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Analysis and FDTD Simulation of a Perfectly Matched Layer for the Drude Metamaterial

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Abstract. In this paper, we are concerned about the stability analysis for a Perfectly Matched Layer (PML) recently developed by Bécache et al. [5] for simulating wave propagation in the Drude metamaterial. This PML is proved to be stable originally in [6] through a modal analysis. Here we establish its stability by the energy method. A FDTD scheme is developed and analyzed. Numerical simulations illustrate the stability of the PML model and its effectiveness in absorbing outgoing waves in the Drude medium.

AMS subject classifications: 65N30, 35L15, 78-08

Key words: Maxwell's equations, perfectly matched layer, FDTD method.

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1 Introduction

One of the challenges to simulate wave propagation in unbounded domains is how to construct effective artificial boundary conditions to absorb the outgoing waves without reflecting them back into the computational domains. A widely adopted technique is the so-called Perfectly Matched Layer (PML) proposed by Bérenger [7] in 1994 for solving the three-dimensional (3D) time-dependent Maxwell's equations. Since 1994, in addition to many PML models proposed and studied further for Maxwell's equations [1,8,10,11,28,35,36], the PML technique has also been extended

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to solve other wave propagation problems, such as acoustics and elastodynamics [2, 5, 16].

In late 1990s, the so-called negative index metamaterials (NIMs) was manufactured successfully [30, 32] and immediately became a very hot research topic as evidenced by numerous papers (cf. [37] and references therein) and books published on metamaterials (e.g., [13, 17, 24] and references therein). Due to the importance of numerical simulation for NIMs, many studies of PMLs in NIMs have been carried out (e.g., [12,15,31]). Cummer [14] first noticed that the classical PMLs fail in NIMs and proposed stable PMLs for the Drude metamaterial with $\omega_e = \omega_m$ (see (2.2) below). Stable PMLs were later extended to the general case $\omega_e \neq \omega_m$ in [15,31]. In [5], Bécache et al. presented a rigorous development of a stable PMLs for the Drude model for the general case $\omega_e \neq \omega_m$. The stability is proved in [6] through a modal analysis. Since the modal analysis is limited to the constant damping coefficients, one of our main goals of this paper is to make an effort in establishing a stability for the practical variable damping functions by using the energy method. To our best knowledge, this is the first energy stability established for this PML model.

Since the PML models (cf. [33, Ch. 7], [34], [24, Ch. 8] and references therein) are much more complicated than the corresponding Maxwell's equations, the stability analysis is quite challenging. For example, the stability for the classical Bérenger PML with variable damping functions is made possible through an equivalent form [4]. Furthermore, developing and analyzing effective numerical methods for solving the PML models is not trivial, and many researchers have made contributions in this direction (e.g., [3, 9, 19–23, 26, 27]). Though Bécache et al. [5] have presented Finite-Difference Time-Domain (FDTD) simulation for their developed metamaterial PML model, but no detail has been given for the FDTD scheme and its analysis. Hence, another major goal of our paper is to fill the gap by developing and analyzing a FDTD scheme for the metamaterial PML model proposed by Bécache et al. [5].

The rest of the paper is organized as follows. In Section 2, we first introduce the 2D metamaterial PML model proposed in [5], and then carry out its stability analysis. In Section 3, we propose a FDTD scheme for this PML model, and establish a discrete stability. Numerical results are presented in Section 4 to demonstrate the stability of this PML model and its effectiveness in absorbing outgoing waves. We conclude the paper in Section 5.

2 The 2-D metamaterial PML model

A general 2-D Transverse Electric (TEz) metamaterial PML model wth 16 unknowns was developed in Bécache et al. Here we focus on the popular $\omega_e = \omega_m$ case whose governing equations can be written as follows (cf. [5, Eq. (48)]): For any $(x,y,t) \in \Omega \times (0,T]$,

$$\partial_t E_x + \omega_e^2 J_x + \epsilon_0^{-1} \sigma_y E_x = \epsilon_0^{-1} \partial_y (H^x + H^y), \tag{2.1a}$$

$$\partial_t J_x - E_x = 0, \tag{2.1b}$$

$$\partial_t E_y + \omega_e^2 J_y + \epsilon_0^{-1} \sigma_x E_y = -\epsilon_0^{-1} \partial_x (H^x + H^y), \tag{2.1c}$$

$$\partial_t J_y - E_y = 0, \tag{2.1d}$$

$$\partial_t H^x + \omega_m^2 K^x + \mu_0^{-1} \sigma_y H^x = \mu_0^{-1} \partial_y E_x, \tag{2.1e}$$

$$\partial_t K^x - H^x = 0, \tag{2.1f}$$

$$\partial_t H^y + \omega_m^2 K^y + \mu_0^{-1} \sigma_x H^y = -\mu_0^{-1} \partial_x E_y, \tag{2.1g}$$

$$\partial_t K^y - H^y = 0, \tag{2.1h}$$

where ϵ_0 and μ_0 are the permittivity and permeability in free space, $\boldsymbol{E} = (E_x, E_y)$ and $H = H^x + H^y$ are the electric field and magnetic field (in split form) respectively, $\boldsymbol{J} = (J_x, J_y)$ and $\boldsymbol{K} = (K^x, K^y)$ are the auxiliary variables, $\sigma_x(x) \geq 0$ and $\sigma_y(y) \geq 0$ are the damping functions in the x and y directions, $\omega_e > 0$ and $\omega_m > 0$ are the electric and the magnetic plasma frequencies in the Drude model described by the following:

$$\epsilon(\omega) = \epsilon_0 \Big(1 - \frac{\omega_e^2}{\omega^2}\Big), \quad \mu(\omega) = \mu_0 \Big(1 - \frac{\omega_m^2}{\omega^2}\Big). \tag{2.2}$$

Here and in the rest of the paper, ω denotes the general wave frequency.

Since no detailed derivation for the above PML model is given in [5], to make our paper self-contained, we now present some details in deriving this model. As mentioned in [5], this PML model can be derived by splitting the magnetic field and using the change of variable technique (cf. [24, Ch. 8]) from the Maxwell's equations given in the frequency domain:

$$i\omega\epsilon(\omega)\tilde{E}_x = \partial_{\tilde{y}}(\tilde{H}^x + \tilde{H}^y), \tag{2.3a}$$

$$i\omega\epsilon(\omega)\tilde{E}_y = -\partial_{\tilde{x}}(\tilde{H}^x + \tilde{H}^y), \tag{2.3b}$$

$$i\omega\mu(\omega)\tilde{H}^x = \partial_{\tilde{y}}\tilde{E}^x, \tag{2.3c}$$

$$i\omega\mu(\omega)\tilde{H}^y = -\partial_{\tilde{x}}\tilde{E}^y, \tag{2.3d}$$

where the magnetic field $\tilde{H}$ is splitted into the sum of $\tilde{H}^x$ and $\tilde{H}^y$, and $\tilde{E}_x$, $\tilde{E}_y$, $\tilde{H}^x$ and $\tilde{H}^y$ are the electric and magnetic fields in the frequency domain.

Let us first show the derivation of (2.1a)-(2.1b) from (2.3a). Applying the mapping

$$\tilde{y}(y) = \int_0^y \frac{1}{\alpha(\xi)} d\xi$$

with

$$\alpha(y)=\Big(1+\frac{\sigma_y}{i\omega\psi(\omega)}\Big)^{-1}, \quad \psi(\omega)=\epsilon_0\Big(1-\frac{\omega_e^2}{\omega^2}\Big), \tag{2.4}$$

and the Drude model permittivity $\epsilon(\omega)$ given in (2.2) to (2.3a), we have

$$i\omega\epsilon_0\Big(1-\frac{\omega_e^2}{\omega^2}\Big)\tilde{E}_x=\alpha(y)\partial_y(\tilde{H}^x+\tilde{H}^y)=\Big(1+\frac{\sigma_y}{i\omega\epsilon_0(1-\frac{\omega_e^2}{\omega^2})}\Big)^{-1}\partial_y(\tilde{H}^x+\tilde{H}^y), \tag{2.5}$$

which can be simplified further as

$$\Big(i\omega+\frac{\omega_e^2}{i\omega}+\epsilon_0^{-1}\sigma_y\Big)\tilde{E}_x=\epsilon_0^{-1}\partial_y(\tilde{H}^x+\tilde{H}^y). \tag{2.6}$$

Now let us introduce an auxiliary variable

$$\tilde{J}_x=\frac{1}{i\omega}\tilde{E}_x. \tag{2.7}$$

Using the assumption that the electromagnetic fields are time-harmonic (i.e., the time-domain fields and the frequency-domain fields satisfy the relation $u(\boldsymbol{x},t)=e^{i\omega t}\tilde{u}(\boldsymbol{x})$), we immediately obtain (2.1b) from (2.7). Furthermore, using (2.7), we can rewrite (2.6) as

$$i\omega\tilde{E}_x+\omega_e^2\tilde{J}_x+\epsilon_0^{-1}\sigma_y\tilde{E}_x=\epsilon_0^{-1}\partial_y(\tilde{H}^x+\tilde{H}^y). \tag{2.8}$$

Applying the time-harmonic assumption to (2.8), we immediately obtain (2.1a).

Similarly, (2.1c)-(2.1d) can be obtained from (2.3b) by using the mapping

$$\tilde{x}(x)=\int_0^x\frac{1}{\alpha(\xi)}d\xi$$

with

$$\alpha(x)=\Big(1+\frac{\sigma_x}{i\omega\psi(\omega)}\Big)^{-1}, \quad \psi(\omega)=\epsilon_0\Big(1-\frac{\omega_e^2}{\omega^2}\Big), \tag{2.9}$$

and introducing the auxilary variable $\tilde{J}_y=\frac{1}{i\omega}\tilde{E}_y$.

By symmetry, (2.1e)-(2.1f) can be derived from (2.3c) by using the mapping

$$\tilde{y}(y)=\int_0^y\frac{1}{\alpha(\xi)}d\xi$$

with

$$\alpha(y)=\Big(1+\frac{\sigma_y}{i\omega\psi(\omega)}\Big)^{-1}, \quad \psi(\omega)=\mu_0\Big(1-\frac{\omega_m^2}{\omega^2}\Big), \tag{2.10}$$

and introducing the auxilary variable $\tilde{K}^x = \frac{1}{i\omega}\tilde{H}^x$. Finally, (2.1g)-(2.1h) can be derived from (2.3d) by using the mapping $\tilde{x}(x) = \int_0^x \frac{1}{\alpha(\xi)} d\xi$ with

$$\alpha(x) = \Big(1 + \frac{\sigma_x}{i\omega\psi(\omega)}\Big)^{-1}, \quad \psi(\omega) = \mu_0\Big(1 - \frac{\omega_m^2}{\omega^2}\Big), \tag{2.11}$$

and introducing the auxilary variable $\tilde{K}^y = \frac{1}{i\omega}\tilde{H}^y$.

Since the PML is used in a rectangular domain outside the physical domain, we consider solving (2.1a)-(2.1h) in a rectangular domain $\Omega = [a,b] \times [c,d]$. To complete the model (2.1a)-(2.1h), we assume that the model problem is subject to the initial conditions

$$E_x(x,y,0) = E_{x0}(x,y), \qquad E_y(x,y,0) = E_{y0}(x,y), \tag{2.12a}$$
$$J_x(x,y,0) = J_{x0}(x,y), \qquad J_y(x,y,0) = J_{y0}(x,y), \tag{2.12b}$$

and

$$H^x(x,y,0) = H_{x0}(x,y), \qquad H^y(x,y,0) = H_{y0}(x,y), \tag{2.13a}$$
$$K^x(x,y,0) = K_{x0}(x,y), \qquad K^y(x,y,0) = K_{y0}(x,y), \tag{2.13b}$$

and the perfect conduct (PEC) boundary condition

$$E_x(x,y,t)|_{y=c,d} = 0, \quad E_y(x,y,t)|_{x=a,b} = 0, \tag{2.14}$$

where E_{x0}, E_{y0}, J_{x0}, J_{y0}, H_{x0}, H_{y0}, K_{x0}, and K_{y0} are some properly given functions.

In the rest of the paper, we denote the L^2 norm over Ω as $||\cdot|| := ||\cdot||_{L^2(\Omega)}$.

Theorem 2.1. *For the solution of* (2.1a)-(2.1h)*, define the energy*

$$\begin{aligned}\mathcal{E}_1(t) = \frac{1}{2}\Big[&\epsilon_0(||E_x||^2 + ||E_y||^2) + \epsilon_0\omega_e^2(||J_x||^2 + ||J_y||^2)\\ &+ \mu_0||H^x + H^y||^2 + \mu_0\omega_m^2||K^x + K^y||^2\Big].\end{aligned} \tag{2.15}$$

Then for any nonnegative functions $\sigma_x(x)$ and $\sigma_y(y)$, we have

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}_1(t) + ||\sigma_y^{\frac{1}{2}} E_x||^2 + ||\sigma_x^{\frac{1}{2}} E_y||^2 + ||\sigma_y^{\frac{1}{2}} H^x||^2\\ &\quad + ||\sigma_x^{\frac{1}{2}} H^y||^2 + ((\sigma_x + \sigma_y)H^x, H^y) = 0.\end{aligned} \tag{2.16}$$

When $\sigma_x = \sigma_y = \sigma \geq 0$ (i.e., a positive constant), the energy is decreasing:

$$\mathcal{E}_1(t) \leq \mathcal{E}_1(0), \quad \forall t \in [0,T]. \tag{2.17}$$

Proof. To make our proof easy to follow, we divide the proof into three major parts.

(I) Multiplying (2.1a) and (2.1b) by $\epsilon_0 E_x$ and $\epsilon_0\omega_e^2 J_x$, respectively, integrating each over Ω, then adding up the results, we have

$$\frac{1}{2}\frac{d}{dt}\left[\epsilon_0||E_x||^2+\epsilon_0\omega_e^2||J_x||^2\right]+||\sigma_y^{\frac{1}{2}}E_x||^2=(\partial_y(H^x+H^y),E_x). \tag{2.18}$$

Multiplying (2.1c) and (2.1d) by $\epsilon_0 E_y$ and $\epsilon_0\omega_e^2 J_y$, respectively, integrating each over Ω, then adding up the results, we have

$$\frac{1}{2}\frac{d}{dt}\left[\epsilon_0||E_y||^2+\epsilon_0\omega_e^2||J_y||^2\right]+||\sigma_x^{\frac{1}{2}}E_y||^2=-(\partial_x(H^x+H^y),E_y). \tag{2.19}$$

(II) Multiplying (2.1e) and (2.1f) by $\mu_0(H^x+H^y)$ and $\mu_0\omega_m^2(K^x+K^y)$, respectively, integrating each over Ω, then adding up the results, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\mu_0||H^x||^2+\mu_0\omega_m^2||K^x||^2\right]+\mu_0(\partial_t H^x,H^y)+\mu_0\omega_m^2(\partial_t K^x,K^y)\\&\quad+\mu_0\omega_m^2[(K^x,H^y)-(K^y,H^x)]+||\sigma_y^{\frac{1}{2}}H^x||^2+(\sigma_y H^x,H^y)\\&=(\partial_y E_x,H^x+H^y).\end{aligned} \tag{2.20}$$

Similarly, multiplying (2.1g) and (2.1h) by $\mu_0(H^x+H^y)$ and $\mu_0\omega_m^2(K^x+K^y)$, respectively, integrating each over Ω, then adding up the results, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\mu_0||H^y||^2+\mu_0\omega_m^2||K^y||^2\right]+\mu_0(\partial_t H^y,H^x)+\mu_0\omega_m^2(\partial_t K^y,K^x)\\&\quad+\mu_0\omega_m^2[(K^y,H^x)-(K^x,H^y)]+||\sigma_x^{\frac{1}{2}}H^y||^2+(\sigma_x H^x,H^y)\\&=-(\partial_x E_y,H^x+H^y).\end{aligned} \tag{2.21}$$

Adding up (2.20) and (2.21), and using the identity

$$\frac{d}{dt}(||H^x||^2+||H^y||^2)+2(\partial_t H^x,H^y)+2(\partial_t H^y,H^x)=\frac{d}{dt}(||H^x+H^y||^2),$$

we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\mu_0||H^x+H^y||^2+\mu_0\omega_m^2||K^x+K^y||^2\right]\\&\quad+||\sigma_y^{\frac{1}{2}}H^x||^2+||\sigma_x^{\frac{1}{2}}H^y||^2+((\sigma_x+\sigma_y)H^x,H^y)\\&=(\partial_y E_x,H^x+H^y)-(\partial_x E_y,H^x+H^y)\\&=-(E_x,\partial_y(H^x+H^y))+(E_y,\partial_x(H^x+H^y)),\end{aligned} \tag{2.22}$$

where in the last step we used integration by parts and the PEC boundary condition (2.14).

(III) Now adding up (2.18), (2.19) and (2.22), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\epsilon_0(||E_x||^2+||E_y||^2)+\mu_0\omega_e^2(||J_x||^2+||J_y||^2)\\&\quad+\mu_0||H^x+H^y||^2+\mu_0\omega_m^2||K^x+K^y||^2\Big]+||\sigma_y^{\frac{1}{2}}E_x||^2+||\sigma_x^{\frac{1}{2}}E_y||^2\\&\quad+||\sigma_y^{\frac{1}{2}}H^x||^2+||\sigma_x^{\frac{1}{2}}H^y||^2+((\sigma_x+\sigma_y)H^x,H^y)=0,\end{aligned} \tag{2.23}$$

which concludes the proof of (2.16).

When $\sigma_x=\sigma_y=\sigma$ is a positive constant, using the identity

$$||\sigma_y^{\frac{1}{2}}H^x||^2+||\sigma_x^{\frac{1}{2}}H^y||^2+((\sigma_x+\sigma_y)H^x,H^y)=||\sigma^{\frac{1}{2}}(H^x+H^y)||^2$$

in (2.23) and using the energy definition (2.15), we immediately have

$$\frac{d\mathcal{E}}{dt}(t)\leq 0,$$

which completes the proof of (2.17). □

We like to remark that for general σ_x and σ_y, we can use the following identity

$$\begin{aligned}&||\sigma_y^{\frac{1}{2}}H^x||^2+||\sigma_x^{\frac{1}{2}}H^y||^2+((\sigma_x+\sigma_y)H^x,H^y)\\=&\frac{1}{2}||(\sigma_x+\sigma_y)^{\frac{1}{2}}(H^x+H^y)||^2+\frac{1}{2}((\sigma_y-\sigma_x)H^x,H^x)+\frac{1}{2}((\sigma_x-\sigma_y)H^y,H^y).\end{aligned} \tag{2.24}$$

But the last two terms of (2.24) are not guaranteed to be nonnegative, i.e., how to obtain a nice stability of the model is still open in the general case.

3 The FDTD scheme and its analysis

To develop our difference scheme, we assume that the physical domain $\Omega=[a,b]\times[c,d]$ is partitioned by a uniform rectangular grid

$$a=x_0<x_1<\cdots<x_{N_x}=b,\quad c=y_0<y_1<\cdots<y_{N_y}=d,$$

and the time interval $[0,T]$ is partitioned into N_t uniform intervals, i.e., we have discrete times $t_k=k\tau$, $\tau=\frac{T}{N_t}$, $k=0,1,\cdots,N_t$, grid points $x_i=ih_x$, $h_x=\frac{b-a}{N_x}$, $i=0,1,\cdots,N_x$

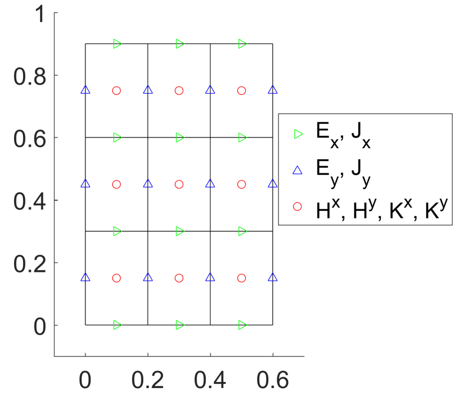

Figure 1: A sample grid showing how the unknown variables are placed.

in the x-direction, and grid points $y_j=jh_y$, $h_y=\frac{d-c}{N_y}$, $j=0,1,\cdots,N_y$ in the y-direction. Note that h_x and h_y can be different.

Following the classic Yee scheme, we choose the unknowns H_x (and K_x) at the mid-points of the horizontal edges, H_y (and K_y) at the mid-points of the vertical edges, and E_z (and J_z) at the element centers (cf. Fig. 1).

To define the fully-discrete scheme, we introduce the following difference and averaging operators: For any discrete function $u^n_{i,j}$,

$$\delta_\tau u^{n+\frac{1}{2}}_{i,j}:=\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}, \qquad \overline{u}^n_{i,j}=\frac{u^{n+\frac{1}{2}}_{i,j}+u^{n-\frac{1}{2}}_{i,j}}{2},$$
$$\delta_x u^n_{i,j}:=\frac{u^n_{i+\frac{1}{2},j}-u^n_{i-\frac{1}{2},j}}{h_x}, \qquad \delta_y u^n_{i,j}:=\frac{u^n_{i,j+\frac{1}{2}}-u^n_{i,j-\frac{1}{2}}}{h_y}.$$

We can develop the following FDTD scheme for solving the system of (2.1a)-(2.1h):

$$\delta_\tau E^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}+\omega_e^2 J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}+\epsilon_0^{-1}\sigma_{y,j}\overline{E}^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}=\epsilon_0^{-1}\delta_y(H^x+H^y)^{n+\frac{1}{2}}_{i+\frac{1}{2},j}, \tag{3.1a}$$
$$\delta_\tau J^n_{x,i+\frac{1}{2},j}-E^n_{x,i+\frac{1}{2},j}=0, \tag{3.1b}$$
$$\delta_\tau E^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}+\omega_e^2 J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}+\epsilon_0^{-1}\sigma_{x,i}\overline{E}^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}=-\epsilon_0^{-1}\delta_x(H^x+H^y)^{n+\frac{1}{2}}_{i,j+\frac{1}{2}}, \tag{3.1c}$$
$$\delta_\tau J^n_{y,i,j+\frac{1}{2}}-E^n_{y,i,j+\frac{1}{2}}=0, \tag{3.1d}$$
$$\delta_\tau H^{x,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\omega_m^2 K^{x,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\mu_0^{-1}\sigma_{y,j+\frac{1}{2}}\overline{H}^{x,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}=\mu_0^{-1}\delta_y E^{n+1}_{x,i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.1e}$$
$$\delta_\tau K^{x,n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}-H^{x,n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}=0, \tag{3.1f}$$

$$\delta_\tau H^{y,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\omega_m^2 K^{y,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\mu_0^{-1}\sigma_{x,i+\frac{1}{2}}\overline{H}^{y,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}=-\mu_0^{-1}\delta_x E^{n+1}_{y,i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.1g}$$

$$\delta_\tau K^{y,n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}-H^{y,n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}=0, \tag{3.1h}$$

where we denote

$$\sigma_{y,j}=\sigma_y(y_j) \quad \text{and} \quad E^n_{x,i+\frac{1}{2},j}\approx E_x(x_{i+\frac{1}{2}},y_j,t_n),$$

i.e., the approximate solution of E_x at point $(x_{i+\frac{1}{2}},y_j,t_n)$. Similar notations are used for other variables.

The given initial conditions (2.12)-(2.13) can be discretized as follows:

$$E^0_{x,i+\frac{1}{2},j}=E_{x0}(x_{i+\frac{1}{2}},y_j), \qquad E^0_{y,i,j+\frac{1}{2}}=E_{y0}(x_i,y_{j+\frac{1}{2}}), \tag{3.2a}$$

$$\overline{J}^0_{x,i+\frac{1}{2},j}=J_{x0}(x_{i+\frac{1}{2}},y_j), \qquad \overline{J}^0_{y,i,j+\frac{1}{2}}=J_{y0}(x_i,y_{j+\frac{1}{2}}), \tag{3.2b}$$

$$\overline{H}^{x,0}_{i+\frac{1}{2},j+\frac{1}{2}}=H_{x0}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}}), \qquad \overline{H}^{y,0}_{i+\frac{1}{2},j+\frac{1}{2}}=H_{y0}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}}), \tag{3.2c}$$

$$K^{x,0}_{i+\frac{1}{2},j+\frac{1}{2}}=K_{x0}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}}), \qquad K^{y,0}_{i+\frac{1}{2},j+\frac{1}{2}}=K_{y0}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}}). \tag{3.2d}$$

The scheme (3.1a)-(3.1h) can be implemented as follows.

When $n=0$, we need to couple the discretized initial conditions (3.2a)-(3.2d) with the scheme (3.1a)-(3.1h). For example, using (3.2a), (3.2b), and (3.1b) with $n=0$, we have

$$J^{\frac{1}{2}}_{i+\frac{1}{2},j}=J_{x0}(x_{i+\frac{1}{2}},y_j)+\frac{\tau}{2}E_{x0}(x_{i+\frac{1}{2}},y_j).$$

Then at every time step,

Step 1: Solve (3.1b) for $J^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j}$, (3.1d) for $J^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}$, (3.1f) for $K^{x,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}$, and (3.1h) for $K^{y,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}$. Note that these can be done in parallel.

Step 2: Solve (3.1a) for $E^{n+1}_{x,i+\frac{1}{2},j}$, (3.1c) for $E^{n+1}_{y,i,j+\frac{1}{2}}$, (3.1e) for $H^{x,n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$, and (3.1g) for $H^{y,n+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$. Note that these can be done in parallel.

In the rest of this section, we will carry out the discrete energy analysis of the scheme (3.1a)-(3.1h). To simplify the notation, we denote the discrete L^2 norm and the corresponding discrete inner product as follows:

$$||E^n_x||^2_*:=h_x h_y\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}|E^n_{x,i+\frac{1}{2},j}|^2, \quad \langle u^n,v^n\rangle=h_x h_y\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}u^n_{x,i+\frac{1}{2},j}v^n_{x,i+\frac{1}{2},j}.$$

Similar notations will be used for other variables, which have different shifts for the indexes. We also introduce the notation $c_v=\frac{1}{\sqrt{\epsilon_0\mu_0}}$ for the wave propagation speed in the free space.

Theorem 3.1. *For the solution of* (3.1a)-(3.1h), *define the discrete energy*

$$\begin{aligned}\mathcal{E}_{dis}(m):=&\frac{\epsilon_0}{2}(||E_x^{m+1}||_*^2+||E_y^{m+1}||_*^2)+\frac{\epsilon_0\omega_e^2}{2}(||J_x^{m+\frac{1}{2}}||_*^2+||J_y^{m+\frac{1}{2}}||_*^2)\\&+\frac{\mu_0}{2}||(H^x+H^y)^{m+\frac{3}{2}}||_*^2+\frac{\mu_0\omega_m^2}{2}||(K^x+K^y)^{m+1}||_*^2.\end{aligned}\tag{3.3}$$

Then for any $m\in[1,N_t-1]$ and nonnegative damping functions $\sigma_x(x)$ and $\sigma_y(y)$, we have the discrete energy identity:

$$\begin{aligned}&\frac{1}{\tau}(\mathcal{E}_{dis}(m)-\mathcal{E}_{dis}(-1))+\sum_{0\le n\le m}(||\sigma_y^{\frac{1}{2}}\overline{E}_x^{n+\frac{1}{2}}||_*^2+||\sigma_x^{\frac{1}{2}}\overline{E}_y^{n+\frac{1}{2}}||_*^2)\\&\quad+\sum_{0\le n\le m}\Big[||\sigma_y^{\frac{1}{2}}\overline{H}^{x,n+1}||_*^2+||\sigma_x^{\frac{1}{2}}\overline{H}^{y,n+1}||_*^2+\langle(\sigma_x+\sigma_y)\overline{H}^{x,n+1},\overline{H}^{y,n+1}\rangle\Big]\\&\quad+\frac{\mu_0\omega_m^2}{2}\Big[\langle(K^x+K^y)^{m+1},(H^x+H^y)^{m+\frac{3}{2}}\rangle-\langle(K^x+K^y)^0,(H^x+H^y)^{\frac{1}{2}}\rangle\Big]\\&\quad+\frac{\epsilon_0\omega_e^2}{2}\Big[\langle J_x^{m+\frac{1}{2}},E_x^{m+1}\rangle-\langle J_x^{-\frac{1}{2}},E_x^0\rangle+\langle J_y^{m+\frac{1}{2}},E_y^{m+1}\rangle-\langle J_y^{-\frac{1}{2}},E_y^0\rangle\Big]\\&=\frac{1}{2}\Big[\langle(H^x+H^y)^{m+\frac{3}{2}},\delta_yE_x^{m+1}-\delta_xE_y^{m+1}\rangle-\langle(H^x+H^y)^{\frac{1}{2}},\delta_yE_x^0-\delta_xE_y^0\rangle\Big].\end{aligned}\tag{3.4}$$

When $\sigma_x=\sigma_y=\sigma$ is a positive constant, under the time step constraint

$$\tau\le\min\Big(\frac{1}{\sqrt{2}\omega_e},\frac{1}{\sqrt{2}\omega_m},\frac{h_x}{8\sqrt{2}c_v},\frac{h_y}{8\sqrt{2}c_v}\Big),\tag{3.5}$$

the following discrete stability for the scheme (3.1a)-(3.1h) *holds true:*

$$\mathcal{E}_{dis}(m)\le 3\mathcal{E}_{dis}(-1).\tag{3.6}$$

Proof. To make our proof easy to follow, we divide the proof into several major parts.

(I) Multiplying (3.1a) by $h_xh_y\epsilon_0\overline{E}_{x,i+\frac{1}{2},j}^{n+\frac{1}{2}}$, then summing up over i and j, we have

$$\begin{aligned}&\frac{\epsilon_0}{2\tau}(||E_x^{n+1}||_*^2-||E_x^n||_*^2)+||\sigma_y^{\frac{1}{2}}\overline{E}_x^{n+\frac{1}{2}}||_*^2+\epsilon_0\omega_e^2\langle J_x^{n+\frac{1}{2}},\overline{E}_x^{n+\frac{1}{2}}\rangle\\&=\langle\delta_y(H^x+H^y)^{n+\frac{1}{2}},\overline{E}_x^{n+\frac{1}{2}}\rangle.\end{aligned}\tag{3.7}$$

Multiplying (3.1b) by $\epsilon_0\omega_e^2 h_x h_y \overline{J}^n_{x,i+\frac{1}{2},j}$, then summing up over i and j, we have

$$\frac{\epsilon_0\omega_e^2}{2\tau}(||J_x^{n+\frac{1}{2}}||_*^2-||J_x^{n-\frac{1}{2}}||_*^2)-\epsilon_0\omega_e^2\langle E_x^n,\overline{J}_x^n\rangle=0. \tag{3.8}$$

Adding (3.7) and (3.8) together, then summing up the result from $n=0$ to any $m\le N_t-1$, and using the following identity

$$\langle J_x^{n+\frac{1}{2}},\overline{E}_x^{n+\frac{1}{2}}\rangle-\langle E_x^n,\overline{J}_x^n\rangle=\frac{1}{2}(\langle J_x^{n+\frac{1}{2}},E_x^{n+1}\rangle-\langle J_x^{n-\frac{1}{2}},E_x^n\rangle),$$

we obtain

$$\begin{aligned}&\frac{\epsilon_0}{2\tau}(||E_x^{m+1}||_*^2-||E_x^0||_*^2)+\sum_{0\le n\le m}||\sigma_y^{\frac{1}{2}}\overline{E}_x^{n+\frac{1}{2}}||_*^2+\frac{\epsilon_0\omega_e^2}{2\tau}(||J_x^{m+\frac{1}{2}}||_*^2-||J_x^{-\frac{1}{2}}||_*^2)\\&\quad+\frac{\epsilon_0\omega_e^2}{2}(\langle J_x^{m+\frac{1}{2}},E_x^{m+1}\rangle-\langle J_x^{-\frac{1}{2}},E_x^0\rangle)\\&=\sum_{0\le n\le m}\langle\delta_y(H^x+H^y)^{n+\frac{1}{2}},\overline{E}_x^{n+\frac{1}{2}}\rangle.\end{aligned}\tag{3.9}$$

Similarly, multiplying (3.1c) by $h_x h_y\epsilon_0\overline{E}^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2}}$, multiplying (3.1d) by $\epsilon_0\omega_e^2 h_x h_y\overline{J}^n_{y,i,j+\frac{1}{2}}$, and summing up the results from $n=0$ to any $m\le N_t-1$, we obtain

$$\begin{aligned}&\frac{\epsilon_0}{2\tau}(||E_y^{m+1}||_*^2-||E_y^0||_*^2)+\sum_{0\le n\le m}||\sigma_x^{\frac{1}{2}}\overline{E}_y^{n+\frac{1}{2}}||_*^2+\frac{\epsilon_0\omega_e^2}{2\tau}(||J_y^{m+\frac{1}{2}}||_*^2-||J_x^{-\frac{1}{2}}||_*^2)\\&\quad+\frac{\epsilon_0\omega_e^2}{2}(\langle J_y^{m+\frac{1}{2}},E_y^{m+1}\rangle-\langle J_y^{-\frac{1}{2}},E_y^0\rangle)\\&=-\sum_{0\le n\le m}\langle\delta_x(H^x+H^y)^{n+\frac{1}{2}},\overline{E}_y^{n+\frac{1}{2}}\rangle.\end{aligned}\tag{3.10}$$

Adding (3.9) and (3.10) together, we have

$$\begin{aligned}&\frac{\epsilon_0}{2\tau}\left[(||E_x^{m+1}||_*^2+||E_y^{m+1}||_*^2)-(||E_x^0||_*^2+||E_y^0||_*^2)\right]+\sum_{0\le n\le m}(||\sigma_y^{\frac{1}{2}}\overline{E}_x^{n+\frac{1}{2}}||_*^2+||\sigma_x^{\frac{1}{2}}\overline{E}_y^{n+\frac{1}{2}}||_*^2)\\&\quad+\frac{\epsilon_0\omega_e^2}{2\tau}\left[(||J_x^{m+\frac{1}{2}}||_*^2+||J_y^{m+\frac{1}{2}}||_*^2)-(||J_x^{-\frac{1}{2}}||_*^2+||J_y^{-\frac{1}{2}}||_*^2)\right]\\&\quad+\frac{\epsilon_0\omega_e^2}{2}\left[(\langle J_x^{m+\frac{1}{2}},E_x^{m+1}\rangle-\langle J_x^{-\frac{1}{2}},E_x^0\rangle)+(\langle J_y^{m+\frac{1}{2}},E_y^{m+1}\rangle-\langle J_y^{-\frac{1}{2}},E_y^0\rangle)\right]\\&=\sum_{0\le n\le m}\left[\langle\delta_y(H^x+H^y)^{n+\frac{1}{2}},\overline{E}_x^{n+\frac{1}{2}}\rangle-\langle\delta_x(H^x+H^y)^{n+\frac{1}{2}},\overline{E}_y^{n+\frac{1}{2}}\rangle\right].\end{aligned}\tag{3.11}$$

(II) Multiplying (3.1e) by $h_xh_y\mu_0(\overline{H}^{x,n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\overline{H}^{y,n+1}_{i+\frac{1}{2},j+\frac{1}{2}})$, then summing up over i and j, we have

$$\begin{aligned}&\frac{\mu_0}{2\tau}(||H^{x,n+\frac{3}{2}}||_*^2-||H^{x,n+\frac{1}{2}}||_*^2)+\mu_0\langle\delta_\tau H^{x,n+1},\overline{H}^{y,n+1}\rangle\\&\quad+\mu_0\omega_m^2\langle K^{x,n+1},\overline{H}^{x,n+1}+\overline{H}^{y,n+1}\rangle+||\sigma_y^{\frac{1}{2}}\overline{H}^{x,n+1}||_*^2+\langle\sigma_y\overline{H}^{x,n+1},\overline{H}^{y,n+1}\rangle\\&=\langle\delta_yE_x^{n+1},\overline{H}^{x,n+1}+\overline{H}^{y,n+1}\rangle.\end{aligned}\tag{3.12}$$

Multiplying (3.1f) by $h_xh_y\mu_0\omega_m^2(\overline{K}^{x,n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}+\overline{K}^{y,n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}})$, then summing up over i and j, we have

$$\begin{aligned}&\frac{\mu_0\omega_m^2}{2\tau}(||K^{x,n+1}||_*^2-||K^{x,n}||_*^2)+\mu_0\omega_m^2\langle\delta_\tau K^{x,n+\frac{1}{2}},\overline{K}^{y,n+\frac{1}{2}}\rangle\\&\quad-\mu_0\omega_m^2\langle H^{x,n+\frac{1}{2}},\overline{K}^{x,n+\frac{1}{2}}+\overline{K}^{y,n+\frac{1}{2}}\rangle=0.\end{aligned}\tag{3.13}$$

Adding (3.12) and (3.13) together, then summing up the result from $n=0$ to any $m\le N_t-1$, and using the following identity

$$\langle K^{x,n+1},\overline{H}^{x,n+1}\rangle-\langle H^{x,n+\frac{1}{2}},\overline{K}^{x,n+\frac{1}{2}}\rangle=\frac{1}{2}(\langle K^{x,n+1},H^{x,n+\frac{3}{2}}\rangle-\langle K^{x,n},H^{x,n+\frac{1}{2}}\rangle),$$

we have

$$\begin{aligned}&\frac{\mu_0}{2\tau}(||H^{x,m+\frac{3}{2}}||_*^2-||H^{x,\frac{1}{2}}||_*^2)+\sum_{0\le n\le m}\mu_0\langle\delta_\tau H^{x,n+1},\overline{H}^{y,n+1}\rangle+\sum_{0\le n\le m}||\sigma_y^{\frac{1}{2}}\overline{H}^{x,n+1}||_*^2\\&\quad+\frac{\mu_0\omega_m^2}{2}(\langle K^{x,m+1},H^{x,m+\frac{3}{2}}\rangle-\langle K^{x,0},H^{x,\frac{1}{2}}\rangle)+\sum_{0\le n\le m}\mu_0\langle\sigma_y\overline{H}^{x,n+1},\overline{H}^{y,n+1}\rangle\\&\quad+\frac{\mu_0\omega_m^2}{2\tau}(||K^{x,m+1}||_*^2-||K^{x,0}||_*^2)+\sum_{0\le n\le m}\mu_0\omega_m^2\langle\delta_\tau K^{x,n+\frac{1}{2}},\overline{K}^{y,n+\frac{1}{2}}\rangle\\&\quad+\sum_{0\le n\le m}\mu_0\omega_m^2(\langle K^{x,n+1},\overline{H}^{y,n+1}\rangle-\langle H^{x,n+\frac{1}{2}},\overline{K}^{y,n+\frac{1}{2}}\rangle)\\&=\sum_{0\le n\le m}\langle\delta_yE_x^{n+1},\overline{H}^{x,n+1}+\overline{H}^{y,n+1}\rangle.\end{aligned}\tag{3.14}$$

By symmetry, from (3.1g) and (3.1h), we have

$$\begin{aligned}&\frac{\mu_0}{2\tau}(||H^{y,m+\frac{3}{2}}||_*^2-||H^{y,\frac{1}{2}}||_*^2)+\sum_{0\le n\le m}\mu_0\langle\delta_\tau H^{y,n+1},\overline{H}^{x,n+1}\rangle+\sum_{0\le n\le m}||\sigma_x^{\frac{1}{2}}\overline{H}^{y,n+1}||_*^2\\&\quad+\frac{\mu_0\omega_m^2}{2}(\langle K^{y,m+1},H^{y,m+\frac{3}{2}}\rangle-\langle K^{y,0},H^{y,\frac{1}{2}}\rangle)+\sum_{0\le n\le m}\mu_0\langle\sigma_x\overline{H}^{y,n+1},\overline{H}^{x,n+1}\rangle\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu_0\omega_m^2}{2\tau}(||K^{y,m+1}||_*^2-||K^{y,0}||_*^2)+\sum_{0\le n\le m}\mu_0\omega_m^2\langle\delta_\tau K^{y,n+\frac{1}{2}},\overline{K}^{x,n+\frac{1}{2}}\rangle\\
&+\sum_{0\le n\le m}\mu_0\omega_m^2(\langle K^{y,n+1},\overline{H}^{x,n+1}\rangle-\langle H^{y,n+\frac{1}{2}},\overline{K}^{x,n+\frac{1}{2}}\rangle)\\
=&-\sum_{0\le n\le m}\langle\delta_x E_y^{n+1},\overline{H}^{x,n+1}+\overline{H}^{y,n+1}\rangle. \qquad (3.15)
\end{aligned}
$$

Adding (3.14) and (3.15), and using the following identities:

$$
\begin{aligned}
&\sum_{0\le n\le m}(\langle\delta_\tau H^{x,n+1},\overline{H}^{y,n+1}\rangle+\langle\delta_\tau H^{y,n+1},\overline{H}^{x,n+1}\rangle)\\
&=\frac{1}{\tau}\sum_{0\le n\le m}(\langle H^{x,n+\frac{3}{2}},H^{y,n+\frac{3}{2}}\rangle-\langle H^{x,n+\frac{1}{2}},H^{y,n+\frac{1}{2}}\rangle)\\
&=\frac{1}{\tau}(\langle H^{x,m+\frac{3}{2}},H^{y,m+\frac{3}{2}}\rangle-\langle H^{x,\frac{1}{2}},H^{y,\frac{1}{2}}\rangle), \qquad (3.16a)\\
&\sum_{0\le n\le m}\Big[(\langle K^{x,n+1},\overline{H}^{y,n+1}\rangle-\langle H^{x,n+\frac{1}{2}},\overline{K}^{y,n+\frac{1}{2}}\rangle)+(\langle K^{y,n+1},\overline{H}^{x,n+1}\rangle-\langle H^{y,n+\frac{1}{2}},\overline{K}^{x,n+\frac{1}{2}}\rangle)\Big]\\
&=\frac{1}{2}\sum_{0\le n\le m}\Big[(\langle K^{x,n+1},H^{y,n+\frac{3}{2}}\rangle-\langle K^{x,n},H^{y,n+\frac{1}{2}}\rangle)+(\langle K^{y,n+1},H^{x,n+\frac{3}{2}}\rangle-\langle K^{y,n},H^{x,n+\frac{1}{2}}\rangle)\Big]\\
&=\frac{1}{2}\Big[(\langle K^{x,m+1},H^{y,m+\frac{3}{2}}\rangle-\langle K^{x,0},H^{y,\frac{1}{2}}\rangle)+(\langle K^{y,m+1},H^{x,m+\frac{3}{2}}\rangle-\langle K^{y,0},H^{x,\frac{1}{2}}\rangle)\Big], \qquad (3.16b)\\
&\sum_{0\le n\le m}(\langle\delta_\tau K^{x,n+\frac{1}{2}},\overline{K}^{y,n+\frac{1}{2}}\rangle+\langle\delta_\tau K^{y,n+\frac{1}{2}},\overline{K}^{x,n+\frac{1}{2}}\rangle)\\
&=\frac{1}{\tau}\sum_{0\le n\le m}(\langle K^{x,n+1},K^{y,n+1}\rangle-\langle K^{x,n},H^{y,n}\rangle)\\
&=\frac{1}{\tau}(\langle K^{x,m+1},K^{y,m+1}\rangle-\langle K^{x,0},K^{y,0}\rangle), \qquad (3.16c)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{\mu_0}{2\tau}(||H^{x,m+\frac{3}{2}}+H^{y,m+\frac{3}{2}}||_*^2-||H^{x,\frac{1}{2}}+H^{y,\frac{1}{2}}||_*^2)\\
&\quad+\frac{\mu_0\omega_m^2}{2\tau}(||K^{x,m+1}+K^{y,m+1}||_*^2-||K^{x,0}+K^{y,0}||_*^2)\\
&\quad+\frac{\mu_0\omega_m^2}{2}\Big[\langle K^{x,m+1}+K^{y,m+1},H^{x,m+\frac{3}{2}}+H^{y,m+\frac{3}{2}}\rangle-\langle K^{x,0}+K^{y,0},H^{x,\frac{1}{2}}+H^{y,\frac{1}{2}}\rangle\Big]\\
&\quad+\sum_{0\le n\le m}\Big[||\sigma_y^{\frac{1}{2}}\overline{H}^{x,n+1}||_*^2+||\sigma_x^{\frac{1}{2}}\overline{H}^{y,n+1}||_*^2+\langle(\sigma_x+\sigma_y)\overline{H}^{x,n+1},\overline{H}^{y,n+1}\rangle\Big]\\
&=\sum_{0\le n\le m}\Big[\langle\delta_y E_x^{n+1},\overline{H}^{x,n+1}+\overline{H}^{y,n+1}\rangle-\langle\delta_x E_y^{n+1},\overline{H}^{x,n+1}+\overline{H}^{y,n+1}\rangle\Big]. \qquad (3.17)
\end{aligned}
$$

(III) To add up the right hand side (RHS) terms of (3.11) and (3.17), we need the following estimate:

$$
\begin{aligned}
&\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1\\0\le n\le m}}\Big[\delta_y(H^x+H^y)^{n+\frac12}_{i+\frac12,j}\cdot\overline{E}^{n+\frac12}_{x,i+\frac12,j}+\delta_y E^{n+1}_{x,i+\frac12,j+\frac12}\cdot(\overline{H}^{x,n+1}_{i+\frac12,j+\frac12}+\overline{H}^{y,n+1}_{i+\frac12,j+\frac12})\Big]\\
=&\frac{1}{2h_y}\sum_{\substack{0\le i\le N_x-1\\0\le n\le m}}\Big[(H^x+H^y)^{n+\frac12}_{i+\frac12,-\frac12}E^{n+1}_{x,i+\frac12,0}-(H^x+H^y)^{n+\frac12}_{i+\frac12,N_y-\frac12}E^{n+1}_{x,i+\frac12,N_y}\Big]\\
&+\frac{1}{2h_y}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}\Big[(H^x+H^y)^{\frac12}_{i+\frac12,j+\frac12}E^{0}_{x,i+\frac12,j}-(H^x+H^y)^{m+\frac32}_{i+\frac12,j+\frac12}E^{m+1}_{x,i+\frac12,j}\Big]\\
&+\frac{1}{2h_y}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}\Big[(H^x+H^y)^{m+\frac32}_{i+\frac12,j+\frac12}E^{m+1}_{x,i+\frac12,j+1}-(H^x+H^y)^{\frac12}_{i+\frac12,j+\frac12}E^{0}_{x,i+\frac12,j+1}\Big]\\
&+\frac{1}{2h_y}\sum_{\substack{0\le i\le N_x-1\\0\le n\le m}}\Big[(H^x+H^y)^{n+\frac12}_{i+\frac12,N_y-\frac12}E^{n}_{x,i+\frac12,N_y}-(H^x+H^y)^{n+\frac12}_{i+\frac12,-\frac12}E^{n}_{x,i+\frac12,0}\Big]\\
=&\frac{1}{2}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}\Big[(H^x+H^y)^{m+\frac32}_{i+\frac12,j+\frac12}\delta_yE^{m+1}_{x,i+\frac12,j+\frac12}-(H^x+H^y)^{\frac12}_{i+\frac12,j+\frac12}\delta_yE^{0}_{x,i+\frac12,j+\frac12}\Big],
\end{aligned}\tag{3.18}
$$

where in the last step we used the PEC boundary conditions (2.14). Similarly, we have

$$
\begin{aligned}
&\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1\\0\le n\le m}}\Big[\delta_x(H^x+H^y)^{n+\frac12}_{i,j+\frac12}\cdot\overline{E}^{n+\frac12}_{y,i,j+\frac12}+\delta_x E^{n+1}_{y,i+\frac12,j+\frac12}\cdot(\overline{H}^{x,n+1}_{i+\frac12,j+\frac12}+\overline{H}^{y,n+1}_{i+\frac12,j+\frac12})\Big]\\
=&\frac{1}{2h_x}\sum_{\substack{0\le j\le N_y-1\\0\le n\le m}}\Big[(H^x+H^y)^{n+\frac12}_{N_x-\frac12,j+\frac12}E^{n+1}_{y,N_x,j+\frac12}-(H^x+H^y)^{n+\frac12}_{-\frac12,j+\frac12}E^{n+1}_{y,0,j+\frac12}\Big]\\
&+\frac{1}{2h_x}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}\Big[(H^x+H^y)^{\frac12}_{i+\frac12,j+\frac12}E^{0}_{y,i,j+\frac12}-(H^x+H^y)^{m+\frac32}_{i+\frac12,j+\frac12}E^{m+1}_{y,i,j+\frac12}\Big]\\
&+\frac{1}{2h_x}\sum_{\substack{0\le i\le N_x-1\\0\le j\le N_y-1}}\Big[(H^x+H^y)^{m+\frac32}_{i+\frac12,j+\frac12}E^{m+1}_{y,i+1,j+\frac12}-(H^x+H^y)^{\frac12}_{i+\frac12,j+\frac12}E^{0}_{y,i+1,j+\frac12}\Big]\\
&+\frac{1}{2h_x}\sum_{\substack{0\le j\le N_y-1\\0\le n\le m}}\Big[(H^x+H^y)^{n+\frac12}_{N_x-\frac12,j+\frac12}E^{n}_{y,N_x,j+\frac12}-(H^x+H^y)^{n+\frac12}_{-\frac12,j+\frac12}E^{0}_{y,0,j+\frac12}\Big]
\end{aligned}
$$

$$=\frac{1}{2}\sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}}\Big[(H^x+H^y)^{m+\frac{3}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\delta_x E^{m+1}_{y,i+\frac{1}{2},j+\frac{1}{2}}-(H^x+H^y)^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}\delta_x E^{0}_{y,i+\frac{1}{2},j+\frac{1}{2}}\Big], \quad (3.19)$$

where in the last step we used the PEC boundary conditions (2.14).

Now adding up (3.11) and (3.17), and using (3.18), (3.19) and the discrete energy definition (3.3), we complete the proof for the discrete energy identity (3.4).

(IV) When $\sigma_x=\sigma_y=\sigma$ is a positive constant, using the identity

$$\begin{aligned}&||\sigma_y^{\frac{1}{2}}\overline{H}^{x,n+1}||_*^2+||\sigma_x^{\frac{1}{2}}\overline{H}^{y,n+1}||_*^2+\langle(\sigma_x+\sigma_y)\overline{H}^{x,n+1},\overline{H}^{y,n+1}\rangle\\ =&||\sigma^{\frac{1}{2}}(\overline{H}^{x,n+1}+\overline{H}^{y,n+1})||_*^2,\end{aligned}$$

and dropping the second and third terms on the left hand side of (3.4), we have

$$\begin{aligned}&\mathcal{E}_{dis}(m)-\mathcal{E}_{dis}(-1)\\ \le&-\frac{\tau\mu_0\omega_m^2}{2}\Big[\langle(K^x+K^y)^{m+1},(H^x+H^y)^{m+\frac{3}{2}}\rangle-\langle(K^x+K^y)^0,(H^x+H^y)^{\frac{1}{2}}\rangle\Big]\\ &-\frac{\tau\epsilon_0\omega_e^2}{2}\Big[\langle J_x^{m+\frac{1}{2}},E_x^{m+1}\rangle-\langle J_x^{-\frac{1}{2}},E_x^0\rangle+\langle J_y^{m+\frac{1}{2}},E_y^{m+1}\rangle-\langle J_y^{-\frac{1}{2}},E_y^0\rangle\Big]\\ &+\frac{\tau}{2}\Big[\langle(H^x+H^y)^{m+\frac{3}{2}},\delta_yE_x^{m+1}-\delta_xE_y^{m+1}\rangle-\langle(H^x+H^y)^{\frac{1}{2}},\delta_yE_x^0-\delta_xE_y^0\rangle\Big]. \quad (3.20)\end{aligned}$$

Now we just need to estimate those right hand side terms of (3.20). By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{\tau\mu_0\omega_m^2}{2}\langle(K^x+K^y)^{m+1},(H^x+H^y)^{m+\frac{3}{2}}\rangle\\ &\le\frac{\mu_0\omega_m^2}{4}||(K^x+K^y)^{m+1}||_*^2+\frac{(\tau\omega_m)^2}{4}\cdot\mu_0||(H^x+H^y)^{m+\frac{3}{2}}||_*^2, && (3.21a)\\ &\frac{\tau\epsilon_0\omega_e^2}{2}\langle J_x^{m+\frac{1}{2}},E_x^{m+1}\rangle\le\frac{\epsilon_0\omega_e^2}{4}||J_x^{m+\frac{1}{2}}||_*^2+\frac{(\tau\omega_e)^2}{4}\cdot\epsilon_0||E_x^{m+1}||_*^2, && (3.21b)\\ &\frac{\tau\epsilon_0\omega_e^2}{2}\langle J_y^{m+\frac{1}{2}},E_y^{m+1}\rangle\le\frac{\epsilon_0\omega_e^2}{4}||J_y^{m+\frac{1}{2}}||_*^2+\frac{(\tau\omega_e)^2}{4}\cdot\epsilon_0||E_y^{m+1}||_*^2. && (3.21c)\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\frac{\tau}{2}\langle(H^x+H^y)^{m+\frac{3}{2}},\delta_yE_x^{m+1}-\delta_xE_y^{m+1}\rangle\\ =&\langle\sqrt{\mu_0}(H^x+H^y)^{m+\frac{3}{2}},\frac{\tau c_v\sqrt{\epsilon_0}}{2}(\delta_yE_x^{m+1}-\delta_xE_y^{m+1})\rangle\\ \le&\frac{\mu_0}{8}||(H^x+H^y)^{m+\frac{3}{2}}||_*^2+4(\tau c_v)^2\epsilon_0(||\delta_yE_x^{m+1}||_*^2+||\delta_xE_y^{m+1}||_*^2)\\ \le&\frac{\mu_0}{8}||(H^x+H^y)^{m+\frac{3}{2}}||_*^2+\frac{16(\tau c_v)^2}{h_y^2}\cdot\epsilon_0||E_x^{m+1}||_*^2+\frac{16(\tau c_v)^2}{h_x^2}\cdot\epsilon_0||E_y^{m+1}||_*^2, \quad (3.22)\end{aligned}$$

where in the last step we used the following estimates

$$\begin{aligned}||\delta_y E_x^{m+1}||_*^2 &= h_x h_y \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} \frac{|E^{m+1}_{x,i+\frac12,j+1}-E^{m+1}_{x,i+\frac12,j}|^2}{h_y^2}\\ &\le \frac{2}{h_y^2}\cdot h_x h_y \sum_{\substack{0\le i\le N_x-1\\ 0\le j\le N_y-1}} (|E^{m+1}_{x,i+\frac12,j+1}|^2+|E^{m+1}_{x,i+\frac12,j}|^2) = \frac{4}{h_y^2}||E_x^{m+1}||_*^2,\end{aligned}$$

and

$$||\delta_x E_y^{m+1}||_*^2 \le \frac{4}{h_x^2}||E_y^{m+1}||_*^2.$$

Substituting (3.21a)-(3.22) and similar estimates for the rest terms into (3.20), we obtain

$$\begin{aligned}&\mathcal{E}_{dis}(m)-\mathcal{E}_{dis}(-1)\\ \le&\Big(\frac{(\tau\omega_e)^2}{2}+\frac{32(\tau c_v)^2}{h_y^2}\Big)\cdot\frac{\epsilon_0}{2}||E_x^{m+1}||_*^2+\Big(\frac{(\tau\omega_e)^2}{2}+\frac{32(\tau c_v)^2}{h_x^2}\Big)\cdot\frac{\epsilon_0}{2}||E_y^{m+1}||_*^2\\ &+\frac{\epsilon_0\omega_e^2}{4}(||J_x^{m+\frac12}||_*^2+||J_x^{m+\frac12}||_*^2)+\Big(\frac{(\tau\omega_m)^2}{2}+\frac{\tau c_v}{2}\Big)\frac{\mu_0}{2}||(H^x+H^y)^{m+\frac32}||_*^2\\ &+\frac{\mu_0\omega_m^2}{4}||(K^x+K^y)^{m+1}||_*^2+\Big(\frac{(\tau\omega_e)^2}{2}+\frac{32(\tau c_v)^2}{h_y^2}\Big)\cdot\frac{\epsilon_0}{2}||E_x^0||_*^2\\ &+\Big(\frac{(\tau\omega_e)^2}{2}+\frac{32(\tau c_v)^2}{h_x^2}\Big)\cdot\frac{\epsilon_0}{2}||E_y^0||_*^2+\frac{\epsilon_0\omega_e^2}{4}(||J_x^{-\frac12}||_*^2+||J_x^{-\frac12}||_*^2)\\ &+\Big(\frac{(\tau\omega_m)^2}{2}+\frac14\Big)\cdot\frac{\mu_0}{2}||(H^x+H^y)^{\frac12}||_*^2+\frac{\mu_0\omega_m^2}{4}||(K^x+K^y)^0||_*^2.\end{aligned}\tag{3.23}$$

If we choose τ satisfying the following (which is equivalent to the time step constraint (3.5)):

$$\frac{(\tau\omega_m)^2}{2}\le\frac14,\quad \frac{(\tau\omega_e)^2}{2}\le\frac14,\quad \frac{32(\tau c_v)^2}{h_y^2}\le\frac14,\quad \frac{32(\tau c_v)^2}{h_x^2}\le\frac14,$$

then the estimate (3.23) can be simplified to

$$\frac12\mathcal{E}_{dis}(m)-\mathcal{E}_{dis}(-1)\le\frac12\mathcal{E}_{dis}(-1),\tag{3.24}$$

which concludes our proof of (3.6). □

4 Numerical results

In this section, we present some numerical results to demonstrate the wave absorbing efficiency of the PML model. We adopt the same examples of [5] and our simulation was carried out by MATLAB installed under Windows 10 on a Dell XPS Notebook (with Intel Core i5-1035G1 1.10GHz CPU and 8GB RAM).

4.1 Wave absorbing by the PML model

To test the effectiveness and stability of the PML model (2.1a)-(2.1h) on absorbing the outgoing waves, we simulate a source wave propagating in a Drude metamaterial of dimension $[-17,17]\times[-17,17]$. The Drude metamaterial region (governed by (2.1a)-(2.1h) with $\sigma_x=\sigma_y=0$) is surrounded by the PML with thickness $d=15h$, where h denotes the mesh size. The incident source wave is imposed in the H_x equation (2.1e) as a source function

$$f(x,y,t)=g(x,y)h(t),$$

where

$$g(x,y)=e^{-5(x^2+y^2)} \quad \text{and} \quad h(t)=-20(t-1)e^{-10(t-1)^2}.$$

In our simulation, we use $\epsilon_0=1$, $\mu_0=1$, $h_x=h_y=h=0.2$ and $\tau=0.1$. The damping function σ_x is a fourth-order polynomial given as follows:

$$\sigma_x(x)=\begin{cases} \sigma_{\max}\left(\dfrac{x-17}{d}\right)^m, & \text{if } x\geq 17, \\ \sigma_{\max}\left(\dfrac{|x+17|}{d}\right)^m, & \text{if } x\leq -17, \\ 0, & \text{elsewhere}, \end{cases} \tag{4.1}$$

where $\sigma_{\max}=-(m+1)\log(R)/(2d)$ with $R=10^{-6}$ and $m=4$. The damping function $\sigma_y(y)$ has exactly the same form as $\sigma_x(x)$. The H_z fields obtained by our scheme (3.1a)-(3.1h) at various time steps are presented in Fig. 2, which shows that both of forward and backward waves are well absorbed by the PML as observed in the paper [5, Fig. 10].

4.2 A refocusing simulation

In this example, we simulate a transmission problem between the vacuum and a Drude medium surrounded by Berenger's PML and the metamaterial PML respectively (see Fig. 3) with thickness $d=15h$ on all sides. In this simulation, we put

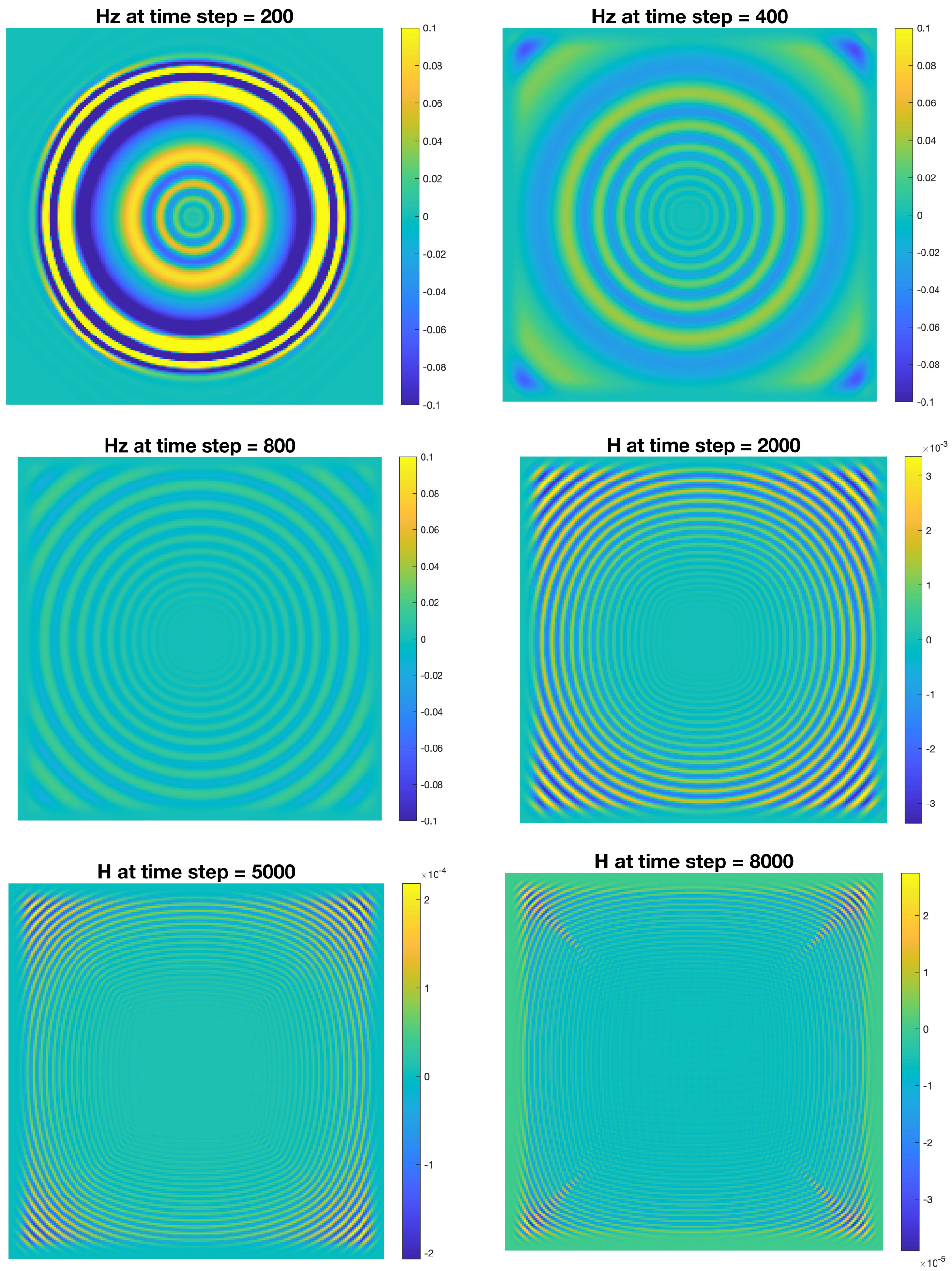

Figure 2: Snapshots of $H=H^x+H^y$ obtained by scheme (3.1a)-(3.1h) with $\tau=0.1$ at 200,400,800,2000, 5000, and 8000 time steps.

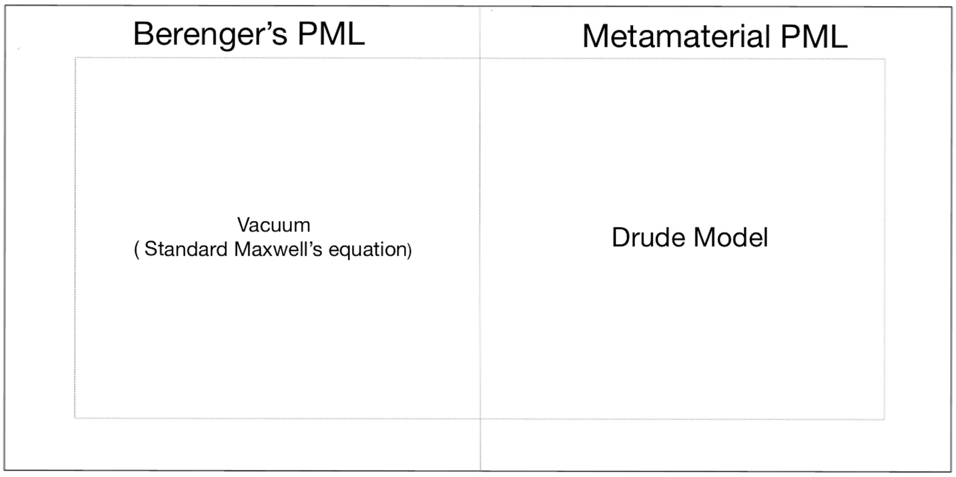

Figure 3: The setup of the refocusing simulation.

a periodic time source $h(t)=\sin(\omega_0 t)$ in the center of vacuum region, while the computational domain is a rectangle $\Omega=[-20,20]\times[0,20]$. To create a refocusing phenomenon, we choose parameters $\epsilon_0=\mu_0=1$, and $\omega_e=\omega_m=\sqrt{2}\omega_0$, which leads to $\epsilon(\omega_0)=\mu(\omega_0)=-1$ by the Drude model, i.e., the effective index of metamaterial is -1. We use the wave frequency $\omega_0=\sqrt{2}$, and the fourth-order damping functions σ_x and σ_y.

For this simulation, we need to solve a coupled problem with different governing equations in different subdomains: on the left subdomain Ω_1, the equations are governed by the 2D Berenger PML model (cf. [7, Eq. (3)] and [24, p. 219]); on the right subdomain Ω_2, the governing equations are the metamaterial PML model (2.1a)-(2.1h). We can unify these models together and rewrite them as follows:

$$\partial_t E_x + D_J J_x + \epsilon_0^{-1}\sigma_y E_x = \epsilon_0^{-1}\partial_y (H^x + H^y), \tag{4.2a}$$
$$D(\partial_t J_x - E_x) = 0, \tag{4.2b}$$
$$\partial_t E_y + D_J J_y + \epsilon_0^{-1}\sigma_x E_y = -\epsilon_0^{-1}\partial_x (H^x + H^y), \tag{4.2c}$$
$$D(\partial_t J_y - E_y) = 0, \tag{4.2d}$$
$$\partial_t H_x + D_K K_y + D_{H1} H^x = D_{E1}\partial_y E_x - D_{E2}\partial_x E_y, \tag{4.2e}$$
$$D(\partial_t K_x - H_x) = 0, \tag{4.2f}$$
$$\partial_t H_y + D_K K_y + D_{H2} H^y = D_{E2}\partial_y E_x - D_{E1}\partial_x E_y, \tag{4.2g}$$
$$D(\partial_t K_y - H_y) = 0, \tag{4.2h}$$

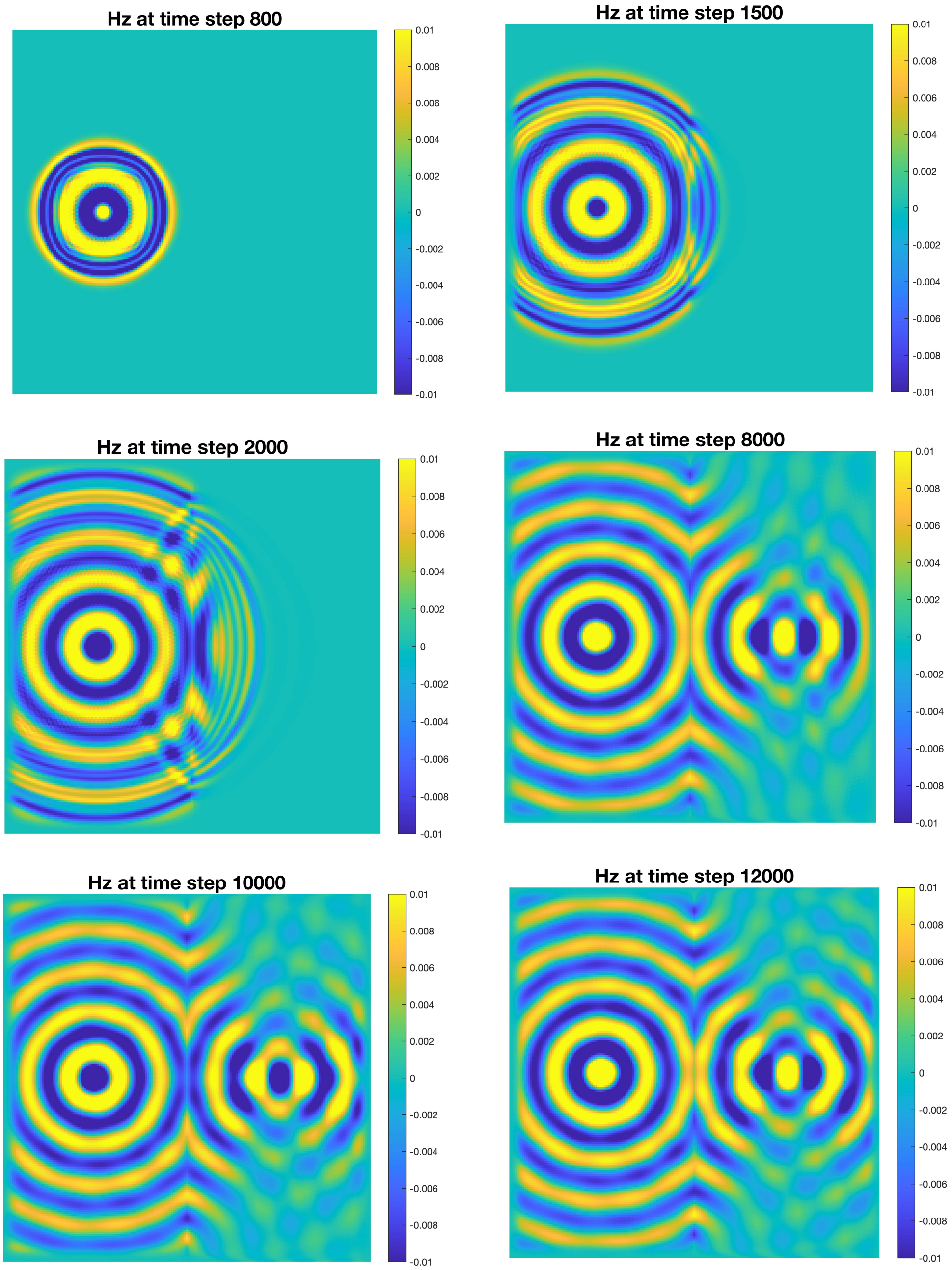

Figure 4: Snapshots of $H=H^x+H^y$ obtained by the scheme (3.1a)-(3.1h) with $\tau=0.01$ at 800,1500,2000,8000,10000,12000 time steps.

where the coefficients are defined as:

$$D=\begin{cases} 0, & (x,y)\in\Omega_1, \\ 1, & (x,y)\in\Omega_2, \end{cases} \quad D_J=\begin{cases} 0, & (x,y)\in\Omega_1, \\ \omega_e^2, & (x,y)\in\Omega_2, \end{cases} \quad D_K=\begin{cases} 0, & (x,y)\in\Omega_1, \\ \omega_m^2, & (x,y)\in\Omega_2, \end{cases}$$

$$D_{H1}=\begin{cases} \epsilon_0^{-1}\sigma_x, & (x,y)\in\Omega_1, \\ \mu_0^{-1}\sigma_y, & (x,y)\in\Omega_2, \end{cases} \quad D_{H2}=\begin{cases} \epsilon_0^{-1}\sigma_y, & (x,y)\in\Omega_1, \\ \mu_0^{-1}\sigma_x, & (x,y)\in\Omega_2, \end{cases}$$

$$D_{E1}=\begin{cases} 0, & (x,y)\in\Omega_1, \\ \mu_0^{-1}, & (x,y)\in\Omega_2, \end{cases} \quad D_{E2}=\begin{cases} \mu_0^{-1}, & (x,y)\in\Omega_1, \\ 0, & (x,y)\in\Omega_2. \end{cases}$$

The source wave $\sin(\omega_0 t)$ is imposed on the H^y field. We use the scheme (3.1a)-(3.1h) with $h_x=h_y=0.2$ (i.e.,$N_x=N_y=200$) and $\tau=0.01$ to obtain the snapshots of the H fields in Fig. 4, which shows a refocusing property as originally obtained in [5, Fig. 12].

5 Conclusions

In this paper, by using the energy method we established a stability for the metamaterial PML model developed by Bécache et al. [5]. This PML was originally proved to be stable through a complicated modal analysis in [6]. But the stability obtained by the energy method offers more practical use for proving numerical stability of the FDTD scheme. Currently, both stability proved in [6] and this paper are limited to the constant damping coefficient case. How to obtain a stability in the practical variable damping functions are still open. We will continue the investigation in the future. More advanced numerical methods such as discontinuous Galerkin methods [18, 25] and edge element methods [24, 29] for this Drude PML model will be explored in the future too.

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