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THE GENERALIZED JACOBIAN OF THE PROJECTION ONTO THE INTERSECTION OF A HALF-SPACE AND A VARIABLE BOX*

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Abstract

This paper is devoted to studying the generalized Jacobian for the projection onto the intersection of a closed half-space and a variable box. This paper derives the explicit formulas of an element in the set of the generalized HS Jacobian for the projection. In particular, we reveal that the generalized HS Jacobian can be formulated as the combination of a diagonal matrix and few rank-one symmetric matrices, which are crucial for future design of efficient second order nonsmooth methods for solving the related optimization problems.

Keywords generalized HS Jacobian; projection; intersection of a half-space and a variable box

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1 Introduction

Given $(x, t) \in \Re^n \times \Re$ and $r > 0$, we consider the following optimization problem:

$$\begin{aligned} \min_{y,\tau} \quad & \frac{1}{2}\|y - x\|^2 + \frac{1}{2}(\tau - t)^2 \\ \text{s.t.} \quad & e_n^{\mathrm{T}} y \le r\tau, \\ & 0 \le y \le \tau e_n, \end{aligned} \tag{P}$$

where e_n denotes the vector of all ones in $\Re^n$. For simplicity, we define $\mathcal{B}_r$ as the intersection of a closed half-space and a variable box:

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$$\mathcal{B}_r := \{(y, \tau) \in \Re^n \times \Re \mid e_n^{\mathrm{T}} y \leq r\tau, 0 \leq y \leq \tau e_n\}. \tag{1}$$

It is obvious (cf. [13]) that $\mathcal{B}_r$ is a polyhedral convex set. It is also easy to see that the optimal solution to (P) is the projection $\Pi_{\mathcal{B}_r}(x, t)$ of (x, t) onto $\mathcal{B}_r$.

Fast solvers for computing the projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ and the explicit formulas of the generalized Jacobian for $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ are needed in designing efficient algorithms such as augmented Lagrangian methods for the optimization problems subject to the epigraph of the vector k-norm functions and the matrix Ky Fan k-norm functions. It is worth mentioning that Ky Fan k-norm functions appear frequently in matrix optimization problems. For instance, the problem of finding the fastest Markov chain on a graph which can be recast as minimizing the Ky Fan 2-norm was studied in [1, 2]. Besides, the structured low rank matrix approximation [3] arising in many areas is a kind of matrix optimization problem involving Ky Fan k-norm. For more applications, we refer the reader to [4, 14] and references therein.

Note that the explicit formulas of the projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ and the fast algorithm proposed for finding the projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ were studied in [12], we mainly focus on the computation of the generalized Jacobian of $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ in this paper. The generalized Jacobian of the projector $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ plays an essential role in the algorithmic design of the second order nonsmooth methods for solving optimization problems with constraints involving $\mathcal{B}_r$. The efficiency and robustness of the algorithms which utilize the second order information like the semismooth Newton augmented Lagrangian methods (Ssnal) for solving the optimization problems have been demonstrated in many works [8–11, 15]. Thus, the efficient computation of generalized Jacobian of the metric projector $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ deserves our research efforts.

As far as we know, the generalized Jacobian of the projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ has not been studied before. The main contribution of this paper is to study the explicit formulas of the generalized HS-Jacobian [7,10] for the projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$. Due to the difficulty of characterizing the B-subdifferential and the Clarke generalized Jacobian of the projection onto the nonempty polyhedral convex set, Han and Sun [7] proposed a particular multi-valued mapping (HS-Jacobian) as an alternative for the generalized Jacobian of the projector onto polyhedral sets. Recently, the authors [10] derived an explicit formula for constructing the generalized HS-Jacobian and thus the computation of the generalized Jacobian was further simplified. Based on their works, we are able to study the generalized HS-Jacobian of $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ in an efficient way.

The remaining parts of this paper are organized as follows. Section 2 reviews some preliminary results on the generalized Jacobian of the projection onto the general polyhedral convex set. This lays the foundation for the study on the generalized HS-Jacobian of the projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ in Section 3. Finally, we conclude the paper in Section 4.

Notation For any positive integer p, we use I_p and e_p to denote the $p \times p$ identity matrix and the p-vector of all ones respectively. We denote the $p \times q$ zero matrix by $\mathbf{0}_{p\times q}$ and p, q will be dropped from $\mathbf{0}_{p\times q}$ when the dimension is clear from the context. Similarly, $E_{p\times q}$ denotes the $p \times q$ matrix of all ones and is abbreviated as E_p when $p = q$. The symbol s_K^p refers to the p-vector whose i-th component is 1 if $i \in K$ and 0 otherwise. For any given vector $z \in \Re^p$, $\mathrm{Diag}(z)$ denotes the $p \times p$ diagonal matrix whose diagonal is given by z. The notation $A^\dagger$ is used to denote the Moore-Penrose inverse of given matrix A.

2 Generalized Jacobians of the Projection onto Polyhedral Convex Sets

Given $B \in \Re^{p\times m}$, $C \in \Re^{q\times m}$, $b \in \Re^p$ and $c \in \Re^q$, we consider the following polyhedral set:

$$\mathcal{D} := \{z \in \Re^m \mid Bz \le b, Cz = c\}.$$

Here, we assume that $\mathrm{rank}(C) = q$, $q \le m$.

Denote the projection of any $z \in \Re^m$ onto $\mathcal{D}$ by $\Pi_{\mathcal{D}}(z)$. Then, the KKT conditions are derived as

$$\begin{cases} \Pi_{\mathcal{D}}(z) - z + B^{\mathrm{T}}\vartheta + C^{\mathrm{T}}\mu = 0, \\ B\Pi_{\mathcal{D}}(z) - b \le 0, \quad \vartheta \ge 0, \quad \vartheta^{\mathrm{T}}[B\Pi_{\mathcal{D}}(z) - b] = 0, \\ C\Pi_{\mathcal{D}}(z) - c = 0, \end{cases} \tag{2}$$

with Lagrange multipliers $\vartheta \in \Re^p$ and $\mu \in \Re^q$. Define the set of multipliers associated with z by

$$M(z) := \{(\vartheta, \mu) \in \Re^p \times \Re^q \mid (z, \vartheta, \mu) \text{ satisfies } (2)\}.$$

Note that $M(z)$ is a nonempty polyhedral set containing no lines. It follows from [13, Corollary 18.5.3] that there exists at least one extreme point in $M(z)$.

Let $\mathcal{I}(z)$ be the active index set:

$$\mathcal{I}(z) := \{i \in \{1, \cdots, p\} \mid (B\Pi_{\mathcal{D}}(z))_i = b_i\}. \tag{3}$$

Denote a collection of index sets by

$$\begin{aligned} \mathcal{K}_{\mathcal{D}}(z) := \big\{ & K \subseteq \{1, \cdots, p\} \mid \exists (\vartheta, \mu) \in M(z) \text{ s.t. } \mathrm{supp}(\vartheta) \subseteq K \subseteq \mathcal{I}(z), \\ & \begin{pmatrix} B_K^{\mathrm{T}} & C^{\mathrm{T}} \end{pmatrix} \text{ is of full column rank} \big\}, \end{aligned}$$

where $\mathrm{supp}(\vartheta)$ is the support of ϑ, that is, the set of indices such that $\vartheta_i \ne 0$, and B_K denotes the submatrix of B with rows indexed by K. Due to the existence of the extreme point of $M(z)$, we knows from [7] that the set $\mathcal{K}_{\mathcal{D}}(z)$ is nonempty.

For given $z \in \mathbb{R}^m$, considering the difficulty of computing the B-subdifferential $\partial_B \Pi_{\mathcal{D}}(z)$ or the Clarke generalized Jacobian $\partial \Pi_{\mathcal{D}}(z)$, we define the following multi-function $\mathcal{P} : \Re^m \rightrightarrows \Re^{m\times m}$ called HS-Jacobian [7] as an alternative for $\partial_B \Pi_{\mathcal{D}}(z)$:

$$\mathcal{P}_{\mathcal{D}}(z) := \left\{ P \in \Re^{m\times m} \mid P = I_m - \left(B_K^{\mathrm{T}} \ \ C^{\mathrm{T}}\right) \left(\begin{pmatrix} B_K \\ C \end{pmatrix} \left(B_K^{\mathrm{T}} \ \ C^{\mathrm{T}}\right) \right)^{-1} \begin{pmatrix} B_K \\ C \end{pmatrix}, K \in \mathcal{K}_{\mathcal{D}}(z) \right\}.$$

For the purpose of our later analysis, we below present some important propositions obtained from [7, 10].

Proposition 2.1 *For any given $z \in \Re^m$, there exists a neighborhood $\mathcal{Z}$ of z such that*

$$\mathcal{K}_{\mathcal{D}}(w) \subseteq \mathcal{K}_{\mathcal{D}}(z), \quad \mathcal{P}_{\mathcal{D}}(w) \subseteq \mathcal{P}_{\mathcal{D}}(z), \quad for\ any\ w \in \mathcal{Z}.$$

When $\mathcal{K}_{\mathcal{D}}(w) \subseteq \mathcal{K}_{\mathcal{D}}(z)$, we have

$$\Pi_{\mathcal{D}}(w) = \Pi_{\mathcal{D}}(z) + P(w - z), \quad for\ any\ P \in \mathcal{P}_{\mathcal{D}}(w).$$

Thus, $\partial_B \Pi_{\mathcal{D}}(z) \subseteq \mathcal{P}_{\mathcal{D}}(z)$.

Proposition 2.2 *For any $z \in \Re^m$, denote*

$$P_0 := I_m - \left(B_{\mathcal{I}(z)}^{\mathrm{T}} \ \ C^{\mathrm{T}}\right) \left(\begin{pmatrix} B_{\mathcal{I}(z)} \\ C \end{pmatrix} \left(B_{\mathcal{I}(z)}^{\mathrm{T}} \ \ C^{\mathrm{T}}\right) \right)^{\dagger} \begin{pmatrix} B_{\mathcal{I}(z)} \\ C \end{pmatrix}, \tag{4}$$

where $\mathcal{I}(z)$ is given as in (3)*. Then, $P_0 \in \mathcal{P}_{\mathcal{D}}(z)$.*

3 The HS Jacobian of the Projection $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$

In this section, we shall study the generalized HS Jacobian of the projection onto the intersection of a closed half-space and a variable box.

For given $(x, t) \in \Re^n \times \Re$, let $(y^*, \tau^*) := \Pi_{\mathcal{B}_r}(x, t)$. Define the index sets associated with (y^*, τ^*) by

$$\begin{aligned} \mathcal{K}_1 &:= \{i \in \{1, \cdots, n\} \mid y_i^* = \tau^*\}, \\ \mathcal{K}_2 &:= \{i \in \{1, \cdots, n\} \mid y_i^* = 0\}, \\ \mathcal{K}_3 &:= \{i \in \{1, \cdots, n\} \mid 0 < y_i^* < \tau^*\}, \end{aligned}$$

with cardinalities k_1, k_2 and k_3 respectively. When $\tau^* \neq 0$, we have

$$\mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_3 = \emptyset \quad \text{and} \quad k_1 + k_2 + k_3 = n.$$

We also denote the submatrices of I_n with the rows in $\mathcal{K}_1$ and $\mathcal{K}_2$ by $I_{\mathcal{K}_1}$ and $I_{\mathcal{K}_2}$ respectively.

Theorem 3.1 *Assume that $r > 0$ and $(x, t) \in \Re^n \times \Re$ in* (P) *are given. Then, the optimal solution $\Pi_{\mathcal{B}_r}(x, t)$ denoted as (y^*, τ^*) of* (P) *is unique and admits a closed-form expression. Moreover, the element N_0 of the generalized HS-Jacobian for $\Pi_{\mathcal{B}_r}(\cdot, \cdot)$ at (x, t) has the following form:*

(i) *If* $e^{\mathrm{T}}y^* \neq r\tau^*$, *then*

$$N_0 = \mathrm{Diag}(s^{n+1}_{\mathcal{K}_3}) + \frac{1}{k_1+1}\begin{pmatrix} s^n_{\mathcal{K}_1} \\ 1 \end{pmatrix}\big((s^n_{\mathcal{K}_1})^{\mathrm{T}} \quad 1\big).$$

(ii) *If* $e^{\mathrm{T}}y^* = r\tau^*$ *and one of the following conditions holds:* ① $k_1 + k_2 < n$, ② $k_1 \neq r$, *then*

$$N_0 = \mathrm{Diag}(s^{n+1}_{\mathcal{K}_3}) + \frac{1}{k_1+1}\begin{pmatrix} s^n_{\mathcal{K}_1} \\ 1 \end{pmatrix}\big((s^n_{\mathcal{K}_1})^{\mathrm{T}} \quad 1\big) - bb^{\mathrm{T}},$$

where

$$b := \alpha_1 \begin{pmatrix} s^n_{\mathcal{K}_1} \\ -k_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -s^n_{\mathcal{K}_2} \\ 0 \end{pmatrix} + \alpha_2 a \tag{5}$$

with

$$\alpha_1 = -\frac{r+1}{\sqrt{|\eta|(k_1+1)}}, \quad \alpha_2 = \sqrt{\frac{k_1+1}{|\eta|}}, \quad a := \begin{pmatrix} e_n \\ -r \end{pmatrix}$$

and

$$\eta = n + r^2 + (n-2r-1)k_1 - (1+k_1)k_2.$$

(iii) *If* $e^{\mathrm{T}}y^* = r\tau^*$, $k_1 + k_2 = n$ *and* $k_1 = r$, *we consider three situations as follows:*

(1) *If* $r^2 - rn + 1 < 0$, *then*

$$N_0 = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{k_1^2}{k_1^2+k_1+1} \end{pmatrix} + \frac{1}{k_1^2+k_1+1}\left(\begin{pmatrix} s^n_{\mathcal{K}_1} \\ 1 \end{pmatrix}\big((s^n_{\mathcal{K}_1})^{\mathrm{T}} \quad 1\big) + k_1 \begin{pmatrix} s^n_{\mathcal{K}_1} \\ 0 \end{pmatrix}\big((s^n_{\mathcal{K}_1})^{\mathrm{T}} \quad 0\big)\right)$$
$$- \left(\frac{r}{r^2-rn+1}\begin{pmatrix} s^n_{\mathcal{K}_2} \\ 0 \end{pmatrix}\big((s^n_{\mathcal{K}_2})^{\mathrm{T}} \quad 0\big) + \frac{1}{n+2+r^2+r}aa^{\mathrm{T}} + \frac{1}{(r+1)(n+1)^2}cc^{\mathrm{T}}\right),$$

where

$$c := \beta_1 \begin{pmatrix} s^n_{\mathcal{K}_1} \\ -k_1 \end{pmatrix} + \beta_2 \begin{pmatrix} -s^n_{\mathcal{K}_2} \\ 0 \end{pmatrix} + \beta_3 a \tag{6}$$

with

$$\beta_1 = \sqrt{\frac{(n+2+r^2+r)(r^2-rn+1)}{-r^2-r-1}}, \quad \beta_2 = \sqrt{\frac{(n+2+r^2+r)(-r^2-r-1)}{r^2-rn+1}},$$

$$\beta_3 = -\sqrt{\frac{(r^2-rn+1)(-r^2-r-1)}{n+2+r^2+r}}.$$

(2) *If* $n = k_1 = r$, *then*

$$N_0 = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{n^2+2n}{(n+1)^2} \end{pmatrix} + \frac{n+2}{(n+1)^2}\begin{pmatrix} e_n \\ 0 \end{pmatrix}\big(e_n^{\mathrm{T}} \quad 0\big) - \frac{1}{(n+1)^3}\big(-e_{n+1}e_{n+1}^{\mathrm{T}} + (n+2)aa^{\mathrm{T}}\big).$$

(3) *If* $n=2$ *and* $r=k_1=k_2=1$, *then*

$$N_0=\begin{cases}\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}, & \textit{if } y^*=\begin{pmatrix}\tau^*\\ 0\end{pmatrix},\\ \begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}, & \textit{otherwise.}\end{cases}$$

(iv) *If* $\Pi_{\mathcal{B}_r}(x,t)=(\boldsymbol{0},0)$, *then* $N_0=\boldsymbol{0}$.

Proof The closed-form solution of (P) and the algorithm for finding the solution were discussed in [12]. We omit the explicit formulas of the projection $\Pi_{\mathcal{B}_r}(x,t)$ here and refer the readers to [12].

First, we equivalently rewrite problem (P) as

$$\begin{aligned}\min_{u}\ & \frac{1}{2}\|u-v\|^2\\ \text{s.t.}\ & Au\le 0,\end{aligned}\tag{P$'$}$$

where

$$u:=\begin{pmatrix}y\\ \tau\end{pmatrix}\in\Re^{n+1},\quad v:=\begin{pmatrix}x\\ t\end{pmatrix}\in\Re^{n+1},\quad A:=\begin{pmatrix}I_n & -e_n\\ -I_n & \mathbf{0}\\ e_n^{\mathrm{T}} & -r\end{pmatrix}\in\Re^{(2n+1)\times(n+1)}.$$

Denote Ω as the nonempty polyhedral convex set $\{u\in\Re^{n+1}\mid Au\le 0\}$. Then, there exists Lagrange multiplier $\lambda\in\Re^{2n+1}$ such that

$$\begin{cases}\Pi_\Omega(v)-v+A^{\mathrm{T}}\lambda=0,\\ A\Pi_\Omega(v)\le 0,\quad \lambda\ge 0,\\ \lambda^{\mathrm{T}}A\Pi_\Omega(v)=0.\end{cases}\tag{7}$$

Let $\mathcal{I}(v)$ be the active set that

$$\mathcal{I}(v):=\{i\in\{1,\cdots,2n+1\}\mid (A\Pi_\Omega(v))_i=0\}.$$

Denote $A_{\mathcal{I}(v)}$ as the submatrix of A consisting of the rows whose indices are in $\mathcal{I}(v)$.

From Proposition 2.2, one can readily obtain that the element N_0 of the generalized HS-Jacobian of the projection $\Pi_{\mathcal{B}_r}(x,t)$ is given by

$$N_0:=I_{n+1}-A_{\mathcal{I}(v)}^{\mathrm{T}}(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger}A_{\mathcal{I}(v)}.\tag{8}$$

(i) If $e^{\mathrm{T}}y^*\ne r\tau^*$, then

$$A_{\mathcal{I}(v)}=\begin{pmatrix}I_{\mathcal{K}_1} & -e_{k_1}\\ -I_{\mathcal{K}_2} & \mathbf{0}\end{pmatrix}.$$

After calculation, we get that

$$A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}} = \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ -I_{\mathcal{K}_2} & \mathbf{0} \end{pmatrix} \begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} & -I_{\mathcal{K}_2}^{\mathrm{T}} \\ -e_{k_1}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} E_{k_1} + I_{k_1} & \mathbf{0} \\ \mathbf{0} & I_{k_2} \end{pmatrix}.$$

It is obvious that $A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}}$ is invertible. By using Sherman-Morrison-Woodbury formula (cf. [5]), we obtain that

$$(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{-1} = \left(I_{k_1+k_2} + \begin{pmatrix} e_{k_1} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} e_{k_1}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \right)^{-1} = I_{k_1+k_2} - \frac{1}{1+k_1} \begin{pmatrix} E_{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Finally, we get that

$$\begin{aligned} N_0 &= I_{n+1} - A_{\mathcal{I}(v)}^{\mathrm{T}}(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{-1}A_{\mathcal{I}(v)} \\ &= I_{n+1} - \begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} & -I_{\mathcal{K}_2}^{\mathrm{T}} \\ -e_{k_1}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \left(I_{k_1+k_2} - \frac{1}{1+k_1} \begin{pmatrix} E_{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ -I_{\mathcal{K}_2} & \mathbf{0} \end{pmatrix} \\ &= I_{n+1} - \left(\begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} I_{\mathcal{K}_1} + I_{\mathcal{K}_2}^{\mathrm{T}} I_{\mathcal{K}_2} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} - \frac{1}{k_1+1} \begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} e_{k_1} \\ 1 \end{pmatrix} \begin{pmatrix} e_{k_1}^{\mathrm{T}} I_{\mathcal{K}_1} & 1 \end{pmatrix} \right) \\ &= \mathrm{Diag}(s_{\mathcal{K}_3}^{n+1}) + \frac{1}{k_1+1} \begin{pmatrix} s_{\mathcal{K}_1}^{n} \\ 1 \end{pmatrix} \begin{pmatrix} (s_{\mathcal{K}_1}^{n})^{\mathrm{T}} & 1 \end{pmatrix}. \end{aligned}$$

Next, we turn to consider the case of $e^{\mathrm{T}}y^* = r\tau^*$. In this case, we have that

$$A_{\mathcal{I}(v)} = \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ -I_{\mathcal{K}_2} & \mathbf{0} \\ e_n^{\mathrm{T}} & -r \end{pmatrix}.$$

Then, it holds that

$$A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}} = \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ -I_{\mathcal{K}_2} & \mathbf{0} \\ e_n^{\mathrm{T}} & -r \end{pmatrix} \begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} & -I_{\mathcal{K}_2}^{\mathrm{T}} & e_n \\ -e_{k_1}^{\mathrm{T}} & \mathbf{0} & -r \end{pmatrix} = \begin{pmatrix} E_{k_1}+I_{k_1} & \mathbf{0} & (r+1)e_{k_1} \\ \mathbf{0} & I_{k_2} & -e_{k_2} \\ (r+1)e_{k_1}^{\mathrm{T}} & -e_{k_2}^{\mathrm{T}} & n+r^2 \end{pmatrix}. \tag{9}$$

(ii) If $e^{\mathrm{T}}y^* = r\tau^*$ and one of the following conditions holds: ① $k_1 + k_2 < n$, ② $k_1 \neq r$, $A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}}$ is nonsingular.

After performing a series of elementary row operations to $A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}}$, the inverse of $A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}}$ has the following form:

$$(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{-1} = \begin{pmatrix} I_{k_1} + \gamma_1 E_{k_1} & \gamma_2 E_{k_1 \times k_2} & \gamma_2 e_{k_1} \\ \gamma_2 E_{k_2 \times k_1} & I_{k_2} + \gamma_3 E_{k_2} & \gamma_3 e_{k_2} \\ \gamma_2 e_{k_1}^{\mathrm{T}} & \gamma_3 e_{k_2}^{\mathrm{T}} & \gamma_3 \end{pmatrix},$$

where

$$\gamma_1 = \frac{k_2 - n + 2r + 1}{\eta}, \quad \gamma_2 = -\frac{r+1}{\eta}, \quad \gamma_3 = \frac{k_1+1}{\eta}$$

and η is given as in (5).

In order to simplify further calculations, we split $(A_{\mathcal{I}(v)}A^{\mathrm{T}}_{\mathcal{I}(v)})^{-1}$ into three parts:

$$(A_{\mathcal{I}(v)}A^{\mathrm{T}}_{\mathcal{I}(v)})^{-1} = \begin{pmatrix} I_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{k_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} -\frac{1}{k_1+1}E_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \alpha_1 e_{k_1} \\ \alpha_2 e_{k_2} \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_1 e^{\mathrm{T}}_{k_1} & \alpha_2 e^{\mathrm{T}}_{k_2} & \alpha_2 \end{pmatrix}$$

with α_1 and α_2 given as in (5). Then, we get that

$$\begin{aligned} N_0 &= I_{n+1} - A^{\mathrm{T}}_{\mathcal{I}(v)}(A_{\mathcal{I}(v)}A^{\mathrm{T}}_{\mathcal{I}(v)})^{-1}A_{\mathcal{I}(v)} \\ &= I_{n+1} - \begin{pmatrix} I^{\mathrm{T}}_{\mathcal{K}_1} & -I^{\mathrm{T}}_{\mathcal{K}_2} & e_n \\ -e^{\mathrm{T}}_{k_1} & \mathbf{0} & -r \end{pmatrix} \left(\begin{pmatrix} I_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{k_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} -\frac{1}{k_1+1}E_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \alpha_1 e_{k_1} \\ \alpha_2 e_{k_2} \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_1 e^{\mathrm{T}}_{k_1} & \alpha_2 e^{\mathrm{T}}_{k_2} & \alpha_2 \end{pmatrix} \right) \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ -I_{\mathcal{K}_2} & \mathbf{0} \\ e^{\mathrm{T}}_n & -r \end{pmatrix} \\ &= I_{n+1} - \left(\begin{pmatrix} I^{\mathrm{T}}_{\mathcal{K}_1} I_{\mathcal{K}_1} + I^{\mathrm{T}}_{\mathcal{K}_2} I_{\mathcal{K}_2} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} - \frac{1}{k_1+1} \begin{pmatrix} I^{\mathrm{T}}_{\mathcal{K}_1} e_{k_1} \\ 1 \end{pmatrix} \begin{pmatrix} e^{\mathrm{T}}_{k_1} I_{\mathcal{K}_1} & 1 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \alpha_1 I^{\mathrm{T}}_{\mathcal{K}_1} e_{k_1} - \alpha_2 I^{\mathrm{T}}_{\mathcal{K}_2} e_{k_2} + \alpha_2 e_n \\ -\alpha_1 k_1 - \alpha_2 r \end{pmatrix} \begin{pmatrix} \alpha_1 I^{\mathrm{T}}_{\mathcal{K}_1} e_{k_1} - \alpha_2 I^{\mathrm{T}}_{\mathcal{K}_2} e_{k_2} + \alpha_2 e_n \\ -\alpha_1 k_1 - \alpha_2 r \end{pmatrix}^{\mathrm{T}} \right) \\ &= \mathrm{Diag}(s^{n+1}_{\mathcal{K}_3}) + \frac{1}{k_1+1} \begin{pmatrix} s^n_{\mathcal{K}_1} \\ 1 \end{pmatrix} \begin{pmatrix} (s^n_{\mathcal{K}_1})^{\mathrm{T}} & 1 \end{pmatrix} - bb^{\mathrm{T}} \end{aligned}$$

with b given as in (5).

(iii) If $e^{\mathrm{T}}y^* = r\tau^*$, $k_1 + k_2 = n$ and $k_1 = r$, the matrix $A_{\mathcal{I}(v)}A^{\mathrm{T}}_{\mathcal{I}(v)}$ in (9) is singular.

First, we note that $A_{\mathcal{I}(v)}A^{\mathrm{T}}_{\mathcal{I}(v)}$ admits the following full rank factorization (cf. [6]):

$$A_{\mathcal{I}(v)}A^{\mathrm{T}}_{\mathcal{I}(v)} = FG \text{ with } F := \begin{pmatrix} E_{k_1} + I_{k_1} & \mathbf{0} \\ \mathbf{0} & I_{k_2} \\ (r+1)e^{\mathrm{T}}_{k_1} & -e^{\mathrm{T}}_{k_2} \end{pmatrix} \text{ and } G := \begin{pmatrix} I_{k_1} & \mathbf{0} & e_{k_1} \\ \mathbf{0} & I_{k_2} & -e_{k_2} \end{pmatrix}.$$

It is not hard to find that

$$F = \begin{pmatrix} e_{k_1} \\ \mathbf{0} \\ r \end{pmatrix} \begin{pmatrix} e_{k_1}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} + G^{\mathrm{T}}. \tag{10}$$

From (10), one can readily obtain that

$$A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}} = G^{\mathrm{T}} (I_n + dd^{\mathrm{T}}) G \quad \text{with} \quad d := \begin{pmatrix} e_{k_1} \\ \mathbf{0} \end{pmatrix}.$$

Note that $I_n + dd^{\mathrm{T}}$ is a symmetric positive definite matrix, we are able to get the factorization $I_n + dd^{\mathrm{T}} = SS^{\mathrm{T}}$ with symmetric positive definite matrix $S \in \Re^{n \times n}$. It is obvious that $F_1 := G^{\mathrm{T}} S \in \Re^{(n+1) \times n}$ is of full column rank. Hence, we get that $A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}} = F_1 F_1^{\mathrm{T}}$.

Instead of calculating $(A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} = G^{\mathrm{T}} (GG^{\mathrm{T}})^{-1} (F^{\mathrm{T}} F)^{-1} F^{\mathrm{T}}$ directly, we compute the Moore-Penrose inverse of $A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}}$ by

$$\begin{aligned} (A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} = (F_1 F_1^{\mathrm{T}})^{\dagger} &= F_1 (F_1^{\mathrm{T}} F_1)^{-1} (F_1^{\mathrm{T}} F_1)^{-1} F_1^{\mathrm{T}} \\ &= G^{\mathrm{T}} S (S^{\mathrm{T}} GG^{\mathrm{T}} S)^{-1} (S^{\mathrm{T}} GG^{\mathrm{T}} S)^{-1} S^{\mathrm{T}} G \\ &= G^{\mathrm{T}} SS^{-1} (GG^{\mathrm{T}})^{-1} S^{-T} S^{-1} (GG^{\mathrm{T}})^{-1} S^{-T} S^{\mathrm{T}} G \\ &= G^{\mathrm{T}} (GG^{\mathrm{T}})^{-1} (I_n + dd^{\mathrm{T}})^{-1} (GG^{\mathrm{T}})^{-1} G. \end{aligned}$$

After careful manipulations, we obtain that

$$(A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} = \begin{pmatrix} I_{k_1} + \sigma_1 E_{k_1} & \sigma_2 E_{k_1 \times k_2} & \sigma_3 e_{k_1} \\ \sigma_2 E_{k_2 \times k_1} & I_{k_2} + \sigma_4 E_{k_2} & \sigma_5 e_{k_2} \\ \sigma_3 e_{k_1}^{\mathrm{T}} & \sigma_5 e_{k_2}^{\mathrm{T}} & \sigma_6 \end{pmatrix}, \tag{11}$$

where

$$\sigma_1 = \frac{-n^2 + (r-3)n - (r^2+3)}{(r+1)(n+1)^2}, \quad \sigma_2 = \frac{n+2+r^2+r}{(r+1)(n+1)^2}, \quad \sigma_3 = \frac{r^2 - rn + 1}{(r+1)(n+1)^2},$$

$$\sigma_4 = \frac{-r^2 - (n+2)(r+1)}{(r+1)(n+1)^2}, \quad \sigma_5 = \frac{-r^2 - r - 1}{(r+1)(n+1)^2}, \quad \sigma_6 = \frac{-r^2 + rn + n}{(r+1)(n+1)^2}.$$

(1) If $r^2 - rn + 1 < 0$, we further reformulate $(A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger}$ as follows

$$\begin{aligned} (A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} &= \begin{pmatrix} I_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{k_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \rho_1 E_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_2 E_{k_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \rho_3 \end{pmatrix} \\ &\quad + \frac{1}{(r+1)(n+1)^2} \begin{pmatrix} \beta_1 e_{k_1} \\ \beta_2 e_{k_2} \\ \beta_3 \end{pmatrix} \begin{pmatrix} \beta_1 e_{k_1}^{\mathrm{T}} & \beta_2 e_{k_2}^{\mathrm{T}} & \beta_3 \end{pmatrix}, \end{aligned}$$

where

$$\rho_1 = \frac{r+1}{-r^2-r-1}, \quad \rho_2 = \frac{r}{r^2-rn+1}, \quad \rho_3 = \frac{1}{n+2+r^2+r},$$

and β_1, β_2, β_3 are defined as in (6). Thus, one has that

$$\begin{aligned}
N_0 &= I_{n+1} - A_{\mathcal{I}(v)}^{\mathrm{T}}(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger}A_{\mathcal{I}(v)} \\
&= I_{n+1} - \begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} & -I_{\mathcal{K}_2}^{\mathrm{T}} & e_n \\ -e_{k_1}^{\mathrm{T}} & \mathbf{0} & -r \end{pmatrix} \left(\begin{pmatrix} I_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{k_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \rho_1 E_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_2 E_{k_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \rho_3 \end{pmatrix} \right. \\
&\quad \left. + \frac{1}{(r+1)(n+1)^2} \begin{pmatrix} \beta_1 e_{k_1} \\ \beta_2 e_{k_2} \\ \beta_3 \end{pmatrix} \begin{pmatrix} \beta_1 e_{k_1}^{\mathrm{T}} & \beta_2 e_{k_2}^{\mathrm{T}} & \beta_3 \end{pmatrix} \right) \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ -I_{\mathcal{K}_2} & \mathbf{0} \\ e_n^{\mathrm{T}} & -r \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{k_1^2}{k_1^2+k_1+1} \end{pmatrix} + \frac{1}{k_1^2+k_1+1} \left(\begin{pmatrix} s_{\mathcal{K}_1}^n \\ 1 \end{pmatrix} \begin{pmatrix} (s_{\mathcal{K}_1}^n)^{\mathrm{T}} & 1 \end{pmatrix} + k_1 \begin{pmatrix} s_{\mathcal{K}_1}^n \\ 0 \end{pmatrix} \begin{pmatrix} (s_{\mathcal{K}_1}^n)^{\mathrm{T}} & 0 \end{pmatrix} \right) \\
&\quad - \left(\frac{r}{r^2-rn+1} \begin{pmatrix} s_{\mathcal{K}_2}^n \\ 0 \end{pmatrix} \begin{pmatrix} (s_{\mathcal{K}_2}^n)^{\mathrm{T}} & 0 \end{pmatrix} + \frac{1}{n+2+r^2+r} aa^{\mathrm{T}} + \frac{1}{(r+1)(n+1)^2} cc^{\mathrm{T}} \right)
\end{aligned}$$

with c given as in (6).

(2) If $r^2 - rn + 1 > 0$, then $r = n = k_1$ and $k_2 = 0$. Note that $(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger}$ can be written as

$$(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} = \begin{pmatrix} I_{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \frac{1}{(n+1)^3} \left(\begin{pmatrix} (-n^2-3n-4)E_{k_1} & \mathbf{0} \\ \mathbf{0} & n-1 \end{pmatrix} + \begin{pmatrix} e_{k_1} \\ 1 \end{pmatrix} \begin{pmatrix} e_{k_1}^{\mathrm{T}} & 1 \end{pmatrix} \right).$$

Thus, N_0 is derived as

$$\begin{aligned}
N_0 &= I_{n+1} - A_{\mathcal{I}(v)}^{\mathrm{T}}(A_{\mathcal{I}(v)}A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger}A_{\mathcal{I}(v)} \\
&= I_{n+1} - \begin{pmatrix} I_{\mathcal{K}_1}^{\mathrm{T}} & e_n \\ -e_{k_1}^{\mathrm{T}} & -r \end{pmatrix} \left(\begin{pmatrix} I_{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right. \\
&\quad \left. + \frac{1}{(n+1)^3} \left(\begin{pmatrix} (-n^2-3n-4)E_{k_1} & \mathbf{0} \\ \mathbf{0} & n-1 \end{pmatrix} + \begin{pmatrix} e_{k_1} \\ 1 \end{pmatrix} \begin{pmatrix} e_{k_1}^{\mathrm{T}} & 1 \end{pmatrix} \right) \right) \begin{pmatrix} I_{\mathcal{K}_1} & -e_{k_1} \\ e_n^{\mathrm{T}} & -r \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{n^2+2n}{(n+1)^2} \end{pmatrix} + \frac{n+2}{(n+1)^2} \begin{pmatrix} s_{\mathcal{K}_1}^n \\ 0 \end{pmatrix} \begin{pmatrix} (s_{\mathcal{K}_1}^n)^{\mathrm{T}} & 0 \end{pmatrix} \\
&\quad - \frac{1}{(n+1)^3} \left(- \begin{pmatrix} s_{\mathcal{K}_1}^n \\ 1 \end{pmatrix} \begin{pmatrix} (s_{\mathcal{K}_1}^n)^{\mathrm{T}} & 1 \end{pmatrix} + \begin{pmatrix} s_{\mathcal{K}_1}^n \\ -k_1 \end{pmatrix} a^{\mathrm{T}} + a\begin{pmatrix} (s_{\mathcal{K}_1}^n)^{\mathrm{T}} & -k_1 \end{pmatrix} + naa^{\mathrm{T}} \right) \\
&= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{n^2+2n}{(n+1)^2} \end{pmatrix} + \frac{n+2}{(n+1)^2} \begin{pmatrix} e_n \\ 0 \end{pmatrix} \begin{pmatrix} e_n^{\mathrm{T}} & 0 \end{pmatrix} - \frac{1}{(n+1)^3} \left(- e_{n+1}e_{n+1}^{\mathrm{T}} + (n+2)aa^{\mathrm{T}} \right).
\end{aligned}$$

(3) If $r^2 - rn + 1 = 0$, we have $r = k_1 = k_2 = 1$ and $n = 2$. Then, one can easily obtain that

$$A_{\mathcal{I}(v)} = \begin{cases} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, & \text{if } y^* = \begin{pmatrix} \tau^* \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

From (11), we get that

$$(A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

Finally, we can readily conclude that

$$N_0 = I_{n+1} - A_{\mathcal{I}(v)}^{\mathrm{T}} (A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{\dagger} A_{\mathcal{I}(v)} = \begin{cases} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, & \text{if } y^* = \begin{pmatrix} \tau^* \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, & \text{otherwise.} \end{cases}$$

(iv) If $\Pi_{\mathcal{B}_r}(x,t) = (\mathbf{0}, 0)$, the generalized Jacobian contains zero matrix since (x,t) is in the polar cone of $\mathcal{B}_r$. In fact, we can obtain $N_0 = \mathbf{0}$ from formula (8) when we choose $\mathcal{K}_1 = \emptyset$, $\mathcal{K}_2 = \{1, \cdots, n\}$ and $\mathcal{K}_3 = \emptyset$. Then, we have

$$A_{\mathcal{I}(v)} = \begin{pmatrix} -I_n & \mathbf{0} \\ e_n^{\mathrm{T}} & -r \end{pmatrix}.$$

Thus, $(A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{-1}$ is derived as

$$(A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{-1} = \begin{pmatrix} I_n & -e_n \\ -e_n^{\mathrm{T}} & n + r^2 \end{pmatrix}^{-1} = \begin{pmatrix} I_n + \frac{1}{r^2} E_n & \frac{1}{r^2} e_n \\ \frac{1}{r^2} e_n^{\mathrm{T}} & \frac{1}{r^2} \end{pmatrix},$$

which implies

$$\begin{aligned} N_0 &= I_{n+1} - A_{\mathcal{I}(v)}^{\mathrm{T}} (A_{\mathcal{I}(v)} A_{\mathcal{I}(v)}^{\mathrm{T}})^{-1} A_{\mathcal{I}(v)} \\ &= I_{n+1} - \begin{pmatrix} -I_n^{\mathrm{T}} & e_n \\ \mathbf{0} & -r \end{pmatrix} \begin{pmatrix} I_n + \frac{1}{r^2} E_n & \frac{1}{r^2} e_n \\ \frac{1}{r^2} e_n^{\mathrm{T}} & \frac{1}{r^2} \end{pmatrix} \begin{pmatrix} -I_n & \mathbf{0} \\ e_n^{\mathrm{T}} & -r \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

The proof is complete.

4 Conclusion

In this paper, we study the generalized Jacobian for the projection of a vector onto the intersection of a closed half-space and a variable box. We derive the explicit formulas of an element in the set of the generalized HS Jacobian for the projection, which can be expressed as the combination of a diagonal matrix and few rank-one symmetric matrices. The explicit formulas of the HS Jacobian we obtained not only lay foundation for the algorithmic design of the second order nonsmooth methods, but also show that the computational cost would be cheap when the second order information are incorporated in the algorithms for the related optimization problems. We leave it as our future work.

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