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Multidimensional simple wave and Monge-Ampère equation: theory and applications

Boling Guo[1], Yueyang Feng [2] and Yaqing Liu[3,*]

[1] *Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China*
[2] *The Graduate School of China Academy of Engineering Physics, Beijing, 100088, China*
[3] *School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China*

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Abstract. This paper explores the properties and mathematical formulations of multidimensional simple waves, extending the well-established theory of one-dimensional simple waves to higher dimensions. The study focuses on the connection between simple waves and the Monge-Ampère equation, particularly in the context of gas dynamics and potential flows. Key aspects include the characterization of simple waves in unsteady and steady flows, the role of characteristic lines, and the application of Hodograph and Legendre transformations to derive solutions. The paper also addresses the challenges and open questions in extending simple wave theory to more complex systems, such as non-reducible systems, radiative heat transfer, and chemical reactions. The research highlights both theoretical advancements and practical applications, providing a foundation for future studies in this area.

AMS subject classifications: 35Q35, 35J96, 35L05

Key words: Simple wave; Monge-Ampère equation; Gas dynamics; Characteristic lines; Hodograph transformation.

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*Corresponding author.
Emails: `gbl@iapcm.ac.cn` (B. Guo), `fengyueyang23@gscaep.ac.cn` (Y. Feng), `liuyaqing@bistu.edu.cn`(Y. Liu)

1 Introduction

Simple waves, as a fundamental model of wave phenomena, have significant applications across multiple fields. In physics, they are used to study classical wave behaviors such as sound and light waves [1], providing a theoretical foundation for understanding wave propagation, reflection, and interference. In engineering, simple wave models are widely applied in seismic wave analysis, underwater sonar detection, and structural vibration control, helping optimize technical solutions and improve system stability [2]. Additionally, fields like meteorology (e.g., atmospheric wave studies) and medical imaging (e.g., ultrasound technology) rely on simple wave principles [3]. The importance of studying simple waves lies in their role as a starting point for analyzing more complex wave phenomena. By using simplified mathematical models, they reveal essential wave characteristics, offering key theoretical support for interdisciplinary technological advancements. A deep understanding of simple waves not only advances fundamental science but also provides practical tools for solving real-world engineering challenges.

The foundational theories developed for one-dimensional simple waves provide a crucial framework for exploring more complex scenarios [4], including multidimensional flows and their connections to nonlinear partial differential equations such as the Monge-Ampère equation. We begin by revisiting the basic properties of one-dimensional simple waves, which serve as the building blocks for extending these concepts to higher dimensions.

1.1 Basic properties of one-dimensional simple waves

In one-dimensional unsteady isentropic flow or planar unsteady flow, for a reducible system of two unknown functions u,v with two variables x and y the system is given by:

$$A_1\frac{\partial u}{\partial x}+B_1\frac{\partial u}{\partial y}+C_1\frac{\partial v}{\partial x}+D_1\frac{\partial v}{\partial y}=0, \tag{1.1}$$

$$A_2\frac{\partial u}{\partial x}+B_2\frac{\partial u}{\partial y}+C_2\frac{\partial v}{\partial x}+D_2\frac{\partial v}{\partial y}=0, \tag{1.2}$$

where A_i,B_i,C_i,D_i are functions of u,v. For one-dimensional unsteady flow, simply replace x with t, y with x. If $J=\frac{\partial(u,v)}{\partial(x,y)}=0$, that is u,v are related. Functions $u=u(x,y)$, $v=v(x,y)$ map a region $(x,y)\in G$ to a family of curves. The solutions or flows of such equations are known as simple waves. These waves arise in many practical mechanical models, including planar detonations, two-dimensional steady flows, and one-dimensional isentropic flows, among others. This property has an intuitive physical interpretation.

(1) The two families of characteristic lines in the region (x,y) of Eqs. (1.1) and (1.2) are

$$c_+ : \frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_1(u,v),\ c_- : \frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_2(u,v), \tag{1.3}$$

where $\lambda_i(u,v)$ is the distinct real roots of $\det\begin{vmatrix} B_1 - A_1\lambda & D_1 - C_1\lambda \\ B_2 - A_2\lambda & D_2 - C_2\lambda \end{vmatrix} = 0$. The two families of characteristic curves corresponding in the (u,v)-plane are

$$\Gamma_+ : \phi(u,v) = \text{const},\ \Gamma_- : \psi(u,v) = \text{const}, \tag{1.4}$$

where ϕ and ψ are Riemann invariants. They are integrals of the characteristic relation

$$T\frac{\mathrm{d}u}{\mathrm{d}v} = S - a\lambda_i\ (i = 1,2), \tag{1.5}$$

where $S = [DA]$, $T = [AB]$, $a = [CA]$ and $[xy] = x_1y_2 - x_2y_1$.

In the case of simple waves, the image of the characteristic lines reduces to a curve. Consequently, one family of characteristic lines maps onto an arc of either τ_+ or τ_-, while the other family collapses to a single point. As a result, u and v remain constant along this latter family of characteristic lines, which must therefore be straight, i.e., $\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda(u,v) = \text{const}$.

For example, the one-dimensional unsteady flow

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{c^2}{\rho}\rho_x = 0, \\ S_t + uS_x = 0, \end{cases} \tag{1.6}$$

where u denoting the velocity, ρ is density, $c^2 = (\frac{\partial P}{\partial \rho})_{S=\text{constant}}$, c is the acoustic speed, $P = P(\rho,S)$ is pressure and S is entropy.

The characteristic determinant is as follows,

$$\begin{vmatrix} u - \frac{\mathrm{d}x}{\mathrm{d}t} & \rho & 0 \\ \frac{c^2}{\rho} & u - \frac{\mathrm{d}x}{\mathrm{d}t} & 0 \\ 0 & 0 & u - \frac{\mathrm{d}x}{\mathrm{d}t} \end{vmatrix} = 0.$$

The characteristic lines are

$$c_+ : \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_1 = u + c, \tag{1.7}$$

$$c_- : \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_2 = u - c. \tag{1.8}$$

The streamlines are

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\lambda_3=u, \tag{1.9}$$

$$\text{along} \quad c_+,\ \Gamma_+: u+l=\phi(u,\rho)=\text{const}, \tag{1.10}$$

$$\text{along} \quad c_-,\ \Gamma_-: u-l=\psi(u,\rho)=\text{const}, \tag{1.11}$$

where $l=\int_{\rho_0}^{\rho}\frac{c}{\rho}\mathrm{d}\rho$. For ideal gas, $\int\frac{c}{\rho}\mathrm{d}\rho=\frac{2}{\gamma-1}c$, where $\gamma>1$ is a constant. ϕ and ψ are called Riemann invariants, sometimes $\gamma=\frac{\varphi}{2}$, $-S=\frac{\psi}{2}$ also called Riemann invariants. For a polytropic gas, the characteristic lines and streamlines of one-dimensional unsteady flow can be solved analytically. The characteristic lines is c_-: $t=\frac{1}{2}e^{-2\mu^{-2}}\int e^{2\mu^{-2}-1}\xi(\beta)\mathrm{d}\beta+C$ and the streamline is $t=-e^{-2\mu^{-2}}\int e^{\mu^{-2}-1}\xi(\beta)\mathrm{d}\beta+C$, where $\mu^2=\frac{\gamma-1}{\gamma+1}$. Thus, $\xi(\beta)$ is the parameter of the point through which the straight characteristic line $x=\xi+(u+c)t,(\xi(\beta),u=u(\beta),c=c(\beta))$ passes.

In the case of planar steady potential flow, its basic equation is

$$\begin{cases} u_y-v_x=0, \\ (c^2-u^2)u_x-uv(v_x+u_y)+(c^2-v^2)v_y=0. \end{cases} \tag{1.12}$$

Its characteristic lines are $c_+:\frac{\mathrm{d}y}{\mathrm{d}x}=\rho_+(u,v)$ and $c_-:\frac{\mathrm{d}y}{\mathrm{d}x}=\rho_-(u,v)$, where $\rho_++\rho_-=-\frac{2uv}{c^2-u^2}$, $\rho_+\rho_-=\frac{c^2-v^2}{c^2-u^2}$. The corresponding characteristic relations are

$$\Gamma_+:\ \frac{\mathrm{d}u}{\mathrm{d}v}=-\rho_-,\ \Gamma_-:\ \frac{\mathrm{d}u}{\mathrm{d}v}=-\rho_+. \tag{1.13}$$

In the case of a simple wave, c_+ is the point characteristic line and its equations can be written as

$$x=a(\sigma)-r\sin\omega(r), \tag{1.14}$$

$$y=b(\sigma)+r\cos\omega(r), \tag{1.15}$$

where a, b, ω are arbitrary parameters of σ and r is coordinate along the line $\sigma=const$, ω is the angle between the straight characteristic line and the positive y-axis. Along c_+, u, v are constants, c_- is referred to as the line crossing the Mach line. The angle A between the streamline and the c_+ characteristic line is called the Mach angle. For polytropic gas flow, the streamlines and characteristic lines can be solved analytically.

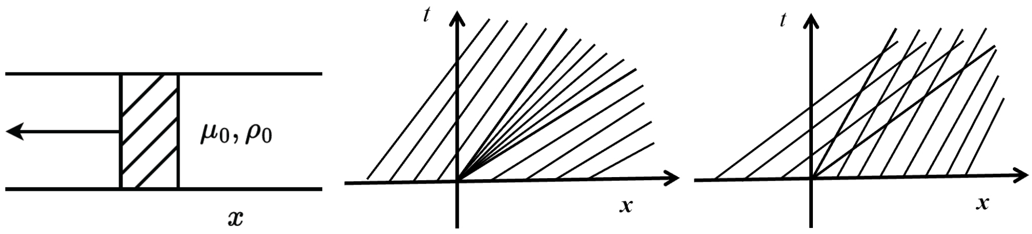
The streamline is $r=R\cos^{-\mu^{-2}}(\omega-\omega_*)$, where R is an appropriate constant.

The crossing Mach line is $r=R\cos^{-\frac{1}{2}\mu^2}\mu(\omega-\omega_*)\sin^{-\frac{1}{2}}\mu(\omega-\omega_*)$, thus, $\mu^2=\frac{\gamma-1}{\gamma+1}$, ω_* are arbitrary constants. In planar steady flow, the images of c^+,c^- characteristic lines in the (u,v)-plane correspond to epicycloid of a sonic circle $u^2+v^*=c_*^2$ and a circle $u^2+v^2=\frac{1}{\mu^2}c_*^2$. Especially for simple waves, there is only one epicycloid.

(2) Another important property of simple waves is that any region adjacent to a constant state must itself be a simple wave region. In other words, a weak discontinuity solution connected to a simple wave state must also be a simple wave solution.

(3) In the entire region, either $\phi(u,v)$, $\psi(u,v)$, or r,s must remain constant. As a result, all physical quantities are constant along the straight characteristic lines. In this sense, this property can also serve as a definition of simple waves.

(4) Central simple waves exist [5], where the straight characteristic line c^+ of the simple wave originates from a single point at a specific time. Simple waves appear in many mechanical systems–a classic example is the gas flow induced by a moving piston. Consider an initial state with uniform velocity $u_0=$const, $\rho_0=$const (Fig. 1(a)). When the piston is gradually withdrawn, the characteristic lines remain straight in the static region, while the c^+ characteristics diverge in the simple wave region. If the piston is suddenly pulled back, a central simple wave (rarefaction wave) forms (Fig. 1(b)). Conversely, if the piston is abruptly pushed forward, a compression wave arises (Fig. 1(c)), leading to a discontinuous solution. The above situations are shown in the following figures.



(a) (b) (c)

Figure 1: (a) Piston. (b) Simple wave. (c) Compression wave.

1.2 The extension of one-dimensional simple waves

The study of one-dimensional simple waves naturally leads to the question of whether multidimensional simple waves exist, for instance, in two-dimensional unsteady flows. If such solutions exist, do they share the same properties as their one-dimensional unsteady or two-dimensional steady counterparts? These questions have garnered significant attention in the field. Partial answers have emerged through both the observation of physical phenomena and rigorous mathematical analysis. It should be particularly noted that N.N. Yaneko [6,7] and D. Naylor [8,9] have conducted systematic, meticulous, and in-depth research and have achieved good results.

First, we give a clear definition of multidimensional simple waves, which should be a mathematical extension of the definition of one-dimensional simple waves. Suppose that there is a system of equations $A_{ijk}(u_1,\cdots,u_m)\frac{\partial u_j}{\partial x_k}=0$, $(i,j,k=1,2,\cdots m)$. If $\frac{\partial(u_1,\cdots,u_m)}{\partial(x_1,\cdots,x_m)}\neq 0$, by applying the hodograph transformation, the system can be converted into a linear set of equations. If the determinant $\frac{\partial(u_1,\cdots,u_m)}{\partial(x_1,\cdots,x_m)}=0$ and its rank is r, it is called a simple wave of rank r. In other words, the functions $u_1,\cdots,u_m$ are related and must satisfy $m-r$ independent functional relationships $\phi_\alpha(u_1,\cdots,u_m)=0$, $(\alpha=1,\cdots,m-r)$. D. Naylor [8] studied the simple wave problem of two-dimensional unsteady flow (in gases), that is, rank $r=2$ and $r=1$ $(i,j,k=3)$, $\mathbf{q}=\mathbf{q}(\lambda,t)$, $a=a(\lambda,t)$, $\lambda=\lambda(x,y,t)$. In [9], he studied the simple wave problem of three-dimensional unsteady flow, that is, $\mathbf{q}=\mathbf{q}(\lambda,\mu)$, $a=a(\lambda,\mu,t)$, $\lambda=\lambda(x,y,t)$, $\mu=\mu(x,y,z,t)$ and conical flow $\mathbf{q}=\mathbf{q}(\mathbf{r}/t)$. Here, it is particularly noteworthy that $a=a(\lambda,\mu,t)$ is related to the independent variable t. For $A_{ijk}(u_1,\cdots,u_m)\frac{\partial u_j}{\partial x_k}$, $i,j=1,\cdots,m$, $k=1,\cdots,n$, $n<m$, when $n=2$, $u_t+A(u)u_x=0$, P.D. Lax [10] obtained a positive conclusion. Other Cauchy problems for simple waves-including their geometric interpretation, applications to detonation waves, and hodograph mapping–have also been extensively studied, yielding significant theoretical and practical results.

In this paper, we extend the classical theory of one-dimensional simple waves to multidimensional settings. We first introduce a rigorous mathematical definition of multidimensional simple waves and establish their fundamental connection with the Monge-Ampère equation. For two- and three-dimensional unsteady potential flows of ranks 2 and 3, respectively, we derive explicit simple-wave solutions by reducing the governing nonlinear systems to solvable Monge-Ampère-type equations via Legendre and Hodograph transformations. The existence of genuinely unsteady central simple waves is also examined. We further develop a Cauchy-integral approach to obtain general integral representations of simple waves in unsteady potential flows, accompanied by detailed analyses of their envelopes, singularities, and physical admissibility. The proposed framework is then applied to construct self-similar solutions for two-dimensional detonation waves propagating around a corner. Finally, we discuss extensions to more complex systems involving radiation, chemical reactions, and non-reducible structures.

2 Main work and research method

Simple waves of the form

$$a_{ijk}(u_1,\cdots,u_m)\frac{\partial u_j}{\partial x_k}=0, \tag{2.1}$$

with rank $r=m-1$ were studied in [4]. These solutions satisfy $u_m=\phi(u_1,u_2,\cdots,u_{m-1})$, implying that the functions $u_\alpha(x_1,\cdots,x_m)$, $(\alpha=1,\cdots,m)$, are constant along characteristic curves. These curves are defined by the system

$$\frac{\mathrm{d}x_i}{\Delta_i}=\frac{\mathrm{d}x_m}{1},i=1,2,\cdots,m-1, \tag{2.2}$$

along which any function $f=f(u_1,\cdots,u_m)$ remains constant. Thus, the relation $\frac{\partial u_i}{\partial x_k}=-\Delta_k\frac{\partial u_i}{\partial x_m}$ holds. Substituting this into Eq. (2.1), we obtain

$$L_i=A_{i\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}=0,\ (i=1,2,\cdots,m,\ \alpha,\beta=1,\cdots,m-1), \tag{2.3}$$

where $A_{i\alpha\beta}=a_{i\alpha\beta}-a_{i\alpha m}\Delta_\beta+a_{im\beta}\varphi_\alpha-a_{imm}\varphi_\alpha\Delta_\beta$. Since Eq. (2.3) holds identically for all x_m, it follows that $\frac{\partial L_i}{\partial x_m}=0$. From these relations, we derive the operator identity $\delta L_i=0$, where $\delta=\frac{\partial}{\partial x_m}+\Delta_r\frac{\partial}{\partial r}$, which expands to $\delta L_i=A^{(1)}_{i\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}=0$, with coefficients given by $A^{(1)}_{i\alpha\beta}=\delta A_{i\alpha\beta}-A_{i\alpha r}\frac{\partial\Delta_\beta}{\partial r}$. This immediately implies the vanishing of higher-order derivatives: $\frac{\partial^s L_i}{\partial x_m^s}=0$ $(s=2,3,\cdots)$ and

$$A^{(s)}_{i\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}=0, \tag{2.4}$$

where $A^{(s)}_{i\alpha\beta}=\delta A^{(s-1)}_{i\alpha\beta}-A^{(s-1)}_{i\alpha r}\frac{\partial\Delta_\beta}{\partial x_r}$.

For Eq. (2.4), the functional dependence of Δ_r determines the equation's character: When Δ_r depends explicitly on $(x_1,\cdots,x_m)$, the equation is quasi-linear in u_α and x_α. If $\Delta_r=\Delta_r(u_1,u_2,\cdots,u_{m-1},x_1,x_2,\cdots,x_m)$, the equation becomes an $(s+1)$-order nonlinear equation in $\frac{\partial u_\alpha}{\partial x_\beta}$. For $\Delta_r=\Delta_r(u_\alpha)$ is only, the equation reduces to a generalized quasi-linear form in u_α. The system (2.4) is compatible because $A^{(0)}_{i\alpha\beta}=A_{i\alpha\beta}$, which ensures it encompasses the simple wave solutions of (2.4). The general solution of (2.3) requires additional constraints on ϕ_α and Δ_α–specifically, appropriate boundary conditions–to determine a unique solution.

2.1 Two-Dimensional Unsteady Potential Flow and Simple Waves

The simple wave solution for two-dimensional unsteady isentropic potential flow (a rank-2 case) was explicitly investigated in [7]. The equations are

$$\begin{cases} \rho(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k})+\frac{\partial P}{\partial x_i}=0, \\ \frac{\partial \theta}{\partial t}+u_k\frac{\partial \theta}{\partial x_k}+(\gamma-1)\theta\frac{\partial u_k}{\partial x_k}=0, \end{cases} \tag{2.5}$$

where $(\gamma-1)\theta=c^2=\frac{\partial P}{\partial \rho}$, $P=\frac{c^2}{\gamma}\rho^\gamma$, and γ is the adiabatic index.

Let $\theta=\theta(u_1,u_2)=\varphi(u_1,u_2)$, substituting it into (2.5), we obtain

$$\begin{cases} \frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}=0, \\ [(\gamma-1)\varphi-\varphi_1^2]\frac{\partial u_1}{\partial x_1}-\varphi_1\varphi_2(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1})+[(\gamma-1)\varphi-\varphi_2^2]\frac{\partial u_2}{\partial x_2}=0, \end{cases} \tag{2.6}$$

where $\varphi_\alpha=\Delta_\alpha-u_\alpha=const$, α=1,2, $\Delta_\alpha=\frac{\partial \Delta}{\partial u_\alpha}$, $\Delta=\varphi+\frac{u_1^2+u_2^2}{2}$. Then $L_s=a_{ij}^s\frac{\partial u_i}{\partial x_j},(i,j,s=1,2)$. Due to $\delta L_1=0$, we obtain the nonlinear equation about $\frac{\partial u_\alpha}{\partial x_\beta}$. From $\delta L_2=0$, we get $KL(\varphi)=0$. Thus, $K=\psi_{11}\psi_{22}-\psi_{12}^2$ and

$$L(\varphi)=(\varphi_{11}+1)[(\gamma-1)\varphi-\varphi_2^2]+2\varphi_1\varphi_2\varphi_{12}+[(\gamma-1)\varphi-\varphi_1^2](\varphi_{22}+1), \tag{2.7}$$

where $\frac{\partial \psi}{\partial x_\alpha}=u_\alpha$, $\varphi_{ij}=\frac{\partial^2\Theta}{\partial u_i\partial u_j}$.

Suppose that $K\neq 0$, the second-order quasi-linear equation of ϕ is obtained,

$$L(\varphi)=0. \tag{2.8}$$

If $K=0$, then $\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)}$, we get a simple wave of rank 1. According to this, we have $\delta^2L_2=0$, that is $KK_1L_2=0$, ($K_1=\Delta_{11}\Delta_{22}-\Delta_{12}^2$, where $\Delta_{kl}=\frac{\partial^2\Delta}{\partial u_k\partial u_l}$). Thus $\delta^sL_2=0$ $(s>1)$.

For the determined solution $\varphi(u_1,u_2)$ of (2.8), we obtain a rank-2 simple wave solution satisfying (2.3). Since $K\neq 0$, we apply the Hodograph transformation to (2.8)

$$\frac{\partial x_\alpha}{\partial u_\beta}=(-1)^{\alpha+\beta}\frac{1}{K}\frac{\partial u_{\xi-\alpha}}{\partial x_{\xi-\beta}},$$

for potential flow, this yields the symmetry condition $\frac{\partial x_1}{\partial u_2}=\frac{\partial x_2}{\partial u_1}$. Defining the potential functions $\chi_\alpha(u_1,u_2)=\frac{\partial x}{\partial u_\alpha}$ be the potential function, then it can be obtained that $\chi(u_1,u_2)$ satisfying

$$[(\gamma-1)\varphi-\varphi_1^2]\frac{\partial^2\chi}{\partial u_1^2}+2\varphi_1\varphi_2\frac{\partial^2\chi}{\partial u_1\partial u_2}+[(\gamma-1)\varphi-\varphi_2^2]\frac{\partial^2\chi}{\partial u_2^2}=0. \tag{2.9}$$

Given $\phi(u_1,u_2)$, (2.9) becomes a linear equation in u_α, from which the potential $\chi(u_1,u_2)$ can be determined. The solution $u_\alpha(x_1,x_2)$ is then obtained through inverse transformation, with the corresponding characteristic speed given by $\Delta=\varphi(U_1,U_2)+\frac{U_1^2+U_2^2}{2}$. On the plane $t=t_0$, each point (x_{10},x_{20}) admits a characteristic line described by

$$\frac{x_1-x_{10}}{\Delta_1[U_1(x_{10},x_{20}),U_2(x_{10},x_{20})]}=\frac{x_2-x_{20}}{\Delta_2(U_1,U_2)}=\frac{t-t_0}{1}.$$

Along the characteristic line passing through the point (x_{10},x_{20},t_0), the quantities $u_\alpha(x,t)=U_\alpha(x_{10},x_{20},t_0)$ and $\theta=\varphi(U_1,U_2)$ remain constant.

For rank-2 simple wave solutions in conical flow, all straight characteristic (mother) lines intersect at a common point in physical space (x_1,x_2,t). This geometric structure is analogous to the degenerate case of one-dimensional central simple waves. The conical flow condition is: $\frac{\partial\Delta_1}{\partial x_2}=\frac{\partial\Delta_2}{\partial x_1}=0$, $\frac{\partial\Delta_1}{\partial x_1}=\frac{\partial\Delta_2}{\partial x_2}=0$, that is, $\Delta_1=\Delta_1(x_1)$, $\Delta_2=\Delta_2(x_2)$, $\Delta_1'=\Delta_2'$. This condition should be compatible with (2.6). It can be obtained that $\Delta_{\alpha\beta}=0$, $(\alpha,\beta=1,2)$, and

$$\varphi=c_0+c_1u_1+c_2u_2-\frac{1}{2}(u_1^2+u_2^2).$$

Generally speaking, the solution $L(\varphi)=0$ does not satisfy the conical flow condition. However, if $\Delta_{\alpha\beta}\neq 0$, a solution to (2.9) exists for any φ.

Theorem 2.1. *If $\phi(u_1,u_2)$ satisfies* (2.8)*, then the corresponding simple wave solution is characterized by two arbitrary functions of a single independent variable, which are determined via integration of* (2.9).

Theorem 2.2. *When the potential function takes the form $\varphi=c_0+c_1u_1+c_2u_2-\frac{1}{2}(u_1^2+u_2^2)$, all traveling wave solutions satisfy conical flow conditions. For general solutions ϕ, the flows are typically non-conical; however, for any such ϕ there exists a solution X to Eq.* (2.9) *that defines a conical flow..*

2.2 Three-Dimensional Unsteady Potential Flow and Simple Waves

The simple waves of rank 3 and rank 2 of three-dimensional unsteady potential flow were mainly studied by use of Legendre transformation which reduces the number of independent variables by itself. The equations are

$$\begin{cases}\frac{\partial u_i}{\partial t}+\sum u_k\frac{\partial u_i}{\partial x_k}+\frac{\partial\theta}{\partial x_i}=0,\ (i=1,2,3)\\ \kappa\frac{\partial\theta}{\partial t}+\theta\sum\frac{\partial u_k}{\partial x_k}+\kappa\sum u_k\frac{\partial\theta}{\partial x_k}=0,\\ \mathrm{rot}\vec{u}=0,\end{cases}\tag{2.10}$$

where $\kappa=\frac{1}{\gamma-1}$, $\theta=c^2$, c is the acoustic speed. The potential flow has the Cauchy integral

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}(u_1^2+u_2^2+u_3^2)+k\theta=F(t), \tag{2.11}$$

where $F(t)$ is an arbitrary function, φ is the velocity potential, and $\frac{\partial\varphi}{\partial x_i}=u_i$ (i=1,2,3).

Introduce the potential function

$$\nabla((u_1,u_2,u_3,t))=\sum x_k u_k-\kappa t\theta-\varphi, \tag{2.12}$$

which establishes a correspondence between the coordinates (x_1,x_2,x_3,t) and (u_1,u_2,u_3,t). The differential of ∇ is given by

$$\mathrm{d}\nabla=\sum(x_k-\kappa t\theta_k)\mathrm{d}u_k-(\frac{\partial\varphi}{\partial t}+\kappa\theta)\mathrm{d}t,$$

from which we obtain the partial derivatives $\frac{\partial\nabla}{\partial u_i}=x_i-\kappa t\theta_i$ and $\frac{\partial\nabla}{\partial t}=-\frac{\partial\varphi}{\partial t}-k\theta$.

From equation (2.11), we obtain the partial derivative:

$$\frac{\partial\nabla}{\partial t}=\frac{1}{2}(u_1^2+u_2^2+u_3^2)-F(t).$$

Integration yields $\nabla=\frac{1}{2}(u_1^2+u_2^2+u_3^2)t+\Phi(u_1,u_2,u_3)+F^0(t)$, where $F^0(t)=-\int F(t)\mathrm{d}t$ and $\Phi(u_1,u_2,u_3)$ is an arbitrary function of integration. The function Φ can be expressed in the specific form $\Phi(u_1,u_2,u_3)=\frac{1}{2}(u_1^2+u_2^2+u_3^2)+\kappa\Pi(u_1,u_2,u_3)$, thus

$$x_i=\kappa\Pi_i+u_i+t(\kappa\theta_i+u_i),\ \Pi_i=\frac{\partial\Pi}{\partial u_i}. \tag{2.13}$$

Given that the Cauchy integral is satisfied, the Euler equations automatically hold. Consequently, only the continuity equation remains to be solved. By using $\frac{\partial\theta}{\partial t}=\sum\theta_k\frac{\partial u_k}{\partial t}$, $\frac{\partial\theta}{\partial x_i}=\sum\theta_k\frac{\partial u_k}{\partial x_i}$, we get

$$A_{ik}\frac{\partial u_k}{\partial x_k}=0, \tag{2.14}$$

where $A_{ik}=\delta_{ik}\theta-k^2\theta_i\theta_k$, $(i,k=1,2,3)$.

Under the assumption that the Jacobian determinant $\Delta=\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\neq 0$, we apply the Hodograph transformation to equation(2.14). Combining this with relation (2.13) yields the quadratic equation in time: $\Gamma_0+\Gamma_1 t+\Gamma_2 t^2=0$, where the coefficients Γ_0, Γ_1, and Γ_2 are determined by the transformation. For any t, we have $\Gamma_0=0$, $\Gamma_1=0$, $\Gamma_2=0$, where $\Gamma_j=\sum_{ik}A_{ik}L_{ik}^j$ $(j=0,1,2)$, L_{ik}^j is the function of Π_{ik}, θ_{ik}.

The condition $\Gamma_2=0$ yields a second-order linear partial differential equation that depends solely on θ. Furthermore, the function Π can be specifically chosen to simultaneously satisfy both $\Gamma_1=0$ and $\Gamma_0=0$. Consider the ansatz $\Pi=\theta+\sum c_k u_k+c_0$ (c_k,c_0 constants). The constants are uniquely determined as the solution to the system $\Gamma_0=0$, $\Gamma_1=0$. A rank-2 simple wave solution is obtained through the ansatz $u_3=\psi(u_1,u_2)$. Introducing the potential function $\nabla=\sum u_k x_k-\varphi$, and applying the Hodograph transformation $(u_1,u_2)\to(x_1,x_2)$, while treating x_3 as arbitrary, we derive the system of equations $\Gamma_i=0$, $(i=0,1,2)$, where $\Gamma_0,\Gamma_1,\Gamma_2$ are functions of u_1,u_2,t and

$$\Gamma_2=R_0\left[\frac{\partial^2\psi}{\partial u_1^2}\frac{\partial^2\psi}{\partial u_2^2}-\left(\frac{\partial^2\psi}{\partial u_1\partial u_2}\right)^2\right]=0, \tag{2.15}$$

where $R_0=F'(t)+\frac{\partial^2\nabla}{\partial t^2}$. Equation (2.15) yields two distinct cases: First case is $R_0=0$. The system reduces to $F(t)+\frac{\partial\nabla}{\partial t}=\lambda(u_1,u_2)$, with solution $\nabla=\lambda(u_1,u_2)t+\chi(u_1,u_2)-\int F(t)dt$, where λ,χ and ψ are arbitrary functions. Second case is degenerate condition. The relation $\frac{\partial^2\psi}{\partial u_1^2}\frac{\partial^2\psi}{\partial u_2^2}-\left(\frac{\partial^2\psi}{\partial u_1\partial u_2}\right)^2=0$ holds, which geometrically corresponds to a developable surface in the parameter space.For solutions beyond the cylindrical case $f(u_1,u_2)$=const, the potential $\nabla(u_1,u_2,t)$must satisfy $\Gamma_1=0$. To ensure compatibility, the function ψ can be chosen appropriately. For instance, the linear ansatz $\psi=\alpha_1u_1+\alpha_2u_2+\alpha_3$, automatically satisfies $\Gamma_1=0$ (since Γ_1 depends on second-order derivatives $\frac{\partial^2\psi}{\partial u_j^2}$, which vanish identically for this choice ψ with respect to u_j). In this case, the remaining condition $\Gamma_0=0$ must still be imposed. Let $\psi_1=\alpha_1$, $\psi_2=\alpha_2$ (where α_1,α_2 are constants). In this case, the characteristic curves degenerate into straight lines (i.e., the characteristic busbar becomes rectilinear). In [8], the authors investigated three-dimensional unsteady potential flows characterized by the velocity field $\mathbf{q}(\lambda,\mu)$, sound speed $c(\lambda,\mu,t)$, and phase variables $\lambda=\lambda(x,y,z,t)$, and $\mu=\mu(x,y,z,t)$. When the sound speed reduces to $a=a(\lambda,\mu)$, the flow is termed a double wave (satisfying$\Phi_{tt}=0$ as defined in [11]). Their analysis further addressed cases conical flows are absent, and the velocity field simplifies to $\mathbf{q}=\mathbf{q}(\lambda)$ (i.e., dependence on a single phase variable).

3 Monge-Ampère Equations and Their Applications in Simple Waves

The basic equations are

$$\text{the continuity equation: } c^2\text{div}\mathbf{q}=q\mathbf{q}\cdot\text{grad}q+2qq_t+\phi_{tt}, \tag{3.1}$$

$$\text{the motion equation: } \frac{q^2}{2}+\frac{c^2}{\gamma-1}+\Phi_t=\frac{C^2}{2}, \tag{3.2}$$

where C is a constant, $c=\left(\frac{\partial P}{\partial \rho}\right)^{\frac{1}{2}}$ is the acoustic speed, where Φ is the Legendre transformation of ϕ, defined as

$$\Phi=\phi-\mathbf{r}\cdot\mathbf{q}, \tag{3.3}$$

$\Phi(\lambda,\mu,t)\leftrightarrow\phi(x,y,z,t)$, grad$\phi=\mathbf{q}$, $\mathbf{q}=\mathbf{q}(x,\mu)$. Calculating total differential of Φ, it can be obtained that

$$\begin{cases}\Phi_\lambda+\mathbf{r}\cdot\mathbf{q}_\lambda=0,\\ \Phi_\mu++\mathbf{r}\cdot\mathbf{q}_\mu=0,\\ \Phi_t=\phi_t,\end{cases} \tag{3.4}$$

where $\mathbf{q}(\lambda,\mu)$ is a curve with two parameters. When $\lambda=const$, $\mu=const$, the velocity field reduces to $\mathbf{q}=const$. By differentiating equation (3.4), we derive explicit expressions for the spatial gradients gradλ, gradμ, and the time derivatives λ_t, μ_t in terms of the quantities $\mathbf{q}_{\alpha\beta}$, $\mathbf{q}_\alpha$, $\Phi_{\alpha t}$, $\Phi_{\alpha\beta}$. Substituting these into (3.1), the governing equation can be rewritten as

$$\begin{aligned}&(a^2\mathbf{q}_\lambda-qq_\lambda\mathbf{q})\cdot\text{grad}\lambda+(c^2\mathbf{q}_\mu-qq_\mu\mathbf{q})\cdot\text{grad}\mu\\ &=\Phi_{tt}+(\Phi_{\lambda t}+2qq_\lambda)\lambda_t+(\Phi_{\mu t}+2qq_\mu)\mu_t.\end{aligned} \tag{3.5}$$

Using the derived expressions for $\nabla\lambda$, $\nabla\mu$, λ_t and μ_t, we obtain a system of nonlinear partial differential equations for the velocity gradients $\mathbf{q}_{\alpha\beta}$, where the coefficients are given by the second derivatives $\Phi_{\alpha\beta}$. Consider the position vector parametrization $\mathbf{r}=\boldsymbol{\rho}(\lambda,\mu,t)+\theta\boldsymbol{n}(\lambda,\mu)$, where $\boldsymbol{\rho}$ satisfies the compatibility conditions $\Phi_\lambda+\boldsymbol{\rho}\cdot\mathbf{q}_\lambda=0$ and $\Phi_\mu+\boldsymbol{\rho}\cdot\mathbf{q}_\mu=0$. The scalar θ is governed by $\theta\boldsymbol{n}\cdot\mathbf{q}=\phi-\Phi-\rho\mathbf{q}$. The normal vector $\mathbf{n}$ is defined via $\mathbf{n}=\mathbf{q}_\lambda\times\mathbf{q}_\mu$, $\mathbf{n}\cdot\mathbf{q}\neq 0$. The vectors $\mathbf{q},\mathbf{q}_\lambda$, and $\mathbf{q}_\mu$ are necessarily coplanar. Substitution of these relations into Eq. (3.5) yields a quadratic expression in the variable θ. Under the assumptions that a^2, Φ, $\mathbf{q}$ are independent of θ. This condition yields three distinct relationships connecting Φ and q. From the term of θ^2, we can obtain $\Phi_{tt}[(\boldsymbol{n}\cdot\mathbf{q}_{\lambda\lambda})(\boldsymbol{n}\cdot\mathbf{q}_{\mu\mu})-(\mathbf{n}\cdot\mathbf{q}_{\lambda\mu})^2]=0$.

When $\Phi_{tt}\neq 0$, then $\mathbf{q}=$const is a developable surface. Especially let $\lambda=u$, $\mu=v$, $\omega=\omega(u,v)$, from (3.5), three Monge-Ampére equations about $\omega(u,v)$, $\Phi(u,v,t)$ can be obtained

$$\Phi_{tt}(\omega_{uu}\omega_{vv}-\omega_{uv}^2)=0. \tag{3.6}$$

$$\begin{aligned}&a^2[(1+\omega_u^2)\omega_{vv}-2\omega_u\omega_v\omega_{vu}+(1+\omega_v^2)\omega_{uu}]\\ &=-\Phi_{tt}[\omega_{uu}\Phi_{vv}-2\omega_{uv}\Phi_{uv}+\omega_{vv}\Phi_{uu}]+\omega_{uu}(\Phi_{vt}+qq_v)^2\\ &\quad-2\omega_{uv}(\Phi_{ut}+qq_u)(\Phi_{vt}+qq_v)+\omega_{vv}(\Phi_{ut}+qq_u)^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&a^2[(1+\omega_u^2)\Phi_{vv}-2\omega_u\omega_v\Phi_{vu}+(1+\omega_v^2)\Phi_{uu}]=\Phi_{tt}(\Phi_{uv}^2-\Phi_{uu}\Phi_{vv})\\ &+\Phi_{uv}(\Phi_{vt}+qq_v)^2-2\Phi_{uv}(\Phi_{ut}+qq_u)(\Phi_{vt}+qq_v)+\Phi_{vv}(\Phi_{ut}+qq_u)^2.\end{aligned} \tag{3.8}$$

Now, we will discuss the two situations respectively.

(I) $\Phi_{tt}\neq 0$.

From (3.6), we get $\omega_{uu}\omega_{vv}-\omega_{uv}^2=0$ and $\vec{q}=\text{const}$ is a developable surface, its general integral is expressed as

$$\omega=uf(\lambda)+vg(\lambda)+h(\lambda). \tag{3.9}$$

Thus, $0=uf'(\lambda)+vg'(\lambda)+h'(\lambda)$, $f(\lambda),g(\lambda),h(\lambda)$ are arbitrary functions, where $\lambda=\lambda(u,v)$, $\omega=\omega(u,v)$. Let $J=\frac{\partial(\lambda,\omega)}{\partial(u,v)}\neq 0$ and $(u,v)\leftrightarrow(\lambda,\omega)$, we have $(1+\omega_u^2)\omega_{vv}-2\omega_u\omega_v\omega_{uv}+(1+\omega_v^2)\omega_{uu}=-AJ^2b^2$, where $b(\lambda)=\left(1+\frac{f_\lambda^2+g_\lambda^2}{(fg_\lambda-gf_\lambda)^2}\right)^{\frac{1}{2}}$.

Substituting (3.9) into (3.7), and exchange Φ with ψ, $\psi=-\Phi-\frac{t}{2}(q^2-C^2)$, $c^2=(\gamma-1)\psi_t$, we have

$$-AJ^2a^2b^2=-\Phi_{tt}(\omega_{uu}\Phi_{vv}-2\omega_{uv}\Phi_{uv}+\omega_{uv}\Phi_{uu})+\omega_{vv}\psi_{ut}^2-2\omega_{uv}\psi_{ut}\psi_{vt}+\omega_{uu}\psi_{vt}^2. \tag{3.10}$$

The characteristic equation of the coefficient of Φ_{tt} is $\omega_{vv}dv^2+2\omega_{uv}dudv+\omega_{uu}du^2=0$.

By the definition of $\lambda(u,v)$ given in (3.9), we know that $(\lambda_u du+\lambda_v dv)^2=0$. By introducing the function $\lambda(u,v)$ as a new variable and performing the transformation $(u,v,t)\leftrightarrow(\lambda,\omega,t)$, we obtain

$$\begin{aligned}
&\omega_{uu}\Phi_{vv}-2\omega_{uv}\Phi_{uv}+\omega_{vv}\Phi_{uu}=-AJ^2\Phi_{\omega\omega},\\
&\omega_{vv}\psi_{ut}^2-2\omega_{uv}\psi_{ut}\psi_{vt}+\omega_{uu}\psi_{vt}^2=-AJ^2\psi_{\omega t}^2,\\
&c^2=\nu^2\psi_t,\ \nu^2=\gamma-1.
\end{aligned} \tag{3.11}$$

By (3.10), we can obtain

$$\psi_{\omega\omega}\psi_{tt}-\psi_{\omega t}^2+b^2(t\psi_{tt}+\nu^2\psi_t)=0, \tag{3.12}$$

which is the Monge-Ampére equation for $\psi(\lambda,\omega,t)$. Since derivatives with respect to λ appears only in the coefficient $b=b(\lambda)$. Eq. (3.12) can be transformed into an Euler-Poisson equation, which admits analytical solutions through standard methods. This structural simplification arises because the λ-dependence is confined to a parametric coefficient rather than differential terms.

Let $\xi=\psi_t$, and $\eta=\psi_\omega$ denote the partial derivatives. The characteristic curves of Eq. (3.12) satisfy

$$b^2\nu^2\xi\omega^2-\mathrm{d}\xi^2=0,\quad \mathrm{d}\eta\mathrm{d}\omega+\mathrm{d}\xi\mathrm{d}t+b^2t\mathrm{d}\omega^2=0. \tag{3.13}$$

Introducing characteristic variables $r(\lambda,\omega,t)$, $s(\lambda,\omega,t)$, we have

$$\begin{aligned}
&b\nu\omega_r-\xi^{-\frac{1}{2}}\xi_r=0,\quad b\nu\omega_s+\xi^{-\frac{1}{2}}\xi_s=0,\\
&\eta_r+b\nu\xi^{\frac{1}{2}}t_r+b^2t\omega_r=0,\quad \eta_s-b\nu\xi^{\frac{1}{2}}t_s+b^2t\omega_s=0.
\end{aligned} \tag{3.14}$$

With the choice $r=b\nu\omega+2\xi^{\frac{1}{2}}$, $s=b\nu\omega-2\xi^{\frac{1}{2}}$, we have

$$\begin{aligned}&(\frac{4\nu}{b})\eta_r+\nu^2(r-s)t_r+2t=0,\\&(\frac{4\nu}{b})\eta_s-\nu^2(r-s)t_s+2t=0.\end{aligned}\tag{3.15}$$

Upon eliminating η and t, we arrive at the Euler–Poisson equation

$$\begin{aligned}&(r-s)t_{rs}-n(t_r-t_s)=0,\\&(r-s)\eta_{rs}-m(\eta_r-\eta_s)=0.\end{aligned}\tag{3.16}$$

Thus, $2n=\frac{\nu^2+2}{\nu^2}=\frac{\gamma+1}{\gamma-1}$, $2m=2n-2=\frac{2-\nu^2}{\nu^2}=\frac{3-\gamma}{\gamma-1}$, and for $\gamma=\frac{7}{5}$, we have $n=3$, $m=2$. Eq. (3.12) can also be solved by means of a contact transformation.

Defining $\tilde{\psi}=\psi-t\psi_t$, $\xi=\psi_t$, we have the identities $\psi_{tt}\tilde{\psi}_{\xi\xi}=-1$, $\psi_{\omega t}\tilde{\psi}_{\xi\xi}=-\tilde{\psi}_{\omega\xi}$, $\psi_{\omega\omega}\tilde{\psi}_{\xi\xi}=\tilde{\psi}_{\omega\omega}\tilde{\psi}_{\xi\xi}-\tilde{\psi}^2_{\omega\xi}$. Using these in (3.12) yields $b^2(\nu^2\xi\tilde{\psi}_{\xi\xi}+\tilde{\psi}_\xi)-\tilde{\psi}_{\omega\omega}=0$.

Proceeding in the same manner as for r and s, we arrive at the Euler-Poisson equation

$$(r-s)\tilde{\psi}_{rs}-m(\tilde{\psi}_r-\tilde{\psi}_s)=0.\tag{3.17}$$

When $m=2$, the general solution of (3.17) is $\tilde{\psi}(r,s,\lambda)=\frac{\partial^2}{\partial r\partial s}\left(\frac{A(r,\lambda)-B(s,\lambda)}{r-s}\right)$, where $A(r,\lambda),B(s,\lambda)$ are arbitrary functions.

Since $c^2=(\gamma-1)\psi_t=(\gamma-1)\xi=\nu^2\xi$, we have $r=b\nu\omega+2\xi^{\frac{1}{2}}=\nu(b\omega+\frac{2c}{\gamma-1})$, $s=\nu[b\omega-\frac{2c}{\gamma-1}]$, where $b=b(\lambda)$. If $\lambda=const$, then, r and s take the same form as the Riemann invariants of one-dimensional unsteady flow: $r=u+\frac{2c}{\gamma-1}$, $s=u-\frac{2c}{\gamma-1}$. In this case r and s remain constant along the characteristic curves defined by $\lambda=const$. Particularly, let $u=\omega\frac{g_\lambda}{fg_\lambda-gf_\lambda}$, $v=\omega\frac{f_\lambda}{fg_\lambda-gf_\lambda}$ and $b(\lambda)=\left(1+\frac{f_\lambda^2+g_\lambda^2}{(fg_\lambda-gf_\lambda)^2}\right)^{\frac{1}{2}}$, we have $b\omega=(u^2+v^2+\omega^2)^{\frac{1}{2}}=q$.

From the above analysis, we can derive several key observations:

(i) For unsteady flow described by the solution form $\mathbf{q}(\lambda,\mu)$, the state surface $\mathbf{q}(u,v)$ constitutes a developable surface. Consequently, the isospeed curves degenerate into a family of straight lines.

(ii) The developable surface admits a general integral solution of the form

$$\begin{aligned}&\omega=uf(\lambda)+vg(\lambda)+h(\lambda),\\&o=uf'(\lambda)+vg'(\lambda)+h'(\lambda).\end{aligned}\tag{3.18}$$

For each constant-value characteristic line $\lambda=$const in parameter space, the corresponding line in physical space consists of points sharing identical flow velocity.

(iii) Along each characteristic line $\lambda = const$, the Riemann invariants r and s remain constant. The general solution for the unsteady degenerate flow problem admits the representation $f(\lambda),g(\lambda)$, $h(\lambda)$, $A(x,r),B(s,\lambda)$, where these functions must be constrained by the requirement that the resulting $\omega(u,v)$ and $\Phi(u,v,t)$ satisfy the Monge-Ampère equation (3.8).

(II) $\Phi_{tt}=0$.

When condition (3.6) is automatically satisfied, the developable surface integral becomes nonexistent, rendering $\psi(\lambda,\omega,t)$ inadmissible as a solution. Instead, the system must comply with constraints (3.7)-(3.8). The Cauchy integral condition Φ_{tt}=0 further reveals that the coefficient a=$a(\lambda,\mu)$ exhibits no temporal dependence. This particular solution structure is classified by Gieze as a double wave.

Given the potential function Φ=$t\Gamma(u,v)$+$\lambda(u,v)$, consider the point $(-\Phi_u,-\Phi_v,0)$ in the (x,y)-plane, The characteristic ray emanating from this point in the direction $(\omega_u,\omega_v,-1)$ satisfies the equation

$$x+z\omega_u+\Phi_u=0,\ y+z\omega_v+\Phi_v=0, \tag{3.19}$$

or

$$t\Gamma_u+\lambda_u+\mathbf{r}\cdot\mathbf{q}_u=0,\ t\Gamma_v+\lambda_v+\mathbf{r}\cdot\mathbf{q}_v=0. \tag{3.20}$$

Especially when λ is a constant, we have

$$t\Gamma_u+x+z\omega_u=0,\ t\Gamma_v+y+z\omega_v=0. \tag{3.21}$$

The system admits a solution where (u,v,ω) depend solely on the similarity variables $(x/t,y/t,z/t)$. his corresponds to a conical flow with the potential ansatz Φ=$t\Gamma(u,v)$. As established in [9], such solutions represent three-dimensional simple waves of the form $\mathbf{q}$=$\mathbf{q}(\lambda)$, where the flow quantities depend on a single characteristic parameter.

The continuity equation (3.1) takes the form

$$(c^2\mathbf{q}_\lambda-qq_\lambda\mathbf{q})\cdot\mathrm{grad}\lambda=\Phi_{tt}+(\Phi_{\lambda\lambda}+2qq_\lambda)\lambda_t, \tag{3.22}$$

where $q=|\mathbf{q}|$. Through elimination of the gradient terms gradλ, λ_t, we obtain the fundamental characteristic equation:

$$c^2\mathbf{q}_\lambda^2=(\Phi_{\lambda t}+qq_\lambda)^2-\Phi_{tt}(\Phi_{\lambda\lambda}+\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda}). \tag{3.23}$$

Through Eq. (3.4), we derive the constraints $\Phi_\lambda+\mathbf{r}\cdot\mathbf{q}_\lambda=0$ and $\Phi=\Phi(\lambda,t)$, demonstrating that the solution constitutes a single-parameter family of planes. The condition $\lambda=$const coupled with the constancy of $\mathbf{q}(\lambda)$ explicitly confirms the flow's essential planarity, as all flow quantities remain invariant within each characteristic plane.

For the case $\Phi_{tt}\neq 0$, Eq. (3.4) yields:

$$F(\lambda,t)+\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda}=0. \tag{3.24}$$

Differentiating this relation gives:

$$\begin{aligned}&(F_\lambda+\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda\lambda})\mathrm{grad}\lambda+\mathbf{q}_{\lambda\lambda}=0,\\&(\Phi_{\lambda\lambda}+\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda})\mathrm{grad}\lambda+\mathbf{q}_{\lambda}=0.\end{aligned} \tag{3.25}$$

Through vector comparison, we obtain the orthogonality condition $\mathbf{q}_\lambda\times\mathbf{q}_{\lambda\lambda}=0$. Integration yields the general solution: $\vec{q}(\lambda)=\mathbf{b}f(\lambda)+\mathbf{c}$ where f is an arbitrary scalar function and $\mathbf{b}$, $\mathbf{c}$ is an arbitrary scalar function and $\mathbf{q}$ lacks any component in the direction $\mathbf{b}\times\mathbf{c}$.

(II) $\Phi_{tt}=0$.

From the characteristic condition (3.23), we obtain the simplified relation: $c^2\mathbf{q}_\lambda^2=(\Phi_{\lambda t}+qq_\lambda)^2$. Integration yields the general potential form: $\Phi(\lambda,t)=t\psi(\lambda)+K(\lambda)$ where ψ,K represent arbitrary functions of the characteristic parameter. Substitution then gives the modified characteristic equation: $c^2\mathbf{q}_\lambda^2=(\psi_\lambda+qq_\lambda)^2$.

The energy equation $\frac{q^2}{2}+\frac{c^2}{\gamma-1}+\Phi_t=\frac{C^2}{2}$ yields the relation $2c^2=\nu^2(C^2-q^2-2\psi)$ when $\gamma=\frac{7}{5}$ $(\nu^2=\frac{2}{5})$. Setting $K=0$ and imposing the characteristic condition $t\psi_t+\mathbf{r}\cdot\mathbf{q}\lambda=0$ with $\lambda=\lambda(x/t,y/t,z/t)$, we obtain the planar wave solution characterized by: (i) the velocity gradient constraint

$$u_\lambda^2+v_\lambda^2+\omega_\lambda^2=25c_\lambda^2. \tag{3.26}$$

And (ii) the plane wave equation $-\psi_\lambda=uu_\lambda+vv_\lambda+\omega\omega_\lambda+5aa_\lambda$ derived from the λ-differentiated form of the energy relation.

Equation (3.26) establishes a constraint relationship among the variables u,v,ω, a, where any three of these quantities may be treated as independent functions.

In their study of two-dimensional unsteady gas flows, Naylor [9] derived an exact velocity potential solution $\phi(x,y,t)$ for planar flow. The analysis demonstrated that the characteristic curves $\mathbf{q}(\lambda,t)$, $\lambda(x,y,t)$=const form straight lines that typically do not converge to a singular envelope point, thereby precluding the existence of central simple waves in unsteady potential flow. The author proved that any scenario admitting a central simple wave must inherently represent either steady flow or radially symmetric flow with constant expansion velocity. Special cases were explicitly solved, including: (i) the condition $h(\lambda)=a=t+\frac{u}{v}=0$ and (ii) uniformly accelerated flow with λ-line envelope $g=b\cos\lambda$ The methodology employed hodograph and Legendre transformations, treating the velocity potential and flow velocity as independent variables. Through $\mathcal{L}$ transformation the governing equations reduced to three simplified Monge-Ampère equations expressed solely in terms of λ,t achieved via the variable transition $(x,y,t)\leftrightarrow(\lambda,\varphi,t)\rightarrow(\lambda,t)$.

Two of the Monge-Ampère equations are parabolic with $\lambda=const$ being one of their characteristic rays. By appropriately selecting λ, the solution of the system of equations is expressed as four time t cubic functions with λ functions as coefficients. These arbitrary functions are determined by the compatibility equations. Owing to the non-uniqueness inherent in the λ-line determined solutions, the resulting flow field is not uniquely specified. Two distinct flow configurations are explicitly derived to demonstrate this multiplicity.

4 Cauchy Integral Method and Exact Solutions for Simple Waves

Similarly to [8], using the Cauchy integral method, the basic equation is transformed into the continuity equation,

$$a^2\mathbf{q}_\lambda^2=(\Phi_{\lambda t}+qq_\lambda+\mathbf{r}\cdot\mathbf{q}_{\lambda t})^2-(\Phi_{\lambda\lambda}+\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda})(\Phi_{tt}+2qq_t+\mathbf{r}\cdot\mathbf{q}_{tt}), \tag{4.1}$$

where $\Phi+\mathbf{r}\cdot\mathbf{q}=\phi$, $\mathbf{q}=\mathrm{grad}\phi$, $\mathbf{q}=\mathbf{q}(\lambda,t),\lambda=\lambda(\mathbf{r},t)$. The governing equations express the higher-order derivatives of $\mathbf{q}$ as linear combinations of its

$$\begin{aligned}&\mathbf{q}_{\lambda\lambda}+A_1\mathbf{q}+B_1\mathbf{q}_\lambda=0,\\&\mathbf{q}_{\lambda t}+A_2\mathbf{q}+B_2\mathbf{q}_\lambda=0,\\&\mathbf{q}_{tt}+A_3\mathbf{q}+B_3\mathbf{q}_\lambda=0,\\&\mathbf{q}_{t}+A_4\mathbf{q}+B_4\mathbf{q}_\lambda=0,\end{aligned} \tag{4.2}$$

where the coefficients $A_i,B_i(i=1,2,3,4)$ are functions of the flow variables $(q(\lambda),\theta(\lambda),t)$. This formulation demonstrates that all second-order derivatives and the first-order time derivative are linearly dependent on $\mathbf{q}$, $\mathbf{q}_\lambda$. By taking the dot product of these equations with the position vector $\mathbf{r}$ and subsequently eliminating the terms $\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda}$, $\mathbf{r}\cdot\mathbf{q}_{\lambda t}$, $\mathbf{r}\cdot\mathbf{q}_{tt}$, and $\mathbf{r}\cdot\mathbf{q}_t$ from equation (4.1), we obtain:

$$X(\lambda,t)+(\Phi-\psi)Y(\lambda,t)+(\Phi-\psi)^2Z(\lambda,t)=0, \tag{4.3}$$

where X,Y,Z are functions of $\Phi(\lambda,t)$ and $\mathbf{q}(\lambda,t)$, their inverse relations exhibit no dependence on ϕ. The independence of ϕ, and λ, then yields

$$X=Y=Z=0,$$

where

$$Z(\lambda,t)=A_2^2-A_1A_3, \tag{4.4}$$

$$\begin{aligned} X(\lambda,t)=&(\Phi_{\lambda,t}+B_2\Phi_\lambda+qq_\lambda)^2-(\Phi_{\lambda\lambda}+B_1\Phi_\lambda)(\Phi_{tt}+B_3\Phi_\lambda+\alpha qq_t)\\ &-(\gamma-1)[\frac{1}{2}(c^2-q^2)-\Phi_t-B_4\Phi_\lambda]\mathbf{q}_\lambda^2, \end{aligned} \tag{4.5}$$

$$\begin{aligned} Y(\lambda,t)=&2A_2(\Phi_{\lambda t}+B_2\Phi_\lambda+qq_\lambda)-A_1(\Phi_{tt}+B_3\Phi_\lambda+2qq_t)\\ &-A_3(\Phi_{\lambda\lambda}+B_1\Phi_\lambda)+(\gamma-1)A_4\mathbf{q}_\lambda^2. \end{aligned} \tag{4.6}$$

For Eq. (4.4), the ansatz $q=\lambda$, $q_t=0$, $z=\log q$, $u=\theta_z,v=\theta_t$ transforms the system into a Monge-Ampère-type equation.

$$\theta_{zz}\theta_{tt}-\theta_{zt}^2+uv^2\theta_{zz}+u(1+u^2)\theta_{tt}-2v(1+u^2)\theta_{zt}-v^2(1+u^2)=0. \tag{4.7}$$

Its characteristics are

$$\begin{aligned} &\mathrm{d}u+u(1+u^2)\mathrm{d}z+m\mathrm{d}t=0,\\ &\mathrm{d}v+m\mathrm{d}z+uv^2\mathrm{d}t=0, \end{aligned} \tag{4.8}$$

where m satisfying $m^2-2mv(1+u^2)+v^2(1+u^2)^2=0$.

Introducing the substitution $m=v(1+u^2)$, integration yields the first integrals:

$$t+\frac{u}{v}=a, \qquad \frac{v^2e^{2z}}{1+u^2}=b, \tag{4.9}$$

where a, b are integration constants. This leads to the intermediate integral:

$$v^2e^{2z}=(1+u^2)f(t+\frac{u}{v}), \tag{4.10}$$

with f denoting an arbitrary function of the characteristic variable. Thus, we have

$$\theta-\lambda=\cos^{-1}[(a-t)b^{\frac{1}{2}}e^{-z}], \tag{4.11}$$

where λ is the integration constant. The parameter λ is chosen to satisfy the orthogonal decomposition: $q\cos(\theta-\lambda)=\beta$, and $q\sin(\theta-\lambda)=\beta_\lambda$ which inherently implies the differential constraint: $q_\lambda\cos(\theta-\lambda)=q\theta_\lambda\sin(\theta-\lambda)$.

The solution takes the functional form: $\beta(\lambda,t)=[h(\lambda)-t]g(\lambda)$, $a=h(\lambda)$, $b^{\frac{1}{2}}=g(\lambda)$, $\lambda=\theta-\cot^{-1}(q\theta_\lambda/q_\lambda)$, where h and g are arbitrary functions of the characteristic parameter λ. The λ-characteristic line, defined by the condition $\Phi_\lambda+\mathbf{r}\cdot\mathbf{q}_\lambda=0$, intersects the streamlines at an angle $\Gamma(x,t)$ satisfying: $\cot\Gamma=q\theta_\lambda/q_\lambda$, $\Gamma=\theta-\lambda$, $q^2=\beta^2+\beta_\lambda^2$.

When $\lambda=q$ in Eq. (4.6), the condition $A_1A_3=A_2^2$ defines a parabolic system. This parabolicity implies that the characteristic curves degenerate to $\lambda(q,t)=cosnt$. Given that λ satisfies the previously specified selection criteria, we derive

$$\Phi_{tt}+2qq_t=(\gamma-1)g(\beta+\beta_{\lambda\lambda}). \tag{4.12}$$

Through direct integration of this system, we obtain

$$\Phi(\lambda,t)=k(\lambda)+f(\lambda)(h-t)+E(\lambda)(h-t)^2+G(\lambda)(h-t)^3, \tag{4.13}$$

where $f(\lambda)$, $k(\lambda)$ are arbitrary, E, G is related to g, h and their derivatives.

The functions $f(\lambda)$, $g(\lambda)$, $h(\lambda)$, and $k(\lambda)$ are functionally dependent and must be constrained to satisfy equation (4.5). Substituting these functions into (4.5) yields

$$(H\Phi_{\lambda t}-H_t\Phi_\lambda+LH^2)^2=(\gamma-1)H^2\Big[gH\Phi_{\lambda\lambda}-(gH)_\lambda\Phi_\lambda+\frac{1}{2}H^2(c^2-q^2-2\Phi_t)\Big], \tag{4.14}$$

where $L(\lambda,t)=\beta_\lambda$, $H=\beta+\beta_{\lambda\lambda}$, $q^2=\beta^2+\beta_\lambda^2$ and $\beta=g(h-t)$.

Generically, the λ-characteristic envelope is nondegenerate, with characteristics exhibiting nonzero divergence and nonvanishing curvature. The envelope conditions

$$\begin{aligned}&\Phi_\lambda+\mathbf{r}\cdot\mathbf{q}_\lambda=0,\\&\Phi_{\lambda\lambda}+\mathbf{r}\cdot\mathbf{q}_{\lambda\lambda}=0,\end{aligned} \tag{4.15}$$

determine the singular solution

$$\mathbf{r}(\lambda,t)=(\Phi_\lambda Q_{\lambda\lambda}-\Phi_{\lambda\lambda}Q_\lambda)|\mathbf{q}_\lambda\times\mathbf{q}_{\lambda\lambda}|^{-1}, \tag{4.16}$$

where $Q=(-q\sin\theta,q\cos\theta)$. Introducing the unit vector field $\mathbf{b}(\lambda)=\mathbf{i}\sin\theta+\mathbf{j}\cos\theta$, the envelope of characteristic lines is given by the position vector: $\mathbf{r}_0(\lambda t)=\Psi_\lambda\mathbf{b}-\Psi\mathbf{b}_\lambda$, where $\Psi=\frac{1}{H}\Phi_\lambda$. Each λ-characteristic line satisfies the orthogonality condition: $\Psi+\mathbf{r}\cdot\mathbf{b}_\lambda=0$.

It can be rigorously demonstrated that for the case of a central simple wave, characterized by the conditions: $\Psi_{\lambda\lambda}+\Psi=S_\lambda=0$ and $\mathbf{r}=\mathbf{r}_0(t)$, the following properties hold

$$\begin{aligned}&\beta=g_0(h_0-t)+a_1\cos\lambda+b_1\sin\lambda,\\&\Psi=(a_0+a_1t)\sin\lambda-(b_0+b_1t)\cos\lambda,\\&\mathbf{q}=\mathbf{q}_0+g_0(h_0-t)\mathbf{b}(\lambda),\\&\Psi_{tt}-g_\lambda=0,\end{aligned} \tag{4.17}$$

which means that describes a fluid undergoing radial motion from the central point, where the radial velocity at any time t is given by $g_0(h_0-t)$. The flow exhibits steady-state properties with: (i) β and Ψ_t being time-independent, and (ii) the velocity field reducing to $\mathbf{q}=\mathbf{q}(\lambda)$. The instantaneous streamlines coincide with the λ-characteristic lines, along which the velocity λ-characteristic lines, along which the velocity $\mathbf{q}$ remains spatially constant.

We derive the position-velocity relation: $\beta(\mathbf{r}-\mathbf{r}_0)=(\beta_\lambda\Psi-\beta\Psi_\lambda-\Phi+\phi)\mathbf{b}$ where $\mathbf{r}$ denotes an arbitrary point on the λ-characteristic line, $\mathbf{r}_0$ represents the envelope position vector, and $\mathbf{b}$ is the unit tangent vector along the λ-line. From this formulation, the instantaneous streamline curvature can be obtained as

$$k=\beta\beta_\lambda(\beta+\beta_{\lambda\lambda})(\beta^2+\beta_\lambda^2)^{-\frac{3}{2}}\mu^{-1}, \tag{4.18}$$

where μ is the distance of $\mathbf{r}-\mathbf{r}_0$, that is $\beta\mu=\beta_\lambda\Psi-\beta\Psi_\lambda-\Phi+\phi$. The trajectory of the mass point is determined by $\mathbf{r}_t=\mathbf{q}$.

If $\Psi(\lambda,t)$, $\beta(\lambda,t)$ are known, the characteristic parameter $\lambda(t)$ is governed by the nonlinear ordinary differential equation:

$$(\Psi_t+\beta_\lambda)\lambda_{tt}=(2\Psi_{\lambda t}+\beta_{\lambda\lambda}-\beta)\lambda_t^2+(\Psi_{\lambda\lambda}+\Psi)\lambda_t^3+(\Phi_{tt}+\beta_{\lambda t})\lambda_t. \tag{4.19}$$

The corresponding particle trajectories are then determined by the position vector:

$$\mathbf{r}=[\Psi_\lambda+(\Psi_t+\beta_\lambda)t_\lambda]\mathbf{b}-\Psi\mathbf{b}_\lambda. \tag{4.20}$$

In their seminal work, Naylor [9] derived exact analytical solutions for three distinct classes of problems. These solutions include:
(I) When the integration constant satisfies $a=t+\frac{u}{v}=h(\lambda)=0$, indicating a degenerate case of the Monge-Ampère equation, we derive

$$\lambda\text{-line:}\qquad y\cos\lambda-x\sin\lambda+\frac{4}{5}bnt^2e^{n\lambda}=0, \tag{4.21}$$

$$\mathbf{q}=bte^{n\lambda}\csc\lambda_0[\mathbf{i}\sin(\lambda-\lambda_0)-\mathbf{j}\cos(\lambda-\lambda_0)], \tag{4.22}$$

where $n=\cot\lambda_0$, $n^2=\frac{7}{2}$.

The envelope of λ-characteristic lines are

$$\begin{aligned} x&=\frac{4}{5}bnt^2e^{n\lambda}\csc\lambda_0\cos(\lambda-\lambda_0),\\ y&=\frac{4}{5}bnt^2e^{n\lambda}\csc\lambda_0\sin(\lambda-\lambda_0). \end{aligned} \tag{4.23}$$

The evolution equation for the characteristic parameter $\lambda(t)$ is governed by the nonlinear ODE

$$nt\lambda_{tt}=6nt^2\lambda_t^3+\frac{29}{2}t\lambda_t^2+u\lambda_t, \tag{4.24}$$

which, upon integration, yields the implicit solution

$$t^2=(a_1e^{-2n\lambda}+a_2e^{-\lambda/2n})e^{-n\lambda}, \tag{4.25}$$

where a_1 and a_2 represent fundamental flow invariants determined by initial/boundary conditions. The mass point trajectory equation is

$$\frac{5}{6}\mathbf{r}=-4n(a_1e^{-2n\lambda}+a_2e^{-\lambda/2n})\mathbf{b}_\lambda+\frac{1}{4}(32a_2e^{-\lambda/2n}-7a_1e^{-2n\lambda})\mathbf{b}, \tag{4.26}$$

and the instantaneous streamline are

$$\begin{aligned} x&=\frac{4}{5}nbt^2\sec 2\lambda_0 e^{n\lambda}\sin(\lambda-2\lambda_0)+Ce^{\lambda/n}\cos\lambda,\\ y&=-\frac{4}{5}nbt^2\sec 2\lambda_0 e^{n\lambda}\cos(\lambda-2\lambda_0)+Ce^{\lambda/n}\sin\lambda. \end{aligned} \tag{4.27}$$

In this configuration, the envelope forms a logarithmic spiral where tangent lines exhibit multiple intersections, resulting in non-unique velocity determination due to the breakdown of the λ-line definition (multi-valued λ). By restricting λ-line segments to single envelope intersections, the velocity field becomes well-defined but with infinite acceleration–characteristic of supersonic flow. The flow topology further bifurcates based on the parameter C: (i) for $C<0$, instantaneous streamlines form convergent spirals without singularities; (ii) for $C>0$, streamlines develop cusps at the envelope, which asymptotically evolves into a source-like curve at infinity, effectively creating a void where fluid is absent or expelled.

(II) For the case of a uniformly accelerated envelope with constant slope, where the acceleration profile is given by $g(\lambda)=b\cos\lambda$ ($b=const$), the corresponding λ-characteristic lines satisfy

$$y-x\tan\lambda+b[\psi+\frac{1}{2}(h^2-t^2)\tan\lambda]=0, \tag{4.28}$$

where

$$\begin{aligned} \psi(\lambda)&=\frac{1}{2}\int(6h_\lambda^2-14hh_\lambda\tan\lambda+5h^2\tan^2\lambda-\frac{c^2}{b^2}\sec^2\lambda)\mathrm{d}\lambda,\\ \frac{h}{h_1}&=\frac{1}{3}(5\sin^2\lambda-3)(\sec\lambda)^{\frac{5}{2}}+\frac{5}{6}\sqrt{2}\tan\lambda.\mathrm{sd}^{-1}(\sqrt{2}\tan\frac{\lambda}{2}), \end{aligned} \tag{4.29}$$

and h_1 is constant.

The envelope of the λ-line is

$$(2bx+c^2+b^2t^2)\sec^2\lambda=6(X+\frac{2}{5}X_\lambda\tan\lambda)^2+(X\tan\lambda-\frac{2}{5}X_\lambda)^2, \tag{4.30}$$

where $X(\lambda)=\frac{5bh_1}{6}[2\tan\lambda(\sec\lambda)^2+\sqrt{2}sd^{-1}(\sqrt{2}\tan\frac{\lambda}{2})]$.

(III) The velocity field $\mathbf{q}(\lambda)$ generates λ characteristic lines described by the transcendental equation: $R\sin(\rho^{-1}z-\psi-\frac{\pi}{2\rho})=\cos z$ where $R=r(\rho bt)^{-1}$ represents the scaled radial coordinate, $\rho^2=\frac{\gamma-1}{\gamma+3}$ defines the compressibility parameter, $q^2=b^2(\sin^2 z+\rho^2\cos^2 z)$ governs the velocity magnitude, (r,ψ) correspond to the polar coordinates in the (x,y) plane.

The envelopes are

$$\begin{aligned}&\psi=\rho^{-1}z-\tan^{-1}(\rho\tan z)-\frac{\pi}{2\rho}+\frac{3\pi}{2},\\&R^2=\cos^2 z+\rho^2\sin^2 z,\end{aligned}\tag{4.31}$$

where z is a parameter.

The modulus $r=R\rho bt$ exhibits linear radial growth. The instantaneous streamline initially forms a right angle with the envelope tangent at $z=0$, then asymptotically approaches parallelism. At the$z=\pi/2$ tangent point, the λ-characteristic originating from $z=0$ generates uniform normal flow with velocity ρb, maintaining this kinematic condition throughout its trajectory.

5 Quasilinear Systems and Hodograph Transformations in Wave Theory

Hodograph transformations are a powerful mathematical tool that involves interchanging dependent and independent variables. These transformations are particularly valuable in solving nonlinear problems, as they enable the conversion of many nonlinear partial differential equations (PDEs) into variable-coefficient linear PDEs [6,12–14]. Such methods transform inherently complex nonlinear systems into forms amenable to systematic analysis and solution.

Consider the following three sets of quasilinear equations:

$$a_i\frac{\partial u}{\partial x}+b_i\frac{\partial u}{\partial y}+c_i\frac{\partial v}{\partial x}+d_i\frac{\partial v}{\partial y}+e_i\frac{\partial w}{\partial x}+f_i\frac{\partial w}{\partial y}=0,\quad (i=1,2,3),\tag{5.1}$$

where (u,v,w) are functions of (x,y) and $a_i,b_i,\dots,f_i$ are functions of (u,v,w). For each solution of system (5.1), there exists a functional relationship $F(u,v,w)=0$. For clarity, we assume the relation admits an explicit solution in the form $w=w(u,v)$.

Let $\mathrm{d}w=p\mathrm{d}u+q\mathrm{d}v$. Substituting this differential form into Eq. (5.1) yields

$$(a_i+pe_i)\frac{\partial u}{\partial x}+(b_i+pf_i)\frac{\partial u}{\partial y}+(c_i+qe_i)\frac{\partial v}{\partial x}+(d_i+qf_i)\frac{\partial v}{\partial y}=0,\quad (i=1,2,3),\tag{5.2}$$

Set $\frac{\partial(u,v)}{\partial(x,y)}\neq 0$, and perform a Hodograph transformation, we get

$$A_i\frac{\partial x}{\partial u}+B_i\frac{\partial x}{\partial v}+C_i\frac{\partial y}{\partial u}+D_i\frac{\partial y}{\partial v}=0,\quad (i=1,2,3),\tag{5.3}$$

where $A_i = d_i + qf_i, B_i = -(b_i + pf_i), C_i = -(c_i + qe_i)$, and $D_i = a_i + pe_i$ are functions of u and v. Assume the rank of (5.3) is 3, we can get

$$\frac{\frac{\partial x}{\partial u}}{\Delta_1} = \frac{\frac{\partial x}{\partial v}}{\Delta_2} = \frac{\frac{\partial y}{\partial u}}{\Delta_3} = \frac{\frac{\partial y}{\partial v}}{\Delta_4} = \mu, \tag{5.4}$$

where μ is the common ratio, and

$$\begin{aligned} &\Delta_1 = \begin{vmatrix} B_1 & C_1 & D_1 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{vmatrix}, \ \Delta_2 = -\begin{vmatrix} A_1 & C_1 & D_1 \\ A_2 & C_2 & D_2 \\ A_3 & C_3 & D_3 \end{vmatrix}, \\ &\Delta_3 = \begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix}, \ \Delta_4 = -\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}. \end{aligned} \tag{5.5}$$

The integrable compatibility condition is

$$\frac{\partial(\Delta_1\mu)}{\partial v} = \frac{\partial(\Delta_2\mu)}{\partial u}. \tag{5.6}$$

Here Eq. (5.6) constitutes a second-order partial differential equation governing the function $w(u,v)$.

If the rank of (4.5) is 2, then the solution of the equation is determined by two arbitrary functions and cannot be determined on $w = w(u,v)$. Consider the Euler equations of one-dimensional unsteady isentropic flow

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0, \quad \frac{\partial v}{\partial x} + bp^{-x}\frac{\partial p}{\partial t} = 0, \quad \frac{\partial b}{\partial t} = 0, \tag{5.7}$$

where t is time, x is the Lagrangian coordinate, v is the velocity of gas particles, p is pressure, and b is a function of entropy. Its two sets of specific solutions are

(i) $b = g(p), v = c_1 t + c_2, p = -c_1 x + c_3$,

(ii) $v = h(b), \int \frac{1}{b} h'(b) \mathrm{d}b = c_4 x + c_5, \frac{1}{1-x} p^{1-x} = -c_4 t + c_6$,

where $g(p)$ and $h(b)$ are arbitrary functions, and $c_i (i=1,\dots,6)$ are arbitrary constants. And the solution of the Cauchy problem is also can be found.

6 Two-Dimensional Detonation Waves Around a Corner

The motion of two-dimensional unsteady detonation waves and detonation products is a very complex issue [15–17]. The simplest case should be self-similar motion. If all boundaries (solid walls, free surfaces, interfaces between different media) are planes passing through the origin, they are parallel to each other in the direction of the z-axis and intersect at an angle in the x, y plane. It is also assumed that the detonation starts at infinity or on a plane parallel to the z-axis or on the z-axis. In this way, a two-dimensional self-similar detonation wave can be obtained. Since both the boundary conditions and detonation conditions are inherently nondimensional, all physical quantities must exhibit self-similar behavior, depending solely on the dimensionless similarity variables $\frac{x}{t}$ and $\frac{y}{t}$.

Introducing the similarity variables $\xi=\frac{x}{t}$, $\eta=\frac{y}{t}$, which have dimensions of velocity, we consequently derive

$$\frac{\partial}{\partial x}=\frac{1}{t}\frac{\partial}{\partial \xi},\ \frac{\partial}{\partial y}=\frac{1}{t}\frac{\partial}{\partial \eta},\ \partial_t=-\frac{1}{t}\left(\xi\frac{\partial}{\partial \xi}+\eta\frac{\partial}{\partial \eta}\right). \tag{6.1}$$

In the (ξ,η) coordinate system, the fluid dynamics equations can be written as follows:

(i) The continuity equation

$$(\mathbf{q}-\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}\rho+\rho\nabla_{\boldsymbol{\xi}}\cdot\mathbf{q}=0. \tag{6.2}$$

(ii) The motion equation

$$(\mathbf{q}-\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}\mathbf{q}=-\frac{1}{\rho}\nabla_{\boldsymbol{\xi}}P. \tag{6.3}$$

(iii) The isentropic equation

$$(\mathbf{q}-\boldsymbol{\xi})\cdot\nabla_{\boldsymbol{\xi}}S=0, \tag{6.4}$$

where $\mathbf{q}$ is the particle velocity of the fluid,

$$\begin{aligned}\boldsymbol{\xi}&=\xi\mathbf{i}+\eta\mathbf{j},\\ \nabla_{\boldsymbol{\xi}}&=\frac{\partial}{\partial\xi}\mathbf{i}+\frac{\partial}{\partial\eta}\mathbf{j}.\end{aligned} \tag{6.5}$$

Introducing the pseudo-velocity $\mathbf{U}=\mathbf{q}-\boldsymbol{\xi}=U\mathbf{i}+V\mathbf{j}$, the fluid dynamics equations can be expressed as

$$\begin{cases}\mathbf{U}\cdot\nabla_{\boldsymbol{\xi}}\rho+\rho\nabla_{\boldsymbol{\xi}}\cdot\mathbf{U}=-2\rho,\\ \mathbf{U}\cdot\nabla_{\boldsymbol{\xi}}\mathbf{U}+\frac{1}{\rho}\nabla_{\boldsymbol{\xi}}P=-\mathbf{U},\\ \mathbf{U}\cdot\nabla_{\boldsymbol{\xi}}S=0.\end{cases} \tag{6.6}$$

If the term on the right side of (6.6) is neglected, then (6.6) is formally the same as the steady flow equations. As long as $\mathbf{U}$ is regarded as the particle velocity and $\boldsymbol{\xi}$ is regarded as the spatial coordinate, owing to the self-similar nature of the equation system's solutions, this is conventionally termed the pseudo-steady flow equation system.

Due to the nature of the equation system and the right side term being unrelated, it is easy to prove that when $c^2 < U^2+V^2$, the equation system (6.6) is hyperbolic, when $c^2 > U^2+V^2$, the equation system (6.6) is elliptic and when $c^2 = U^2+V^2$, the equation system (6.6) is parabolic.

When the flow becomes supersonic $c^2<U^2+V^2$, the system admits characteristic lines with associated Riemann invariants. The characteristic equations equivalent to (6.6) take the form

$$\begin{cases} \mathrm{d}S=0, \\ \frac{\mathrm{d}\eta}{\mathrm{d}\xi}=\frac{V}{U}, \\ \mathrm{d}(\frac{1}{2}U^2+\frac{1}{2}V^2+i)=-(U^2+V^2)\frac{\mathrm{d}\xi}{U}=-(U^2+V^2)\frac{\mathrm{d}\eta}{V}, \end{cases} \tag{6.7}$$

and

$$\begin{cases} \frac{\mathrm{d}\eta}{\mathrm{d}\xi}=\frac{1}{U^2-c^2}\left(UV\pm c\sqrt{U^2+V^2-c^2}\right), \\ \mathrm{d}U\pm\frac{\sqrt{U^2+V^2-c^2}}{U^2+V^2}\frac{\mathrm{d}P}{\mathrm{d}\rho}=-\frac{1}{U^2+V^2}(V\mathrm{d}\xi-U\mathrm{d}\eta), \end{cases} \tag{6.8}$$

where i is the fluid's enthalpy and $dS=0$ represents the isentropic condition indicating that entropy is conserved along the streamline.

Introducing the Mach number m and $\tan m=\frac{c}{\sqrt{U^2+V^2-c^2}}$, then the last two characteristic equations can be written as follows,

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi}=\tan(\theta\pm m), \tag{6.9}$$

$$\mathrm{d}U\pm\frac{\cot m}{U^2+V^2}\frac{\mathrm{d}P}{\mathrm{d}\rho}=-\frac{\mathrm{d}l}{\sqrt{U^2+V^2}}, \tag{6.10}$$

where $dl=\sin\theta d\xi-\cos\theta d\eta$ is the arc length element along the characteristic line, θ is the angle between the streamline and the coordinate axis. Under certain special conditions, the flow is both irrotational and isentropic, which means it is a steady flow with constant entropy and no vorticity. At this point, we have

$$\nabla_{\boldsymbol{\xi}}\times\mathbf{U}=0. \tag{6.11}$$

Introducing the potential function φ such that $\mathbf{U}=\nabla_{\boldsymbol{\xi}}\varphi$, which allows the motion equations to be integrated into the form

$$\varphi+\frac{1}{2}(\nabla_{\boldsymbol{\xi}}\varphi)^2+i=\text{const}. \tag{6.12}$$

Assuming a polytropic equation of state and under the condition of constant entropy, we have

$$\frac{\mathrm{d}\rho}{\rho}=\frac{2}{k-1}\frac{\mathrm{d}c}{c}, \tag{6.13}$$

$$i=\frac{1}{k-1}c^2. \tag{6.14}$$

The equation can be represent as an equation of c^2,

$$\frac{1}{k-1}\nabla_{\boldsymbol{\xi}}\varphi\cdot\nabla_{\boldsymbol{\xi}}c^2+c^2\nabla^2_{\boldsymbol{\xi}}\varphi=-2c^2. \tag{6.15}$$

By direct computation, we have

$$\begin{cases}(c^2-U^2)\frac{\partial^2\varphi}{\partial\xi^2}-2UV\frac{\partial^2\varphi}{\partial\xi\partial\eta}+(c^2-V^2)\frac{\partial^2\varphi}{\partial\eta^2}=U^2+V^2-2c^2,\\ U=\frac{\partial\varphi}{\partial\xi},\ V=\frac{\partial\varphi}{\partial\eta}.\end{cases} \tag{6.16}$$

Under isentropic irrotational supersonic conditions, the characteristic equation is the same as discussed above.

7 Main Results

The main research results can be summarized as follows:

(1) For two-dimensional unsteady potential flow (in gases), the simple wave has the form $\mathbf{q}=\mathbf{q}(\lambda,t)$, where $\lambda=\lambda(x,y,t)$, and $\lambda=\text{const}$ is composed of a series of straight lines. These straight lines are generally not degenerated into a single envelope composed of. Along these lines, $\mathbf{q}=\text{const}$. The 'central simple wave' generally does not exist in unsteady flow.

(2) For three-dimensional unsteady potential flows, we have derived simple wave solutions of both rank 2 and rank 3. The velocity field $\mathbf{q}$ admits solutions of the form $\mathbf{q}=\mathbf{q}(\lambda,\mu)a(\lambda,\mu,t)$ where $\lambda=\lambda(x,y,z,t)$ and $\mu(x,y,z,t)$ are simple wave solutions. In particular, for the case of $\lambda=\mu$, $\mu=v$, $w=w(u,v)$, the following points were obtained:

(i) $\mathbf{q}(u,v)$ is a developable surface and its generators are composed of straight equal velocity lines;

(ii) Selecting the appropriate developable surface (straight-line surface) parameter λ, then on the characteristic line $\lambda=\text{const}$, the generalized Riemann invariants r,s are constants. For the degenerate case, the conditions for conical flow (i.e., similar to one-dimensional central simple waves) are obtained.

(3) Building upon the theoretical foundation of one-dimensional simple waves, we developed a qualitative research framework for multidimensional flows. This approach introduces two key classifications: (i) the k-th kind Riemann invariant, which

generalizes the classical characteristic quantities, and (ii) the k-th kind simple wave, extending the unsteady wave solution paradigm to higher-dimensional configurations. That is, the function $v=v(u_1,u_2,\cdots,u_n)$ that satisfies the condition $\gamma_k\cdot\nabla v=0$ for all u (where γ_k is the k-th kind characteristic vector of $A(u)$, $\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=0$) is called the k-th kind of Riemann invariant of the equation group. And for all the k-th kind of Riemann invariants that are constants on a region in the (x,t) plane, it is called the k-th kind of simple wave. He proved the following results.

(i) The solution connected with the normal state is a simple wave.

(ii) For the k-th kind of simple wave, the k-th kind of characteristic line is a straight line and the solution is a constant along this straight characteristic line.

The Cauchy problem for simple wave uniform motion was specifically solved for the case when $n=3$, resulting in the derivation of its compatibility condition, which was subsequently applied to one-dimensional non-isentropic flow.

8 Conclusions and Discussions

In summary, the multi-dimensional simple wave has been studied in many aspects and achieved some important results. However, from the perspective of the development trend of the entire research work, there is not enough researches closely surrounding the actual problems of physics and mechanics. The research achievements in this area are also quite scarce. The following are several tasks that still need to be undertaken in research.

(1) For non-reducible systems such as those with free terms (one-dimensional unsteady flows of spheres and cylinders), when $a_{ijk}(x,u_i)\frac{\partial u_j}{\partial x_k}=0$, $a_{ijk}(x,u)$ with independent variable x, how to determine the simple wave?

(2) For $a_{ijk}(u_i)\frac{\partial u_j}{\partial x_k}=0$ $(i,j=1,2,\cdots m;\ k=1,2,\cdots n)$ and general m, n, how to determine the simple wave? For the general three-dimensional unsteady aerodynamic equations $(n=4,m=5)$, when $n<m$ or $m<n$, how to determine the simple wave?

(3) How to determine the simple wave for a system of fluid dynamics equations with radiative heat transfer?

(4) Ho to determine the simple wave problem for a system of mixed equations with certain chemical reactions?

(5) From a physico-mathematical perspective, how should the Cauchy problem and boundary value problem for solitary waves be properly formulated to ensure well-posedness? What types of boundary conditions would inherently preclude the existence of solitary wave solutions?

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