(s)\mathrm{d}s \right)^{\frac{1}{\beta}}.$$

Integrating the above inequality from t to u, we have

$$-L_1y(t) \ge y^{\frac{\gamma}{\beta}}(\sigma(u)) \int_t^u r_2^{-\frac{1}{\beta}}(x) \left(\int_x^u q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}x,$$

that is,

$$-y'(t) \ge y^{\frac{\gamma}{\alpha\beta}}(\sigma(u)) r_1^{-\frac{1}{\alpha}}(t) \left(\int_t^u r_2^{-\frac{1}{\beta}}(x) \left(\int_x^u q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}x \right)^{\frac{1}{\alpha}}.$$

Taking $\gamma=\alpha\beta$ into account, we have

$$-y'(t) \ge y(\sigma(u)) r_1^{-\frac{1}{\alpha}}(t) \left(\int_t^u r_2^{-\frac{1}{\beta}}(x) \left(\int_x^u q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}x \right)^{\frac{1}{\alpha}}. \tag{2.31}$$

Setting $u=\sigma(t)$ in (2.31), we get

$$-y'(t) \ge y(\sigma(\sigma(t))) r_1^{-\frac{1}{\alpha}}(t) \left(\int_t^{\sigma(t)} r_2^{-\frac{1}{\beta}}(x) \left(\int_x^{\sigma(t)} q(s)\mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}x \right)^{\frac{1}{\alpha}},$$

that is,

$$y'(t)+y(\sigma(\sigma(t)))r_1^{-\frac{1}{\alpha}}(t)\left(\int_t^{\sigma(t)}r_2^{-\frac{1}{\beta}}(x)\left(\int_x^{\sigma(t)}q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}x\right)^{\frac{1}{\alpha}}\le 0. \tag{2.32}$$

However, condition (2.29) implies the oscillation of (2.32), (see, e.g., [5, Theorem 2.4.1]). It means that (1.1) cannot have a positive solution y in the class S_2, which is a contradiction. Thus, $S_2=\emptyset$.

Finally, noting that (2.1) is necessary for the validity of (2.28), it follows immediately from Remark 2.1 that $S_3=S_4=\emptyset$. The proof is complete.

The following results are simple consequences of the above theorem and Corollary 2.1.

Theorem 2.6 *Assume* (H_1)-(H_4). *If* $\gamma=\alpha\beta$, (2.11) *and* (2.28) *hold, then all positive solutions of* (1.1) *satisfy* (2.15) *for any* $k>0$ *and* t *large enough.*

Theorem 2.7 *Assume* (H_1)-(H_4). *If* $\gamma=\alpha\beta$, (2.19) *and* (2.29) *hold, then* (1.1) *is oscillatory.*

Remark 2.2 If

$$\liminf_{t\to\infty}\int_t^{\sigma(t)}r_1^{-\frac{1}{\alpha}}(v)\left(\int_v^{\sigma(t)}r_2^{-\frac{1}{\beta}}(u)\left(\int_u^{\sigma(t)}q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}v>\frac{1}{\mathrm{e}}, \tag{2.33}$$

holds, we have the validity of (2.29). Thus, the conclusions of Theorems 2.5 and 2.7 remain valid if condition (2.29) is replaced by (2.33).

Theorem 2.8 *Assume* (H_1)-(H_4). *If* $\gamma=\alpha\beta$, (2.23) *and* (2.33) *hold, then* (1.1) *is oscillatory.*

Theorem 2.9 *Assume* (H_1)-(H_4). *If* $\gamma=\alpha\beta$, (2.1), (2.27) *and* (2.33) *hold, then* (1.1) *is oscillatory.*

In order to prove the following conclusions, we recall an auxiliary result which is taken from Wu et al. [6, Lemma2.3].

Lemma 2.4[6, Lemma 2.3] *Let* $g(u)=Au-B(u-C)^{\frac{\alpha+1}{\alpha}}$, *where* $B>0$, A *and* C *are constants, and* α *is a quotient of odd positive numbers. Then* g *attains its maximum value on* $\mathbb{R}$ *at* $u^*=C+(\frac{\alpha A}{(\alpha+1)B})^{\alpha}$ *and*

$$\max_{u\in\mathbb{R}}g(u)=g(u^*)=AC+\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\cdot\frac{A^{\alpha+1}}{B^{\alpha}}, \tag{2.34}$$

for $t\ge t_2$.

Theorem 2.10 *Assume* (H_1)-(H_4) *and* $\gamma=\alpha\beta$. *If* (2.3) *and* (2.33) *hold, and also there exists a function* $\rho\in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\left\{\frac{\pi_1^\alpha(t)}{\rho(t)}\int_T^t\!\left(\rho(u)r_2^{-\frac{1}{\beta}}(u)\left(\int_T^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\left(\frac{\pi_1(\sigma(u))}{\pi_1(u)}\right)^{\alpha}-\frac{r_1(u)(\rho'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(u)}\right)\mathrm{d}u\right\}>1, \tag{2.35}$$

for any $T\in[t_0,\infty)$, *then* (1.1) *is oscillatory.*

Proof On the contrary, suppose that y is a nonoscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that $y(t)>0$ and $y(\sigma(t))>0$ for $t\in[t_1,\infty)\subseteq[t_0,\infty)$. Then we know that y eventually belongs to one of the four classes in Lemma 2.1. We will consider each of them separately.

Assume $y\in S_1$. Define the generalized Riccati substitution

$$w:=\rho\left(\frac{L_1y}{y^{\frac{\gamma}{\beta}}}+\frac{1}{\pi_1^{\frac{\gamma}{\beta}}}\right)=\rho\left(\frac{L_1y}{y^\alpha}+\frac{1}{\pi_1^\alpha}\right)\quad\text{on } [t_1,\infty). \tag{2.36}$$

Taking (2.17) into account, we see that $w\geq 0$ on $[t_1,\infty)$. Differentiating (2.36), we arrive at

$$\begin{aligned}
w'&=\frac{\rho'}{\rho}w+\rho\frac{(L_1y)'}{y^\alpha}-\alpha\rho\frac{(L_1y)\cdot y'}{y^{\alpha+1}}+\rho(-\alpha)\frac{-1}{\pi_1^{\alpha+1}\cdot r_1^{\frac{1}{\alpha}}}\\
&=\frac{\rho'}{\rho}w+\rho\frac{(L_1y)'}{y^\alpha}-\frac{\alpha\rho}{r_1^{\frac{1}{\alpha}}}\left(\frac{L_1y}{y^\alpha}\right)^{\frac{\alpha+1}{\alpha}}+\frac{\alpha\rho}{r_1^{\frac{1}{\alpha}}\pi_1^{\alpha+1}}\\
&=\frac{\rho'}{\rho}w+\rho\frac{(L_1y)'}{y^\alpha}-\frac{\alpha}{(r_1\rho)^{\frac{1}{\alpha}}}\left(w-\frac{\rho}{\pi_1^\alpha}\right)^{\frac{\alpha+1}{\alpha}}+\frac{\alpha\rho}{r_1^{\frac{1}{\alpha}}\pi_1^{\alpha+1}}.
\end{aligned} \tag{2.37}$$

Similar to the proof of Theorem 2.3, we arrive at (2.25). Using (2.16) in (2.25), we deduce that the inequality

$$\begin{aligned}
(L_1y)'(t)&\leq -y^{\frac{\gamma}{\beta}}(\sigma(t))r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_2}^t q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\\
&\leq -y^\alpha(\sigma(t))r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_2}^t q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\\
&\leq -\left(\frac{\pi_1(\sigma(t))}{\pi_1(t)}\right)^{\alpha}r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_2}^t q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}y^\alpha(t)
\end{aligned} \tag{2.38}$$

holds for $t\geq t_2$, where $t_2\in[t_1,\infty)$ is large enough. Considering (2.37) and (2.38), it follows that

$$\begin{aligned}
w'(t)\leq&-\rho(t)\frac{\pi_1^\alpha(\sigma(t))}{\pi_1^\alpha(t)}r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_2}^t q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}+\frac{\rho'(t)}{\rho(t)}w(t)\\
&-\frac{\alpha}{(r_1(t)\rho(t))^{\frac{1}{\alpha}}}\left(w(t)-\frac{\rho(t)}{\pi_1^\alpha(t)}\right)^{\frac{\alpha+1}{\alpha}}+\frac{\alpha\rho(t)}{r_1^{\frac{1}{\alpha}}(t)\pi_1^{\alpha+1}(t)}.
\end{aligned}$$

Let

$$A := \frac{\rho'(t)}{\rho(t)}, \quad B := \frac{\alpha}{(r_1(t)\rho(t))^{\frac{1}{\alpha}}}, \quad C := \frac{\rho(t)}{\pi_1^\alpha(t)}.$$

Using (2.34) with the above inequality, we have

$$\begin{aligned} w'(t) \le & -\rho(t) r_2^{-\frac{1}{\beta}}(t) \left(\int_{t_2}^t q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \frac{\pi_1^\alpha(\sigma(t))}{\pi_1^\alpha(t)} \\ & + \frac{\rho'(t)}{\pi_1^\alpha(t)} + \frac{r_1(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(t)} + \frac{\alpha\rho(t)}{r_1^{\frac{1}{\alpha}}(t)\pi_1^{\alpha+1}(t)} \\ = & -\rho(t) r_2^{-\frac{1}{\beta}}(t) \left(\int_{t_2}^t q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \frac{\pi_1^\alpha(\sigma(t))}{\pi_1^\alpha(t)} \\ & + \left(\frac{\rho}{\pi_1^\alpha}\right)'(t) + \frac{r_1(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(t)}. \end{aligned} \tag{2.39}$$

Integrating (2.39) from t_2 to t, we obtain

$$\begin{aligned} & \int_{t_2}^t \left(\rho(u) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \left(\frac{\pi_1(\sigma(u))}{\pi_1(u)}\right)^\alpha - \frac{r_1(u)(\rho'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(u)}\right)\mathrm{d}u \\ & - \frac{\rho(t)}{\pi_1(t)} + \frac{\rho(t_2)}{\pi_1(t_2)} \le w(t_2) - w(t). \end{aligned}$$

Taking the definition of w into account, we get

$$\begin{aligned} & \int_{t_2}^t \left(\rho(u) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \left(\frac{\pi_1(\sigma(u))}{\pi_1(u)}\right)^\alpha - \frac{r_1(u)(\rho'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(u)}\right)\mathrm{d}u \\ \le & \rho(t_2)\frac{L_1 y(t_2)}{y^\alpha(t_2)} - \rho(t)\frac{L_1 y(t)}{y^\alpha(t)}. \end{aligned} \tag{2.40}$$

On the other hand, using (2.17), it follows that

$$-\frac{\rho(t)}{\pi_1^\alpha(t)} \le \rho(t)\frac{L_1 y(t)}{y^\alpha(t)} \le 0.$$

Substituting the above estimate into (2.40), we get

$$\int_{t_2}^t \left(\rho(u) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \left(\frac{\pi_1(\sigma(u))}{\pi_1(u)}\right)^\alpha - \frac{r_1(u)(\rho'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(u)}\right)\mathrm{d}u \le \frac{\rho(t)}{\pi_1^\alpha(t)}. \tag{2.41}$$

Multiplying (2.41) by $\pi_1^\alpha(t)/\rho(t)$ and taking the lim sup on both sides of the resulting inequality, we obtain a contradiction with (2.35). Thus, $S_1 = \emptyset$.

Assume $y \in S_2$. Similar to the proof of Theorem 2.5, one arrives at a contradiction with (2.33). Thus, $S_2 = \emptyset$.

In following, we show $S_3 = S_4 = \emptyset$. Since (2.3) holds due to (H_2), then the function

$$\int_{t_0}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u$$

is unbounded, and so (2.2) holds. The rest of proof proceeds in the same manner as that of Theorem 2.1. The proof is complete.

Depending on the appropriate choice of the function ρ, we can use Theorem 2.10 in a wide range of applications for studying the oscillation of (1.1). Thus, by choosing $\rho(t) = \pi_1^{\alpha}(t)$, $\rho(t) = \pi_1(t)$ and $\rho(t) = 1$, we obtain the following results, respectively.

Corollary 2.2 *Assume* (H_1)-(H_4) *and* $\gamma = \alpha\beta$. *Moreover, assume that* (2.3) *and* (2.33) *hold. If*

$$\limsup_{t\to\infty}\int_{T}^{t}\left(r_2^{-\frac{1}{\beta}}(u)\left(\int_{T}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\pi_1^{\alpha}(\sigma(u)) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{r_1^{\frac{1}{\alpha}}(u)\pi_1(u)}\right)\mathrm{d}u > 1, \tag{2.42}$$

for any $T \in [t_0, \infty)$, *then* (1.1) *is oscillatory.*

Corollary 2.3 *Assume* (H_1)-(H_4) *and* $\gamma = \alpha\beta$. *Moreover, assume that* (2.3) *and* (2.33) *hold. If*

$$\limsup_{t\to\infty}\pi_1^{\alpha-1}(t)\int_{T}^{t}\left(r_2^{-\frac{1}{\beta}}(u)\left(\int_{T}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\frac{\pi_1^{\alpha}(\sigma(u))}{\pi_1^{\alpha-1}(u)} - \frac{1}{(\alpha+1)^{\alpha+1}r_1^{\frac{1}{\alpha}}(u)\pi_1^{\alpha}(u)}\right)\mathrm{d}u > 1, \tag{2.43}$$

for any $T \in [t_0, \infty)$, *then* (1.1) *is oscillatory.*

Corollary 2.4 *Assume* (H_1)-(H_4) *and* $\gamma = \alpha\beta$. *Moreover, assume that* (2.3) *and* (2.33) *hold. If*

$$\limsup_{t\to\infty}\pi_1^{\alpha}(t)\int_{T}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{T}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\left(\frac{\pi_1(\sigma(u))}{\pi_1(u)}\right)^{\alpha}\mathrm{d}u > 1, \tag{2.44}$$

for any $T \in [t_0, \infty)$, *then* (1.1) *is oscillatory.*

Remark 2.3 The conclusions of Theorem 2.10 and Corollaries 2.2-2.4 remain valid if condition (2.3) is replaced by (2.1).

Lemma 2.5 *Assume* (H_1)-(H_4) *and* $\gamma = \alpha\beta$. *Furthermore, assume that* (2.1) *holds. Suppose that* (1.1) *has a positive solution* $y \in S_1$ *on* $[t_1, \infty) \subseteq [t_0, \infty)$ *and that* λ *and* μ *are constants satisfying*

$$0 \le \lambda + \mu < \alpha, \tag{2.45}$$

$$0 \le \lambda \le r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \pi_1^{\alpha}(\sigma(t))\pi_1(t)r_1^{\frac{1}{\alpha}}(t) \tag{2.46}$$

and

$$0 \le \mu \le \alpha \left(\int_{t_1}^{t} r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u\right)^{\frac{1}{\alpha}} \pi_1(\sigma(t)). \tag{2.47}$$

Then there exists a $t_* \in [t_1, \infty)$ *such that*

$$\frac{y}{\pi_1^{1-\frac{\lambda}{\alpha}}} \uparrow \tag{2.48}$$

and

$$\frac{y}{\pi_1^{\frac{\mu}{\alpha}}} \downarrow \tag{2.49}$$

on $[t_*, \infty)$.

Proof Assume $y \in S_1$. Similar to the proof of Theorem 2.3, we arrive at (2.25). Considering (1.1), (2.17) and (2.37), we see that

$$\begin{aligned}
&(-(L_1y)\cdot \pi_1^{\lambda})'(t)\\
=& -(L_1y)'(t)\pi_1^{\lambda}(t) + L_1y(t)\lambda\pi_1^{\lambda-1}(t)r_1^{-\frac{1}{\alpha}}(t)\\
\ge& r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} y^{\frac{\gamma}{\beta}}(\sigma(t))\pi_1^{\lambda}(t) + \lambda L_1y(t)\pi_1^{\lambda-1}(t)r_1^{-\frac{1}{\alpha}}(t)\\
=& r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} y^{\alpha}(\sigma(t))\pi_1^{\lambda}(t) + \lambda L_1y(t)\pi_1^{\lambda-1}(t)r_1^{-\frac{1}{\alpha}}(t)\\
\ge& -r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} L_1y(\sigma(t))\pi_1^{\alpha}(\sigma(t))\pi_1^{\lambda}(t) + \lambda L_1y(t)\pi_1^{\lambda-1}(t)r_1^{-\frac{1}{\alpha}}(t)\\
\ge& -r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} L_1y(t)\pi_1^{\alpha}(\sigma(t))\pi_1^{\lambda}(t) + \lambda L_1y(t)\pi_1^{\lambda-1}(t)r_1^{-\frac{1}{\alpha}}(t)\\
=& -L_1y(t)\pi_1^{\lambda}(t)\left(r_2^{-\frac{1}{\beta}}(t)\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \pi_1^{\alpha}(\sigma(t)) - \frac{\lambda}{r_1^{\frac{1}{\alpha}}(t)\pi_1(t)}\right)\\
\ge& 0.
\end{aligned}$$

Thus, $-(L_1y)\pi_1^{\lambda}$ is nondecreasing eventually for $y \ge t_2$, where $t_2 \in [t_1, \infty)$ is large enough. Furthermore, using this property, we get

$$y(t) \ge -\int_{t}^{\infty} r_1^{-\frac{1}{\alpha}}(s)(L_1y)^{\frac{1}{\alpha}}(s)\mathrm{d}s$$

$$= -\int_t^\infty r_1^{-\frac{1}{\alpha}}(s)\frac{\pi_1^{\frac{\lambda}{\alpha}}(s)}{\pi_1^{\frac{\lambda}{\alpha}}(s)}(L_1 y)^{\frac{1}{\alpha}}(s)\mathrm{d}s$$

$$\geq -\big((L_1 y)\cdot \pi_1^{\lambda}\big)^{\frac{1}{\alpha}}(t)\int_t^\infty \frac{1}{r_1^{\frac{1}{\alpha}}(s)\pi_1^{\frac{\lambda}{\alpha}}(s)}\mathrm{d}s. \tag{2.50}$$

It is easy to verify that

$$\int_t^\infty \frac{1}{r_1^{\frac{1}{\alpha}}(s)\pi_1^{\frac{\lambda}{\alpha}}(s)}\mathrm{d}s = \frac{\pi_1^{1-\frac{\lambda}{\alpha}}(t)}{1-\frac{\lambda}{\alpha}}, \tag{2.51}$$

and thus, we get

$$y(t) \geq -\frac{(L_1 y)^{\frac{1}{\alpha}}(t)\cdot \pi_1(t)}{1-\frac{\lambda}{\alpha}} = -\frac{r_1^{\frac{1}{\alpha}}(t)y'(t)\pi_1(t)}{1-\frac{\lambda}{\alpha}}. \tag{2.52}$$

Therefore,

$$\left(\frac{y}{\pi_1^{1-\frac{\lambda}{\alpha}}}\right)'(t) = \frac{r_1^{\frac{1}{\alpha}}(t)y'(t)\pi_1(t) + (1-\frac{\lambda}{\alpha})y(t)}{r_1^{\frac{1}{\alpha}}(t)\pi_1^{2-\frac{\lambda}{\alpha}}(t)} \geq 0,$$

thus $y/\pi_1^{1-\frac{\lambda}{\alpha}}$ is nondecreasing.

Next, we will prove the last monotonicity. Similar to the proof of Theorem 2.3, we arrive at (2.26), that is

$$-r_1(t)(y'(t))^\alpha \geq y^\alpha(\sigma(t))\int_{t_1}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u.$$

Using (2.16) with the above inequality, we have

$$-r_1(t)(y'(t))^\alpha \geq \frac{\pi_1^\alpha(\sigma(t))}{\pi_1^\alpha(t)}y^\alpha(t)\int_{t_2}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u,$$

that is,

$$y(t) \leq -r_1^{\frac{1}{\alpha}}(t)y'(t)\frac{\pi_1(t)}{\pi_1(\sigma(t))}\left(\int_{t_2}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{-\frac{1}{\alpha}},$$

for $t \geq t_2$, where $t_2 \geq t_1$. Using the above relation in the equality

$$\left(\frac{y}{\pi_1^{\frac{\mu}{\alpha}}}\right)'(t) = \frac{y'(t)}{\pi_1^{\frac{\mu}{\alpha}}(t)} + \frac{\frac{\mu}{\alpha}y(t)}{\pi_1^{\frac{\mu}{\alpha}+1}(t)r_1^{\frac{1}{\alpha}}(t)},$$

and taking the condition (2.47) into account, we get

$$\begin{aligned}\left(\frac{y}{\pi_1^{\frac{\mu}{\alpha}}}\right)'(t) &\le \frac{y'(t)}{\pi_1^{\frac{\mu}{\alpha}}(t)} - \frac{\frac{\mu}{\alpha}y'(t)}{\pi_1^{\frac{\mu}{\alpha}(t)}\pi_1(\sigma(t))}\left(\int_{t_2}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{-\frac{1}{\alpha}}\\ &= \frac{y'(t)}{\pi_1^{\frac{\mu}{\alpha}}(t)}\left(1-\frac{\mu}{\alpha\pi_1(\sigma(t))}\left(\int_{t_2}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_2}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{-\frac{1}{\alpha}}\right)\\ &\le \frac{y'(t)}{\pi_1^{\frac{\mu}{\alpha}}(t)}\left(1-\frac{\mu}{\alpha\pi_1(\sigma(t))}\left(\int_{t_1}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u\right)^{-\frac{1}{\alpha}}\right)\\ &\le 0.\end{aligned}$$

Thus, $y/\pi_1^{\frac{\mu}{\alpha}}$ is nonincreasing. The proof is complete.

Theorem 2.11 *Assume* (H_1)-(H_4) *and* $\gamma = \alpha\beta$. *Furthermore, suppose that* (2.33) *holds and* λ *and* μ *are constants satisfying* (2.45)-(2.47). *If*

$$\limsup_{t\to\infty} \pi_1^{\lambda}(t)\pi_1^{\alpha-\lambda-\mu}(\sigma(t))\int_{t_1}^t \pi_1^{\mu}(\sigma(u))r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u > \left(1-\frac{\lambda}{\alpha}\right)^{\alpha}, \tag{2.53}$$

for any $t_1 \ge t_0$, *then* (1.1) *is oscillatory.*

Proof Suppose on the contrary that y is a nonoscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that $y(t) > 0$ and $y(\sigma(t)) > 0$ for $t \in [t_1,\infty) \subseteq [t_0,\infty)$. Then we know that y eventually belongs to one of the four classes in Lemma 2.1. We will consider each of them separately.

Before proceeding further, note that (2.11) and

$$\int_{t_0}^{\infty} q(s)\pi_1^{\gamma}(\sigma(s))\mathrm{d}s = \infty \tag{2.54}$$

are necessary for (2.19) to be valid. To verify this, it suffices to see that (H_2) implies

$$\pi_1^{\frac{\lambda}{\alpha}}(t)\pi_1^{1-\frac{\lambda}{\alpha}-\frac{\mu}{\alpha}}(\sigma(t)) \le \pi_1^{1-\frac{\lambda}{\alpha}}(t) \to 0, \quad t\to\infty. \tag{2.55}$$

From the above inequality, we conclude that the function

$$\int_{t_0}^t \pi_1^{\mu}(\sigma(u))r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u$$

and consequently

$$\int_{t_1}^t r_2^{-\frac{1}{\beta}}(u)\left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\mathrm{d}u$$

must be unbounded.

Assume $y \in S_1$. Similar to the proof of Theorem 2.3, we arrive at (2.26), that is

$$
\begin{aligned}
-r_1(t)(y'(t))^\alpha &\ge -r_1(t_1)(y'(t_1))^\alpha + \int_{t_1}^t y^{\frac{\gamma}{\beta}}(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \\
&\ge \int_{t_1}^t y^\alpha(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u. \qquad (2.56)
\end{aligned}
$$

Using the conclusions of Lemma 2.5 that $y/\pi_1^{\frac{\mu}{\alpha}}$ is nonincreasing and $y/\pi_1^{1-\frac{\lambda}{\alpha}}$ is nondecreasing, we obtain

$$
\begin{aligned}
-r_1(t)(y'(t))^\alpha &\ge \int_{t_1}^t \frac{y^\alpha(\sigma(u))}{\pi_1^\mu(\sigma(u))} \pi_1^\mu(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \\
&\ge \left(\frac{y(\sigma(t))}{\pi_1^{\frac{\mu}{\alpha}}(\sigma(t))}\right)^\alpha \int_{t_1}^t \pi_1^\mu(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \\
&= \left(\frac{y(\sigma(t))\pi_1^{1-\frac{\lambda}{\alpha}}(\sigma(t))}{\pi_1^{\frac{\mu}{\alpha}}(\sigma(t))\pi_1^{1-\frac{\lambda}{\alpha}}(\sigma(t))}\right)^\alpha \int_{t_1}^t \pi_1^\mu(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u \\
&\ge \left(\frac{y(t)\pi_1^{1-\frac{\lambda}{\alpha}-\frac{\mu}{\alpha}}(\sigma(t))}{\pi_1^{1-\frac{\lambda}{\alpha}}(t)}\right)^\alpha \int_{t_1}^t \pi_1^\mu(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u.
\end{aligned}
\qquad (2.57)
$$

Using (2.52) in the above inequality, we have

$$
\begin{aligned}
-r_1(t)(y'(t))^\alpha \ge -r_1(t)(y'(t))^\alpha &\left(\frac{\pi_1^{\frac{\lambda}{\alpha}}(t)\pi_1^{1-\frac{\lambda}{\alpha}-\frac{\mu}{\alpha}}(\sigma(t))}{1-\frac{\lambda}{\alpha}}\right)^\alpha \\
&\cdot \int_{t_1}^t \pi_1^\mu(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u,
\end{aligned}
$$

that is,

$$
1 \ge \left(\frac{\pi_1^{\frac{\lambda}{\alpha}}(t)\pi_1^{1-\frac{\lambda}{\alpha}-\frac{\mu}{\alpha}}(\sigma(t))}{1-\frac{\lambda}{\alpha}}\right)^\alpha \int_{t_1}^t \pi_1^\mu(\sigma(u)) r_2^{-\frac{1}{\beta}}(u) \left(\int_{t_1}^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}} \mathrm{d}u.
$$

Taking the limsup on both sides of the above inequality, we reach a contradiction with (2.53). Thus, $S_1 = \emptyset$.

Accounting to Remark 2.2 with (2.33), we have $S_2 = \emptyset$. Also, using Theorem 2.1, we arrive at $S_3 = S_4 = \emptyset$. The proof is complete.

Theorem 2.12 *Assume* (H_1)-(H_4) *and* $\gamma = \alpha\beta$. *Furthermore, suppose that* (2.3) *and* (2.33) *hold, and* $\lambda \in [0, \alpha)$ *is a constant satisfying* (2.46). *If there exists a function* $\rho \in C^1([t_0, \infty), (0, \infty))$ *and* $T \in [t_0, \infty)$ *such that*

$$\limsup_{t\to\infty}\left\{\frac{\pi_1^\alpha(t)}{\rho(t)}\int_T^t\left(\rho(u)r_2^{-\frac{1}{\beta}}(u)\left(\int_T^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\left(\frac{\pi_1(\sigma(u))}{\pi_1(u)}\right)^{\alpha-\lambda}-\frac{r_1(u)(\rho'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(u)}\right)\mathrm{d}u\right\}>1, \tag{2.58}$$

then (1.1) *is oscillatory.*

Proof For the proof of this theorem, it suffices to use (2.48) instead of (2.16) in (2.25) in the proof of Theorem 2.10.

Corollary 2.5 *Assume* (H_1)-(H_4) *and* $\gamma=\alpha\beta$. *Furthermore, suppose that* (2.3) *and* (2.33) *hold and* $\lambda\in[0,\alpha)$ *is a constant satisfying* (2.46). *If*

$$\limsup_{t\to\infty}\int_T^t\left(r_2^{-\frac{1}{\beta}}(u)\left(\int_T^u q(s)\mathrm{d}s\right)^{\frac{1}{\beta}}\pi_1^{\alpha-\lambda}(\sigma(u))\pi_1^\lambda(u)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{r_1^{\frac{1}{\alpha}}(u)\pi_1(u)}\right)\mathrm{d}u>1, \tag{2.59}$$

for any $T\in[t_0,\infty)$, *then* (1.1) *is oscillatory.*

3 Examples

In this section, we illustrate the strength of our results using two Euler-type differential equations, as two examples.

Example 3.1 Consider the third-order advanced differential equation

$$\left(t^3((t^4(y'(t))^{\frac{5}{3}})')^{\frac{1}{7}}\right)'+t^6y^{\frac{9}{5}}(2t)=0,\quad t\geq 1. \tag{3.1}$$

It is easy to verify that condition (2.1) is satisfied. Using Theorem 2.1, we obtain that equation (3.1) has property A.

Example 3.2 Consider the third-order advanced differential equation

$$\left(t^n((t^my'(t))')^{\frac{1}{3}}\right)'+q_0t^{\frac{m}{3}+n-\frac{5}{3}}y^{\frac{1}{3}}(\delta t)=0,\quad t\geq 1, \tag{3.2}$$

where $m>1$, $n>\frac{1}{3}$, $q_0>0$ and $\delta\geq 1$.

Clearly, $r_1(t)=t^m$, $r_2(t)=t^n$, $\alpha=1$, $\beta=\frac{1}{3}$, $\gamma=\alpha\beta=\frac{1}{3}$, $q(t)=q_0t^{\frac{m}{3}+n-\frac{5}{3}}$, $\sigma(t)=\delta t$, and

$$\pi_1(t)=\int_t^\infty r_1^{-\frac{1}{\alpha}}(s)\mathrm{d}s=\int_t^\infty s^{-m}\mathrm{d}s=\frac{t^{1-m}}{m-1}.$$

From Theorem 2.1 (On the asymptotic properties of nonoscillatory solutions), it is easy to verify that condition (2.1) holds. Thus, any nonoscillatory, say positive solution of equation (3.2) converges to zero as $t\to\infty$, without any additional requirement.

In following, we consider the oscillation of equation (3.2).

After some computations, we note that conditions (2.23), (2.28) and (2.33) reduce to

$$27q_0^3\delta^{1-m} > (m+3n-2)^3(m-1)^2, \tag{3.3}$$

$$27q_0^3\ln\delta > \frac{(m+3n-2)^3(m-1)}{\mathrm{e}}, \tag{3.4}$$

and

$$\begin{aligned}&27q_0^3\bigg\{\frac{\delta^{m+3n-2}-1}{(m+3n-2)(3n-1)}+\frac{\ln\delta}{m-1}\\&+\frac{27\big(\delta^{\frac{2m+6n-4}{3}}-1\big)}{(m-6n+1)(2m+6n-4)}-\frac{27\big(\delta^{\frac{m+3n-2}{3}}-1\big)}{(2m-3n-1)(m+3n-2)}\\&-\frac{\delta^{m-1}-1}{m-1}\left(\frac{1}{3n-1}+\frac{9}{m-6n+1}-\frac{9}{2m-3n-1}+\frac{1}{m-1}\right)\bigg\}\\>&\frac{(m+3n-2)^3}{\mathrm{e}},\end{aligned} \tag{3.5}$$

respectively.

Theorem 2.5 and Remark 2.2 imply if both (3.4) and (3.5) hold, then equation (3.2) is oscillatory.

Since condition (2.19) is not satisfied, the related result from Theorem 2.7 can not be applied.

Theorems 2.8 and 2.9 can deduce that oscillation of equation (3.2) is guaranteed by conditions (3.3) and (3.5).

4 Summary

In this paper, we studied the third-order differential equation (1.1) with noncanonical operators. First, we established one-condition criteria for property A of (1.1). Next, we presented various two-condition criteria ensuring oscillation of all solutions of (1.1). Finally, our results are applicable on Euler-type equations of the forms (3.1) and (3.2). It remains open how to generalize these results for higher-order noncanonical equations with deviating arguments.

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