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A note on hypergraph extensions of Mantel's theorem

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Abstract. Chao and Yu introduced an entropy method for hypergraph Turán problems, and used it to show that the family of $\lfloor k/2\rfloor$ k-uniform tents have Turán density $k!/k^k$. Il'kovič and Yan [5] improved this by reducing to a subfamily of $\lceil k/e\rceil$ tents. In this note, enhancing Il'kovič-Yan's result, we give a significantly shorter entropy proof, with optimal bounds within this framework.

AMS subject classifications: 05C35, 05C65

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1 Introduction

The *Turán number* $\mathrm{ex}(n,\mathcal{F})$ of a family $\mathcal{F}$ of k-uniform hypergraphs (*k-graphs* for short) denotes the maximum number of edges in an n-vertex k-graph not containing any member of $\mathcal{F}$ as its subgraph. Its *Turán density* is given by $\pi(\mathcal{F}) := \lim_{n\to\infty}\mathrm{ex}(n,\mathcal{F})/\binom{n}{k}$. For integer n, let $[n]:=\{1,2,\ldots,n\}$. The study of Turán number and Turán density of graphs and hypergraphs is one of the central topics in extremal combinatorics. While the celebrated theorems of Turán [11] and Erdős-Stone-Simonovits [2] completely characterize Turán densities for all graph families, the hypergraph setting remains largely open, with exact densities known only for

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very few cases (see [6]). Significant progress has been made through sustained investigations of hypergraph extensions of Mantel's theorem concerning $\mathrm{ex}(n,K_3)$. These extensions have identified many families of k-graphs with Turán density $k!/k^k$. This particular value has attracted special attention, owing in part to Erdős' famous conjecture regarding whether $k!/k^k$ is a jump for k-graphs (see [4] for details). To proceed, for each $i\in[k]$, we define the $(k-i,i)$-tent $\Delta_{(k-i,i)}$ to be the k-graph with vertex set $[2k-1]$ and edge set

$$\big\{\{1,2,\dots k\},\{1,\dots,i,k+1,\dots,2k-i\},\{i+1,\dots,k+1,2k-i+1,\dots,2k-1\}\big\}. \tag{1.1}$$

More generally, for any partition $\lambda=(\lambda_1,\dots,\lambda_\ell)$ of k with $\lambda_1\ge\dots\ge\lambda_\ell\ge1$, the λ-tent Δ_λ denotes the k-graph with $(k-1)\ell+1$ vertices and $\ell+1$ edges $e,e_1,\dots,e_\ell$ defined as follows:

- There exists a vertex v (called *apex*) such that $e_i\cap e_j=\{v\}$ for all $1\le i<j\le\ell$;
- The subsets $e\cap e_1,\dots,e\cap e_\ell$ form a partition of e, where $|e\cap e_i|=\lambda_i$ for each $i\in[\ell]$.

Frankl and Füredi [3] were the first to determine the exact Turán number for a hypergraph, where they showed $\pi(\Delta_{(2,1)})=3!/3^3$. Pikhurko [10] proved $\pi(\Delta_{(3,1)})=4!/4^4$ and the exact Turán number for large n. Generalizing a result of Mubayi [8], Mubayi and Pikhurko [9] determined $\pi(\Delta_{(1,1,\dots,1)})=k!/k^k$. Recently, Chao and Yu [1] developed an innovative entropy-based approach to hypergraph Turán problems. Applying this method, they [1] proved the following generalization of Mantel's theorem: for every $k\ge2$, the family $\mathcal{F}_k:=\{\Delta_{(k-i,i)}:1\le i\le\lfloor k/2\rfloor\}$ satisfies $\pi(\mathcal{F}_k)=k!/k^k$. This implies that for any $1\le q\le\lceil k/2\rceil$, $\pi(\Delta_{(q,1,\dots,1)})=k!/k^k$, generalizing the above results of [3,8,9]. Combining the entropy method with other techniques, Liu [7] determined the exact Turán number of $\mathcal{F}_k$ for large n. For each $1\le s\le\lfloor k/2\rfloor$, define

$$\mathcal{F}_k^{\le s}=\{\Delta_{(k-i,i)}:1\le i\le s\}. \tag{1.2}$$

Very recently, Il'kovič and Yan [5] improved both of these results in [1,7] by determining $\pi(\mathcal{F}_k^{\le\lceil k/e\rceil})=k!/k^k$ for $k\ge4$ and the exact Turán number of $\mathcal{F}_k^{\le\lceil k/e\rceil}$ for large n.

In this note, building on the entropy method of Chao and Yu [1], we provide a much short proof of the above Turán density results with a slightly better bound. Our main result is as follows. Throughout this note, for any integer $k\ge2$, define

$$t(k) \text{ to be the largest integer } s \text{ such that } 1/s+1/(s+1)+\dots+1/k>1. \tag{1.3}$$

Theorem 1.1. *For every integer* $k\ge2$, $\pi(\mathcal{F}_k^{\le t(k)})=k!/k^k$.

Note that the integer $t=t(k)$ satisfies $1<\sum_{i=t}^{k}\frac{1}{i}\le\int_{t-1}^{k}\frac{1}{x}\mathrm{d}x=\ln\frac{k}{t-1}$, which implies $t(k)\le\lceil k/e\rceil$. So Theorem 1.1 generalizes the result of Il'kovič and Yan [5] that $\pi(\mathcal{F}_k^{\le\lceil k/e\rceil})=k!/k^k$ for all $k\ge4$. Table 1 compares $\lceil k/e\rceil$ and $t(k)$ for $4\le k\le19$.

Table 1: Consider the range $4\leq k\leq 19$. The asterisk * denotes instances where $t(k)<\lceil k/e\rceil$.

k	$\lceil k/e\rceil$	$t(k)$	k	$\lceil k/e\rceil$	$t(k)$	k	$\lceil k/e\rceil$	$t(k)$	k	$\lceil k/e\rceil$	$t(k)$
4	2	2	8	3	3	12	5	5	16	6	6
5	2	2	9^*	4	3	13	5	5	17^*	7	6
6^*	3	2	10	4	4	14^*	6	5	18	7	7
7	3	3	11^*	5	4	15	6	6	19	7	7

As observed in [1], results such as Theorem 1.1 can be extended to determine exact Turán densities for certain λ-tents Δ_λ. The following corollary provides the complete statement of this extension. For any partition $\lambda=(\lambda_1,\ldots,\lambda_\ell)$ of k, let $\Sigma_\lambda:=\left\{\sum_{i\in A}\lambda_i : A\subseteq[\ell]\right\}$ denote its subset sum set.

Corollary 1.1. *Let $k\geq 2$, and let $t=t(k)$ be defined as in* (1.3)*. If λ is a partition of k satisfying $[t]\subseteq\Sigma_\lambda$, then $\pi(\Delta_\lambda)=k!/k^k$.*

In particular, this shows that $\pi(\Delta_{(6,2,1)})=9!/9^9$ and for any $q\leq k-t(k)$, $\pi(\Delta_{(q,1,\ldots,1)})=k!/k^k$.

2 The entropy method of Chao-Yu

In this section, we give a brief overview of the entropy method developed by Chao and Yu [1], emphasizing the results most relevant to our presentation.

We begin by introducing some fundamental concepts of entropy. For any discrete random variable X, we denote by $p_X(x)$ the probability $\mathbb{P}(X=x)$. The *Shannon entropy* of X is defined as

$$\mathbb{H}(X):=-\sum_{x\in\mathrm{supp}(X)}p_X(x)\cdot\log_2 p_X(x),$$

where $\mathrm{supp}(X)$ denotes the support of X, i.e., the set of all x with $p_X(x)>0$.

Let $X,Y,X_1,X_2,\ldots,X_n$ be discrete random variables. We write $\mathbb{H}(X_1,X_2,\ldots,X_n)$ for the entropy of the joint distribution of the random tuple $(X_1,X_2,\ldots,X_n)$. The *conditional entropy* of X given Y is defined as

$$\mathbb{H}(X\,|\,Y):=\mathbb{H}(X,Y)-\mathbb{H}(Y).$$

Let G be a k-graph. A k-tuple of random vertices $(X_1,\ldots,X_k)\in V(G)^k$ is a *random edge with uniform ordering* on G, if $(X_1,\ldots,X_k)$ is symmetric[†], and $\{X_1,\ldots,X_k\}$ is

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[†] That is, the distribution of $(X_{\sigma_1},\ldots,X_{\sigma_k})$ is always the same for any permutation σ of $[k]$.

always an edge of G. Using this concept, Chao and Yu [1] established a novel characterization of Turán density. For two k-graphs F and G, a *homomorphism* from F to G is a mapping $f\colon V(F)\to V(G)$ such that for every edge e in F, its image $f(e)$ is an edge in G. Given a family $\mathcal{F}$ of k-graphs, we say a k-graph G is *$\mathcal{F}$-hom-free*, if there exists no homomorphism from any member $F\in\mathcal{F}$ to G.

Lemma 2.1 (Chao-Yu, [1, Corollary 5.6])**.** *For any family $\mathcal{F}$ of k-graphs, $\pi(\mathcal{F})$ is the supremum of $2^{\mathbb{H}(X_1,\ldots,X_k)-k\mathbb{H}(X_1)}$ for any random edge with uniform ordering $(X_1,\ldots,X_k)$ on any $\mathcal{F}$-hom-free k-graph G.*

The *ratio sequence* $(x_1,\ldots,x_k)$ of a random edge $(X_1,\ldots,X_k)$ with uniform ordering on G is given by $x_i=2^{\mathbb{H}(X_i|X_{i+1},\ldots,X_k)-\mathbb{H}(X_i)}$ for all $i\in[k]$. It is evident to see that $2^{\mathbb{H}(X_1,\ldots,X_k)-k\mathbb{H}(X_1)}=\prod_{i=1}^{k}x_i$ and $x_k=1$. The following result on ratio sequences is implicit in the proof of Lemma 7.2 of [1].

Lemma 2.2 (Chao-Yu, [1, Lemma 7.2])**.** *Let $1\le t\le\lfloor k/2\rfloor$. Let G be any $\mathcal{F}_k^{\le t}$-hom-free k-graph, and let $(x_1,\ldots,x_k)$ be the ratio sequence of any random edge with uniform ordering on G. Then $x_i+x_j\le x_{i+j}$ holds for all i,j with $i\in[t]$ and $1\le i+j\le k$.*

We also need the following well-known property on Turán densities.

Lemma 2.3. *Let $\mathcal{F},\mathcal{G}$ be two family of k-graphs. If for every $G\in\mathcal{G}$, there exists $F\in\mathcal{F}$ such that there exists a homomorphism from F to G, then $\pi(\mathcal{F})\le\pi(\mathcal{G})$.*

3 Proof of the results

The proof of our main theorem relies on the following key lemma. We note that the statement with $t(k)$ replaced by $\lceil k/e\rceil$ was proved by Il'kovič and Yan in [5, Theorem 3.1]. Here we provide a much shorter proof with a slightly better bound.

Lemma 3.1. *Let $k\ge 2$ and $t:=t(k)$ be defined as in* (1.3)*. Suppose that $y_1,\ldots y_k$, are some nonnegative real numbers with $y_i+y_j\le y_{i+j}$ for all $i\in[t]$ and $1\le i+j\le k$. Then $\prod_{i=1}^{k}y_i\le\frac{k!}{k^k}y_k^k$.*

We begin by showing how Lemma 3.1 implies Theorem 1.1.

Proof of Theorem 1.1. First, observe that any complete k-partite k-graph is $\mathcal{F}_k^{\le t}$-hom-free, which yields the lower bound $\pi(\mathcal{F}_k^{\le t})\ge k!/k^k$. Now, let G be any $\mathcal{F}_k^{\le t}$-free k-graph, and let $(x_1,\ldots,x_k)$ be the ratio sequence of any random edge with uniform ordering on G. Applying Lemmas 2.2 and 3.1 under the condition $x_k=1$, we deduce that $\prod_{i\in[k]}x_i\le k!/k^k$. By Lemma 2.1, this immediately gives $\pi(\mathcal{F}_k^{\le t})\le k!/k^k$, thereby completing the proof. $\square$

Using homomorphism properties of λ-tents, we derive Corollary 1.1 from Theorem 1.1 as follows.

Proof of Corollary 1.1. Let $k\geq 2$ and $t:=t(k)$ be defined as in (1.3). Let $\lambda=(\lambda_1,\dots,\lambda_\ell)$ be a partition of k such that $[t]\subseteq\Sigma_\lambda$. Recall that Δ_λ has an apex vertex v and edges $e,e_1,\dots,e_\ell$ satisfying $|e_j\cap e|=\lambda_j$ for each $j\in[\ell]$. The lower bound $\pi(\Delta_\lambda)\geq k!/k^k$ follows easily by the fact that any complete k-partite k-graph is Δ_λ-hom-free. For the upper bound, we claim that for any fixed $i\in[t]$, there exists a homomorphism from Δ_λ to the $(k-i,i)$-tent $\Delta_{(k-i,i)}\in\mathcal{F}_k^{\leq t}$. Since we have $[t]\subseteq\Sigma_\lambda$, there is a subset $A\subseteq[k]$ such that $\sum_{j\in A}\lambda_j=i$. According to the edge set (1.1) of $\Delta_{(k-i,i)}$, where $V(\Delta_{(k-i,i)})=[2k-1]$, we can define a map $f:e\cup\{v\}\to[k+1]$ by setting $f(v)=k+1$ and bijectively mapping e to $[k]$ such that $f(\bigcup_{j\in A}(e\cap e_j))=[i]$ and $f(\bigcup_{j\in[\ell]\setminus A}(e\cap e_j))=[k]\setminus[i]$. Since each vertex in $V(\Delta_\lambda)\setminus(e\cup\{v\})$ has degree exactly one, it is easy to extend f to a homomorphism $\tilde{f}:V(\Delta_\lambda)\to V(\Delta_{(k-i,i)})$, which preserves all edges. This proves the claim. By Lemma 2.3 and Theorem 1.1, we obtain the desired upper bound $\pi(\Delta_\lambda)\leq\pi(\mathcal{F}_k^{\leq t})=k!/k^k$. $\square$

It remains to prove Lemma 3.1. We start with the following technical lemma.

Lemma 3.2. *Let $k\geq 2$, and let $t=t(k)$ be defined as in* (1.3)*. Consider the multiset*

$$T=\left\{1^{(k!)},\ 2^{(k!/2)},\ \dots,\ k^{(k!/k)}\right\},$$

where each $i\in[k]$ appears with multiplicity $k!/i$. Then there exist $k!$ multisets $A_1,\dots,A_{k!}$ satisfying

1. *The family $\{A_i\}_{i=1}^{k!}$ forms a partition of the multiset T;*
2. *For each $i\in[k!]$, the summation (counting with multiplicities) $\omega(A_i):=\sum_{x\in A_i}x$ equals k;*
3. *For each $i\in[k!]$, A_i contains at most one element greater than t.*

Proof. Define $m:=\left(\sum\limits_{i=t+1}^{k}\frac{1}{i}\right)\cdot k!$ and $N:=\left(\sum\limits_{i=1}^{k}\frac{1}{i}\right)\cdot k!$. So $m\leq k!$, and the multiset T contains N elements, among which exactly m elements are strictly greater than t. We order the elements of $T=\{x_1,\dots,x_N\}$ such that $k=x_1\geq x_2\geq\cdots\geq x_m=t+1>t=x_{m+1}\geq\cdots\geq x_N=1$.

To obtain the desired multisets $A_1,\dots,A_{k!}$, we apply induction to construct a nested sequence $A_i^{(0)}\subseteq A_i^{(1)}\subseteq\cdots\subseteq A_i^{(N-m)}=A_i$ for each $i\in[k!]$, satisfying the following for all $0\leq s\leq N-m$:

- The family $\{A_i^{(s)}\}_{i=1}^{k!}$ forms a partition of the multiset $\{x_1,x_2,\dots,x_{m+s}\}$;

- For each $i\in[k!]$, the summation $\omega(A_i^{(s)})\le k$;
- Each $A_i^{(s)}$ contains at most one element greater than t.

We initialize the construction by defining $A_i^{(0)}=\begin{cases}\{x_i\} & \text{for } 1\le i\le m,\\ \emptyset & \text{for } m<i\le k!\end{cases}$.

To proceed, we assume that for some $0\le s\le N-m-1$, the collection $\{A_i^{(s)}\}_{i=1}^{k!}$ has been constructed to satisfy the inductive hypotheses.

Let the value of x_{m+s+1} be $z\ge 1$. We claim that there exists an index $p\in[k!]$ such that $\omega(A_p^{(s)})\le k-z$. Suppose, for contradiction, that $\omega(A_i^{(s)})\ge k+1-z$ for all $i\in[k!]$. By hypothesis, the sets $A_i^{(s)}$ form a partition of $\{x_1,\dots,x_{m+s}\}$, and $T\backslash\{x_1,\dots,x_{m+s},x_{m+s+1}\}$ contains exactly $\frac{k!}{j}$ copies of the element j for every $1\le j\le z-1$. So we have

$$k\cdot k!=\omega(T)=\sum_{i=1}^{k!}\omega(A_i^{(s)})+\sum_{i=m+s+1}^{N}x_i\ge k!\cdot(k+1-z)+z+\sum_{j=1}^{z-1}\frac{k!}{j}\cdot j=k\cdot k!+z>k\cdot k!,$$

which yields the desired contradiction. This proves the claim.

We can then define $A_i^{(s+1)}=\begin{cases}A_p^{(s)}\cup\{x_{m+s+1}\} & \text{if } i=p,\\ A_i^{(s)} & \text{otherwise}\end{cases}$. One can readily verify that these sets satisfy all inductive conditions. Finally, setting $A_i=A_i^{(N-m)}$ for all $i\in[k!]$. We observe that $\omega(A_i)\le k$ for each $i\in[k!]$ and $\sum\limits_{i=1}^{k!}\omega(A_i)=\omega(T)=k\cdot k!$. Consequently, it implies $\omega(A_i)=k$ for all $i\in[k!]$, thus proving Lemma 3.2. $\square$

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let T and $A_1,A_2,\dots,A_{k!}$ be the multisets given by Lemma 3.2. For any indices $1\le i_1,i_2,\dots,i_r\le t$ and $j\in[k]$ satisfying $i_1+i_2+\cdots+i_r+j\le k$, using the lemma condition, we have the chain of inequalities

$$y_{i_1}+y_{i_2}+\cdots+y_{i_r}+y_j\le y_{i_1}+y_{i_2}+\cdots+y_{i_{r-1}}+y_{i_r+j}\le\cdots\le y_{i_1+i_2+\cdots+i_r+j}.$$

Since each A_i contains at most one element greater than t, we obtain the bound $\sum_{j\in A_i}y_j\le y_{\omega(A_i)}=y_k$ for each $i\in[k!]$. Summing over all $i\in[k!]$, we further derive that

$$\sum_{i=1}^{k}\frac{y_i}{i}=\frac{1}{k!}\sum_{i=1}^{k}\frac{k!}{i}y_i=\frac{1}{k!}\sum_{j\in T}y_j=\frac{1}{k!}\sum_{i=1}^{k!}\left(\sum_{j\in A_i}y_j\right)\le y_k.$$

Applying the AM-GM inequality to this, we have $\prod_{i=1}^{k} y_i = k! \prod_{i=1}^{k} \frac{y_i}{i} \leq k! \left(\frac{\sum_{i=1}^{k} \frac{y_i}{i}}{k} \right)^k \leq \frac{k!}{k^k} y_k^k$. □

We remark that the constant $t(k)$ in Lemma 3.1 cannot be reduced, even by one. The construction is essentially identical to that of Lemma 3.10 in [5], so we omit the detailed proof. This establishes that the specific value $t(k)$ in Theorem 1.1 is indeed optimal for the underlying entropy method.

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References

[1] Ting-Wei Chao and Hung-Hsun Hans Yu. When entropy meets Turán: new proofs and hypergraph Turán results, 2024.

[2] Paul Erdős and Miklós Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1(51-57):51, 1966.

[3] Peter Frankl and Zoltán Füredi. A new generalization of the Erdős-Ko-Rado theorem. *Combinatorica*, 3(3):341–349, 1983.

[4] Peter Frankl and Vojtěch Rödl. Hypergraphs do not jump. *Combinatorica*, 4(2):149–159, 1984.

[5] Daniel Il'kovič and Jun Yan. An improved hypergraph Mantel's Theorem. *arXiv preprint arXiv:2503.14474*, 2025.

[6] Peter Keevash. Hypergraph Turán problems. *Surveys in combinatorics*, 392:83–140, 2011.

[7] Xizhi Liu. On a hypergraph Mantel theorem. *arXiv preprint arXiv:2501.19229*, 2025.

[8] Dhruv Mubayi. A hypergraph extension of Turán's theorem. *J. Combin. Theory Ser. B*, 96:122–134, 2006.

[9] Dhruv Mubayi and Oleg Pikhurko. A new generalization of Mantel's theorem to k-graphs. *Journal of Combinatorial Theory, Series B*, 97(4):669–678, 2007.

[10] Oleg Pikhurko. An exact Turán result for the generalized triangle. *Combinatorica*, 28:187–208, 2008.

[11] Paul Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.