Ann. Appl. Math.
doi: 10.4208/aam.OA-2025-0013

Vol. **41**, No. 3, pp. 302-312
August 2025

The 3-coloring of planar graphs with adjacent triangles

Zuosong Liang[*], Xin Xue

School of Mathematics, Guangxi Minzu University, Nanning 530006, China

Received 18 May 2025

Abstract. Erdős raised the following problem according to Steinberg's conjecture: Is there an integer such that every planar graph without cycles of length from 4 to k is 3-colorable. By far, the result about the problem was improved to $k \leq 7$ by Borodin et al. However, by permitting the existence of adjacent triangles except K_4, for an arbitrary integer $k \geq 5$, there exists a planar graph without cycles of length from 5 to k such that G is not 3-colorable. Let d denote the minimum distance between two diamonds in G, where a diamond is the union of two adjacent triangles. In this paper, we prove that a planar graph G with $d \geq 2$ and without cycles of length from 5 to 18 is 3-colorable. The reader is invited to find the smallest integer k such that a planar graph G with $d \geq 2$ and without cycles of length from 5 to k is 3-colorable.

AMS subject classifications: 05C15, 05C69

Key words: Three Color Problem; adjacent triangles; cycle; planar graph.

1 Introduction

The Three Color Problem of planar graphs is to ask whether a planar graph can be colored with three colors such that no two adjacent vertices are assigned the same color. In 1976, Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable. In 2017, Steinberg's conjecture was disproved by constructing a counterexample to the conjecture [6]. However, the question whether every planar graph without cycles of length from 4 to 6 is 3-colorable is still open. Erdős raised

---

[*] Corresponding author.

Email: `liangzuosong@126.com` (Z. Liang)

the following relaxation of Steinberg's conjecture: Is there an integer such that every planar graph without cycles of length from 4 to k is 3-colorable. Abbott and Zhou confirmed that such k exists and $k \leq 11$ [1]. This result was later improved to $k \leq 9$ independently by Borodin [2] and by Sanders and Zhao [8] , and to $k \leq 7$ by Borodin et al [5]. It was proved by Borodin et al [3] and independently by Xu [9] that every planar graph having neither 5- and 7-cycles nor adjacent 3-cycles is 3-colorable which further improved the results above. In 2017, Jin et al proved that plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable [7].

However, for an arbitrary integer $k \geq 5$, we can construct a planar graph G without K_4 and cycles of length from 5 to k such that G is not 3-colorable as follow. Let D_i be the union of two adjacent triangles $[w_iu_iv_i]$ and $[u_iv_iw_{i+1}]$ which share one public edge for $i=1,2,...,k$. Let G be the graph obtained from the union of D_i's by adding the edge w_1w_{k+1}. Obviously, G has neither K_4 nor cycles of length from 5 to $2k$ and G is not 3-colorable. Hence, it is an interesting thing to research the 3-coloring of planar graphs with adjacent triangles.

A *diamond* D is the union of two adjacent triangles $T_1=[wuv]$ and $T_2=[tuv]$ which share one public edge uv. Let d denote the minimum distance between two diamonds in G. In this paper, we prove that a planar graph G with $d \geq 2$ and without cycles of length from 5 to 18 is 3-colorable. The reader is invited to find the smallest integer k such that a planar graph G with $d \geq 2$ and without cycles of length from 5 to k is 3-colorable.

2 The 3-coloring of planar graphs with adjacent triangles

First, we introduce some notations as follow. A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *plane graph* is a planar graph G together with an embedding of G into the plane. Let G be a plane graph and C be a cycle of G. By $Int(C)$ (or $Ext(C)$), we denote the subgraph of G induced by the vertices lying inside (or outside) C. The cycle C is *separating* if neither $Int(C)$ nor $Ext(C)$ is empty. A plane graph G partitions the plane into a number of arcwise-connected open sets which are called the *faces* of G. Each plane graph has exactly one unbounded face, called the *exterior face* of G. Let a *k-face* be a face of degree k. Let a k^+-face be a face of degree at least k. The distance between two subgraphs H_1 and H_2 in G is the length of the shortest path between H_1 and H_2. By Euler's formula and the idea of discharging charge, we directly prove the following result.

Theorem 2.1. *Let G be a planar graph G with $d\geq 2$ and without cycles of length from* 5 *to* 18*. Then G is* 3*-colorable.*

Proof. First, G has no K_4 since a K_4 contains two different diamonds of distance less than 2. Suppose that G is a smallest counterexample to Theorem 2.1. Then G is 2-connected and $\delta(G)\geq 3$. In addition, G has no separating triangle. So every triangle is a 3-face of G in the embedding on the plane. Since G has no cycles of length from 5 to 18, a 3-face is not adjacent to a 4-face and there is no two adjacent 4-faces. Now we distinguish the vertices of degree 3 into the following types. A vertex v of degree 3 is called a vertex of type 1 if the faces around v are all 4^+-faces. Specially, if there is a 4-face around v, the other two faces are 19^+-faces. A vertex v of degree 3 is called a vertex of type 2 if the faces around v are two 19^+-faces and one 3-face. A vertex v of degree 3 is called a vertex of type 3 if the faces around v are two 3-faces and one 19^+-face.

By Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$, we further have that $\sum_{v\in V(G)}(d(v)-6)+\sum_{f\in F(G)}(2d(f)-6)=-12$. Give initial charge $ch(x)$ to each element $x\in V(G)\cup F(G)$, where $ch(x)=(2d(x)-6)$ for $x\in F(G)$ and $ch(x)=(d(x)-6)$ for $x\in V(G)$. We further have that $\sum_{x\in V(G)\cup F(G)}ch(x)=-12$.

Now we discharge the charge of the elements of $V\cup F$ according to the following rules.

Rule 1.1 (Face-to-vertex). For each face f of degree 19 or more, transfer a charge of $\frac{5}{2}$ to each of the vertices of type 3 on f and $\frac{3}{2}$ to each of the other vertices on f. For each face f of degree 4, transfer a charge of $\frac{1}{2}$ to each of the vertices on f.

Rule 1.2 (Vertex-to-vertex). For every vertex v of degree 4, v transfers a charge of $\frac{1}{2}$ to every neighbor of type 3. For every vertex v of degree at leas 5 and there are at least two vertices of type 3 in $N(v)$, v transfers a charge of $\frac{1}{2}$ to every neighbor of type 3. For every vertex v of degree at least 5 and there is only one vertex of type 3 in $N(v)$, v transfers a charge of 1 to the unique neighbor of type 3.

Rule 2 (5^+-vertex-to-face). For every vertex v of degree 5 or more, transfer a charge of $\frac{1}{2}$ to every incident face f of degree 19 or more.

Rule 3 (4-vertex-to-face). For every vertex v of degree 4, if there is no vertex of type 3 in $N(v)$, transfer a charge of $\frac{1}{2}$ to every incident 19^+-face. If there exists only one vertex of type 3 in $N(v)$ and there are three 19^+-faces around v, transfer a charge of $\frac{1}{2}$ to every incident 19^+-face. If there exists only one vertex of type 3 in $N(v)$ and there are two 19^+-faces around v, transfer a charge of $\frac{1}{4}$ to every incident 19^+-face. If there exist two vertices of type 3 in $N(v)$ and there are three 19^+-faces around v, transfer a charge of $\frac{1}{2}$ to every incident 19^+-face. If there exist two vertices of type 3 in $N(v)$ and there are two 19^+-faces and one 4-face around v, transfer a charge of $\frac{1}{4}$ to every incident 19^+-face. If there exist two vertices of type 3 in $N(v)$ and there are two 3-faces around v, transfer no charge to the incident faces.

Rule 4 (3-vertex-to-face). For every vertex v of type 1, if there is a 4-face around v, transfer a charge of $\frac{1}{4}$ to every incident 19^+-face. If there is no 4-face around v, transfer a charge of $\frac{1}{2}$ to every incident 19^+-face.

Rule 5 (3-vertex-to-face). For every vertex v of type 3, assume that v is in two adjacent triangles $T_1=[wuv]$ and $T_2=[tuv]$. Let l be the number of neighbors of v which have the degree at least 4. If $l\geq 2$, v transfers a charge of $\frac{1}{2}(l-1)$ to the incident 19^+-face. If $l=1$ and $d(u)\geq 5$, v transfers a charge of $\frac{1}{2}$ to the incident 19^+-face.

Let $ch^*(x)$ denote the value for every $x\in V(G)\cup F(G)$ after the adjusting on the counterexample G according Rules 1-5. We will check that $ch^*(x)\geq 0$ for every $x\in V(G)\cup F(G)$ as follow and thus get a contradiction.

Assume $v\in V(G)$. Let k be the number of faces of degree at least 19, l be the number of faces of degree 4 and m be the number of faces of degree 3 around v. Then we have that $k+l+m=d(v)$. If there exist two adjacent 3-faces $[vuw]$ and $[vut]$ around v, then there is a diamond containing v and we have that $k\geq l+m-1$ since the distance between any two diamonds of G is at least 2 and a 3-face is not adjacent to a 4-face. If there is not adjacent 3-faces around v, we have that $k\geq l+m$. Thus we get that $k\geq\lceil\frac{d(v)-1}{2}\rceil$.

Let n_v be the number of neighbors of v which are vertices of type 3. Further, we have that $n_v\leq 2$. Suppose not, there would be at least two diamonds containing v, a contradiction to our assumption. If $n_v=2$, the unique case is that there exists one 3-face $[vuw]$ such that both u and w are vertices of type 3. In this case, u and w are also in anther 3-face $[uwt]$ such that t is not a neighbor of v. Then there is no adjacent 3-faces around v which have a public edge since the distance between any two diamonds of G is at least 2. If $n_v=1$, there are two cases. One case is that there exist two adjacent 3-faces $[vuw]$ and $[vut]$ around v such that u is of a vertex of type 3. The other case is that there exists a 3-face $[vuw]$ such that only one in $\{u,w\}$ is a vertex of type 3. Then u and w are also in anther 3-face $[uwt]$ such that t is not a neighbor of v and the other one in $\{u,w\}$ is a vertex of degree at least 4.

So, if $d(v)\geq 5$, v transfers exactly 1 to the neighbor-vertices of type 3 by Rule 1 since $n_v\leq 2$. Thus we have that $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-1-k\times\frac{1}{2}\geq d(v)+k-7\geq d(v)+\lceil\frac{d(v)-1}{2}\rceil-7\geq 0$ by Rules 1 and 2. If $d(v)=4$ and $n_v=0$, we have that $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-k\times\frac{1}{2}=d(v)-6+k\geq d(v)-6+\lceil\frac{d(v)-1}{2}\rceil\geq 0$ by Rules 1 and 3. If $d(v)=4$ and $n_v=1$ and $k=3$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-1\times\frac{1}{2}-k\times\frac{1}{2}\geq 0$ by Rules 1 and 3. If $d(v)=4$ and $n_v=1$ and $k=2$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-1\times\frac{1}{2}-k\times\frac{1}{4}\geq 0$ by Rules 1 and 3. If $d(v)=4$, $n_v=2$ and $k=3$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-2\times\frac{1}{2}-k\times\frac{1}{2}\geq 0$ by Rules 1 and 3. If $d(v)=4$, $n_v=2$ and $k=2$ and $l=1$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}+l\times\frac{1}{2}-2\times\frac{1}{2}-k\times\frac{1}{4}\geq 0$ by Rules 1 and 3. If $d(v)=4$, $n_v=2$, $k=2$ and $m=2$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-2\times\frac{1}{2}\geq 0$ by Rules 1 and 3.

If v is a vertex of type 1 and $k=3$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}-k\times\frac{1}{2}\geq 0$ by Rules 1 and 4. If v is a vertex of type 1 and $k=2$ and $l=1$, $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}+l\times\frac{1}{2}-k\times\frac{1}{4}\geq 0$ by Rules 1 and 4. If v is a vertex of type 2, we can easily see that $k=2$ and thus $ch^*(v)\geq d(v)-6+k\times\frac{3}{2}\geq 0$ by Rule 1.

If v is a vertex of type 3, we will discuss the value of $ch^*(v)$ as follow. Let u, w and t be the neighbors of v. Without loss of generality, assume that $[wuv]$ and $[tuv]$ be the two faces incident to v. Then w is not adjacent to t. If $d(u)\geq 4$ or $d(w)\geq 4$ or $d(t)\geq 4$, then $ch^*(v)\geq d(v)-6+\frac{5}{2}+\frac{1}{2}=0$ by Rule 1. So we may assume that $d(u)=d(w)=d(t)=3$, then we can easily get a 3-coloring of G from a 3-coloring of $G-N[v]$, a contradiction.

Assume f is a face of G. If $d(f)=3$ or $d(f)=4$, we can easily see that $ch^*(f)\geq 0$ by Rule 1. If $d(f)\geq 20$, there are at most $\lfloor\frac{d(f)}{4}\rfloor$ vertices of type 3 on f since the distance of two diamonds is at least 2 and thus the distance of two vertices of type 3 on f is at least 4. Thus $ch^*(f)\geq 2d(f)-6-\frac{5}{2}\times\lfloor\frac{d(f)}{4}\rfloor-\frac{3}{2}\times\lceil\frac{3d(f)}{4}\rceil\geq 0$ when $d(f)\geq 22$. If $20\leq d(f)\leq 21$, there are at most $\lfloor\frac{d(f)}{4}\rfloor=5$ vertices of type 3 on f. If there are at most 4 vertices of type 3 on f, then $ch^*(f)\geq 2d(f)-6-\frac{5}{2}\times 4-\frac{3}{2}\times(d(f)-4))\geq 0$. So we may assume there are exactly 5 vertices of type 3 on f when $20\leq d(f)\leq 21$.

If $d(f)=20$ and there are 5 vertices of type 3 on f, it is the case in Fig.1 in which the five vertices are v_2, v_5, v_8, v_{11} and v_{14}. We consider the five vertices w_1, w_2, w_3, w_4 and w_5. Obviously, w_i is not in a diamond of G and thus has no neighbor of type 3 by our assumption. If there are at least two vertices in $\{w_1,w_2,w_3,w_4,w_5\}$ of degree at least four, $ch^*(f)\geq 2d(f)-6-\frac{5}{2}\times 5-\frac{3}{2}\times 15+2\times\frac{1}{2}\geq 0$ by Rules 1 and 3.

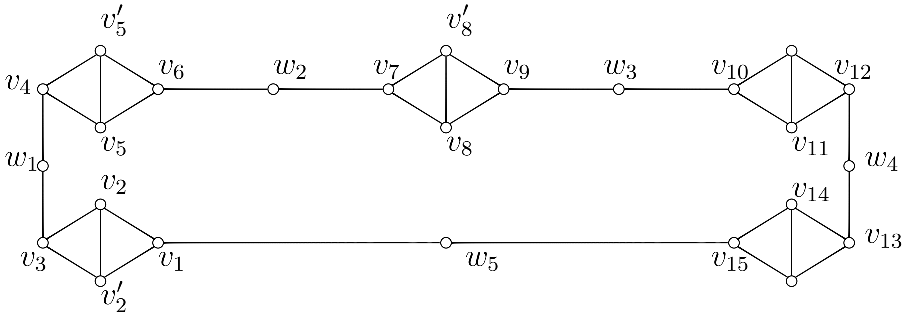

Figure 1: A 20-face f with 5 vertices of type 3

If $d(w_i)=3$ for $i=1,2,3,4,5$, we consider the forms of $G[N[f]]$ as follow. First, if $d(v_1)=d(v_3)=d(v_4)=d(v_6)=d(v_7)=d(v_9)=d(v_{10})=d(v_{12})=d(v_{13})=d(v_{15})=3$, all the vertices in $\{w_1,w_2,w_3,w_4,w_5\}$ are of type 1 and thus $ch^*(f)\geq 0$ by Rules 1 and 4. Without the loss of generality, assume that $d(v_1)\geq 4$. If $d(v_1)\geq 5$ or $d(v_1)=4$ and there are three 19^+-faces around v_1, then v_1 transfers a charge of $\frac{1}{2}$ to the incident face f by Rules 2 and 3. Further, we may assume that $d(v_3)=d(v_2')=3$. Suppose not, the number of neighbors of v_2 with the degree at least 4 is at least 2 and we get that $ch^*(f)\geq 0$ by Rules 1, 2, 3 and 5. Further, we may assume that w_1 is of type 2. If w_1 is of type 1 and there is no 4-face around w_1, w_1 transfers a charge of $\frac{1}{2}$ to the incident face f and we get that $ch^*(f)\geq 0$. If there is a 4-face around w_1, w_1 transfer a charge of $\frac{1}{4}$ to the incident face f. Then $d(v_4)\geq 4$ and v_4 transfer a charge of at least $\frac{1}{4}$ to the incident face f by Rule 3 and hence $ch^*(f)\geq 0$. Thus we may assume that w_1 is of type 2, $d(v_4)=4$ and there are two non-adjacent 3-faces around v_4 as in Fig.2. In the same way, we can also assume that w_2, w_3 and w_4 are vertices of type 2, $d(v_7)=d(v_{10})=d(v_{13})=4$ and $d(v_5')=d(v_6)=d(v_8')=d(v_9)=d(v_{11}')=d(v_{12})=d(v_{14}')=d(v_{15})=3$ (see Fig.2). In this case, we consider the graph G' obtained from G by deleting the edge $v_{15}w_5$. By the minimality, G' has a 3-coloring ϕ such that $\phi(v_{15})=\phi(w_5)$. If $\phi(v_{15})\neq\phi(w_5)$, then G is 3-colorable, a contradiction. We may assume that $\phi(v_{15})=\phi(w_5)=1$. Further, we have that $\phi(v_{13})=1$ since both v_{15} and v_{13} receive the same color. Then we claim that $\phi(v_{12})=\phi(v_{10})=1$. If $\phi(v_{12})=\phi(v_{10})=3$, then $\phi(w_4)=2$. By setting $\phi(w_4)=1$, $\phi(v_{15})=\phi(v_{13})=2$, $\phi(v_{14})=1$ and $\phi(v_{14}')=3$, we get a 3-coloring of G. If $\phi(v_{12})=\phi(v_{10})=2$, then $\phi(w_4)=3$. By setting $\phi(w_4)=1$, $\phi(v_{15})=\phi(v_{13})=3$, $\phi(v_{14})=1$ and $\phi(v_{14}')=2$, we still get a 3-coloring of G. In the same way, we get that $\phi(v_7)=\phi(w_9)=1$. Further we get that $\phi(v_4)=\phi(w_6)=\phi(v_3)=\phi(v_1)=1$. Then v_1 and w_5 receive the same color in the 3-coloring of G', a contradiction. So we may assume that $d(v_i)\leq 4$ and, if $d(v_i)=4$, there are two 19^+-faces around v_i for $i=1,3,4,6,7,9,10,12,13,15$.

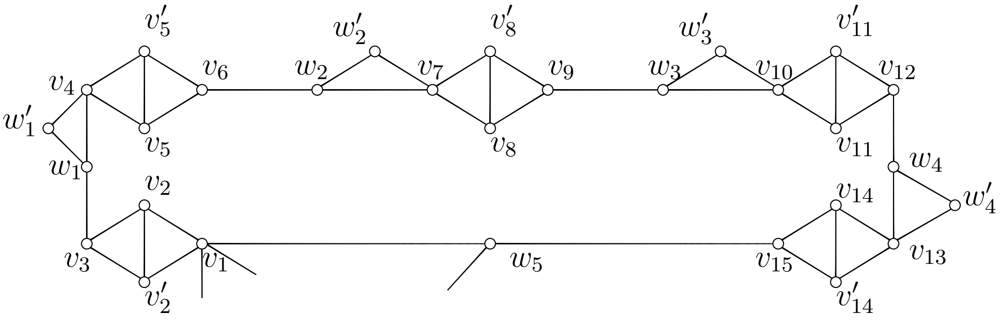

Figure 2: A 20-face such that $d(v_1)\geq 5$ or $d(v_1)=4$ and there are three 19^+-faces around v_1

If $d(v_1)=4$ and there is one 4-face and two 19^+-faces around v_1, then w_5 is a vertex of type 1 and there is a 4-face around w_5. At this time, both v_1 and w_5 transfer a charge of $\frac{1}{4}$ to the incident faces by R3 and R4. Then we may assume that $d(v_3)=d(v_2')=3$. Suppose not, the number of neighbors of v_2 with the degree at least 4 is at least 2 and we will get $ch^*(f)\geq 0$ by rules. Further, we may assume that w_1 is of type 2. If w_1 is of type 1 and there is no 4-face around w_1, w_1 transfer a charge of $\frac{1}{2}$ to the incident face f and thus $ch^*(f)\geq 0$. If there is a 4-face around w_1, w_1 transfer a charge of $\frac{1}{4}$ to the incident face f. Then $d(v_4)=4$ and v_4 transfer a charge of $\frac{1}{4}$ to the incident face f and $ch^*(f)\geq 0$. Hence, we assume that w_1 is of type 2 and we get that $d(v_4)=4$. In the same way, we may assume that w_2, w_3 and w_4 are vertices of type 2, $d(v_7)=d(v_{10})=d(v_{13})=4$ and $d(v_5')=d(v_6)=d(v_8')=d(v_9)=d(v_{11}')=d(v_{12})=d(v_{14}')=d(v_{15})=3$ (see Fig.3). In this case, we consider the graph G' obtained from G by deleting the edge $v_{15}w_5$. As the proof in the case $d(v_1)\geq 5$ or $d(v_1)=4$ and there are three 19^+-faces around v_1, we can still get a contradiction.

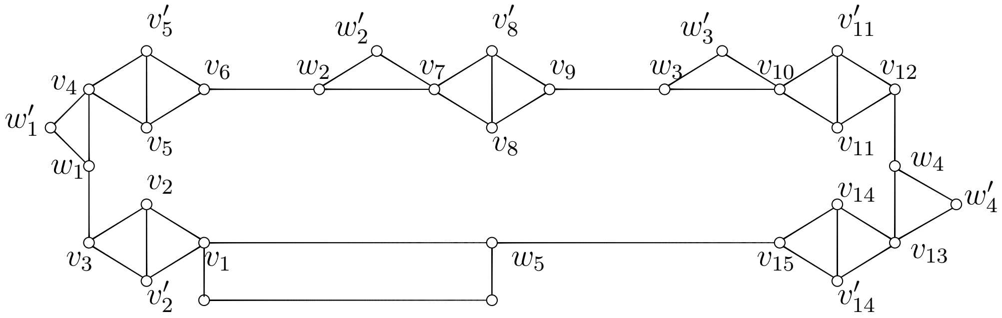

Figure 3: A 20-face f such that $d(v_1)=4$ and there is a 4-face around v_1

So we may assume that $d(v_i)\leq 4$ and, if $d(v_i)=4$, there are two 3-faces and two 19^+-faces around v_i for $i=1,3,4,6,7,9,10,12,13,15$ by symmetry. Let $d(v_1)=4$ and there are two 3-faces around v_1, then w_5 is a vertex of type 2. Further, if $d(v_2')\geq 4$, then $N(v_1)$ has only one vertex of type 3 and v_1 transfer a charge of $\frac{1}{4}$ to the incident face f by Rule 3. In addition, the number of neighbors of v_2 with the degree at least 4 is at least 2. So we may assume that $d(v_3)=3$ and thus v_2 transfer a charge of $\frac{1}{2}$ to the incident face f. In the following, as the proof above, we just consider the form of $G[N[f]]$ in Fig.4 and we still get a contradiction. In the following, we may assume that $d(v_2')=3$. If $d(v_3)=4$, by our assumption, there are two 3-faces around v_3 and v_2 transfers a charge of $\frac{1}{2}$ to the incident face f. If v_5 or v_8 or v_{11} or v_{14} transfer a charge of $\frac{1}{2}$ to the incident face f, we have that $ch^*(f)\geq 0$. So we just consider the situation in which v_5, v_8, v_{11} and v_{14} transfer no charge to the incident face f. Then, for every diamond of two adjacent triangles $[v_{i-1}v_iv_i']$ and $[v_iv_i'v_{i+1}]$ for $i=5,8,11,14$, it must be $d(v_{i-1})+d(v_{i+1})=7$ by Rule 5 as follow. If $d(v_{i-1})+d(v_{i+1})=6$ and $d(v_i)\leq 4$, we can get a 3-coloring of G from the 3-coloring of $G-N[v_i]$. If $d(v_{i-1})+d(v_{i+1})=6$ and $d(v_i)\geq 5$, then v_i transfer $\frac{1}{2}$ to the incident face f by Rule 5. Then such $G[N[f]]$

does not exist under the assumption that $d(v_{i-1})+d(v_{i+1})=7$ for $i=5,8,11,14$ and $d(w_1)=d(w_2)=d(w_3)=d(w_4)=d(w_5)=3$. So we assume that $d(v_3)=3$. By symmetry, we assume that, for every diamond of two adjacent triangles $[v_{i-1}v_iv_i']$ and $[v_iv_i'v_{i+1}]$ for $i=5,8,11,14$, it must be $d(v_{i-1})+d(v_{i+1})\leq 7$. Then we just consider the form of $G[N[f]]$ in Fig.5 by the assumption that there are two 3-faces around v_i if $d(v_i)=4$ for $i=1,3,4,6,7,9,10,12,13,15$. Then we can also get a contradiction by considering the graph $G-w_5v_{15}$ as above.

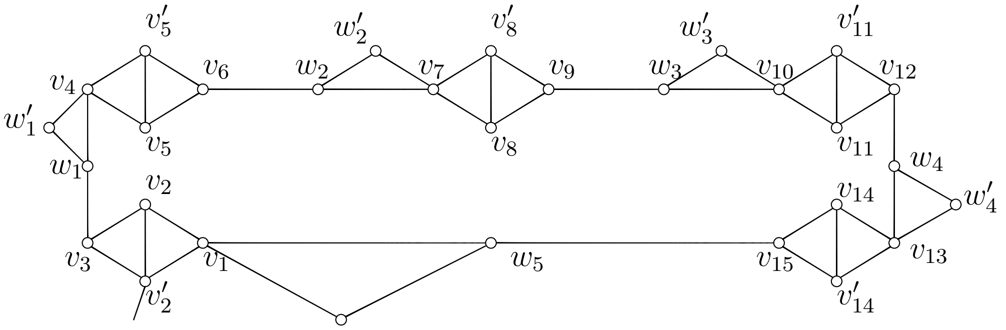

Figure 4: A 20-face f such that $d(v_1)=4$ and there are two 3-faces around v_1 and $d(v_2')\geq 4$

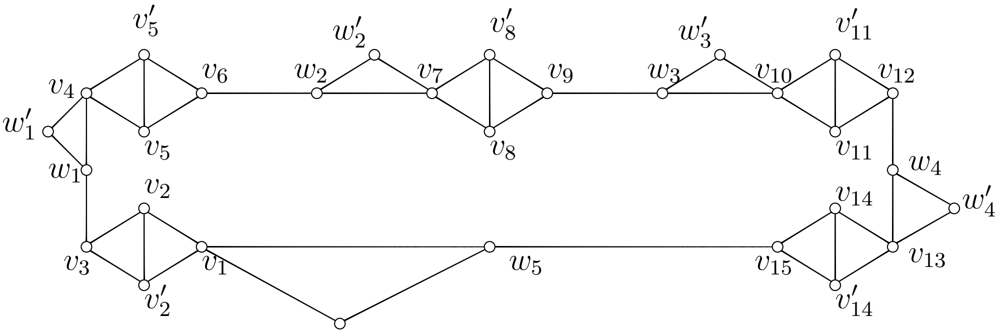

Figure 5: A 20-face f such that $d(v_1)=4$ and there are two 3-faces around v_1 and $d(v_2')=d(v_3)=3$

If there is only one vertex in $\{w_1,w_2,w_3,w_4,w_5\}$ with the degree at least four, assume that $d(w_1)=d(w_2)=d(w_3)=d(w_4)=3$ and $d(w_5)\geq 4$. We can also get a contradiction using the idea as above.

In the following, we consider the case that $d(f)=21$ and there are 5 vertices of type 3 on f. Then, by symmetry, the form of f is Fig.6 in which the five vertices are v_2, v_5, v_8, v_{11}, v_{14}. We consider the six vertices w_1, w_2, w_3, w_4, w_5 and w_6. Obviously, w_i is not in a diamond of G and thus has no neighbor of type 3. So, if there is at least one vertex in $\{w_1,w_2,w_3,w_4,w_5,w_6\}$ which has degree at least four,

$ch^*(f)\geq 2d(f)-6-\frac{5}{2}\times 5-\frac{3}{2}\times 16+1\times\frac{1}{2}\geq 0$ by Rules 1 and 3. So we may assume that every vertex in $\{w_1,w_2,w_3,w_4,w_5,w_6\}$ has degree three. Then every vertex in $\{w_1,w_2,w_3,w_4,w_5,w_6\}$ is a vertex of type 1 or 2. If w_1 is a vertex of type 1 and there is no 4-face around w_1, we get that $ch^*(f)\geq 0$. If w_1 is a vertex of type 1 and there is a 4-face around w_1, then either $d(v_3)\geq 4$ or $d(v_4)\geq 4$, we still get that $ch^*(f)\geq 0$. If w_6 is a vertex of type 1 and there is no 4-face around w_6, we get that $ch^*(f)\geq 0$. If w_6 is a vertex of type 1 and there is a 4-face around w_6, then either $d(v_1)\geq 4$ or w_5 is a vertex of type 1, we still get that $ch^*(f)\geq 0$. So we may assume that every vertex in $\{w_1,w_2,w_3,w_4,w_5,w_6\}$ is of type 2. In addition, for $i\in\{1,4,7,10,13\}$, if both v_i and v_{i+2} have degree at least four, we still get that $ch^*(f)\geq 0$. So we assume that $d(v_i)=3$ or $d(v_{i+2})=3$ for $i\in\{1,4,7,10,13\}$. Then, according to our assumption, such graph does not exist.

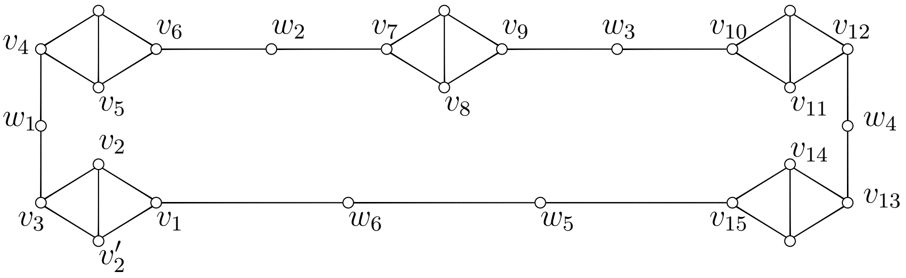

Figure 6: A 21-face f with 5 vertices of type 3

If $d(f)=19$, there are at most $\lfloor\frac{d(f)}{4}\rfloor=4$ vertices of type 3 on f since the distance of two diamonds is at least 2 and thus the shortest distance of two vertices of type 3 on f is at least 4. If there are at most 3 vertices of type 3 on f, we have that $ch^*(f)\geq 2d(f)-6-\frac{5}{2}\times 3-\frac{3}{2}\times 16\geq 0$. Without the loss of generality, assume that there are exactly 4 vertices of type 3 on f. If there exist two vertices of type 3 on f such that the shortest distance between them is 7 along f, the unique form of f is Fig.7. We consider the degree of w_1. Obviously, w_i is not in a diamond for $i=1,2,3,4,7$ by our assumption that the distance of two diamonds is at least 2. So, if w_i is not a vertex of type 2 for $i=1,2,3,4,7$, we can easily get $ch^*(f)\geq 0$. Then we assume that w_i is a vertex of type 2 for $i=1,2,3,4,7$. In addition, we can also assume that $d(v_i)+d(v_{i+2})\leq 7$ for $i=1,4,7,10$. Under the assumption, by symmetry, we just consider the form of $G[N[f]]$ in Fig.8. Then, no matter $d(w_6)=3$ or $d(w_6)=4$, we can get that $ch^*(f)\geq 0$. If there exists no two vertices of type 3 on f such that the shortest distance between them is 7 along f, we can also get $ch^*(f)\geq 0$ or a contradiction. Finally, we get that $ch^*(x)\geq 0$ for every $x\in V(G)\cup F(G)$ and finish the proof. $\square$

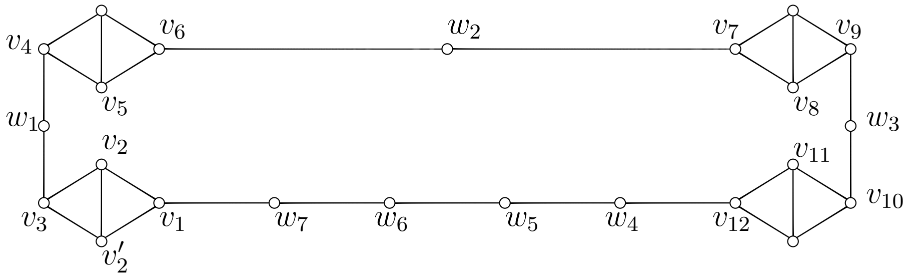

Figure 7: A 19-face f such that the distance between v_2 and v_{11} is 7

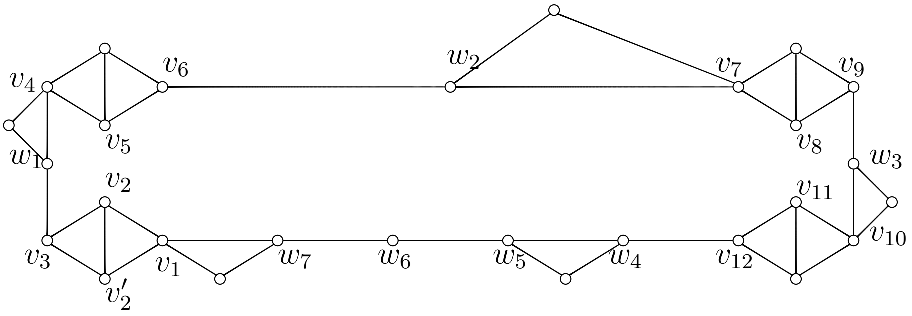

Figure 8: A 19-face f such that the distance between v_2 and v_{11} is 7 and w_i is a vertex of type 2 for $i=1,2,3,4,7$.

3 Conclusion

In this paper, we prove that a planar graph G without cycles of length from 5 to 18 and without two diamonds of distance less than 2 is 3-colorable, where a diamond is the union of two adjacent triangles. The reader is invited to find the smallest integer k such that a planar graph G without cycles of length from 5 to k and without two diamonds of distance less than 2 is 3-colorable.

Acknowledgments

All of the authors have not disclosed any competing interests. This research was partially supported by National Nature Science Foundation of China (No 12361067), Nature Science Foundation of Guangxi Province (No. 2024GXNSFAA010506), Guangxi college students' innovation and entrepreneurship project(S202410608013) and School-level talent programs (2022KJQD04 and2023MDKJ001).

References

[1] H. Abbott and B. Zhou, On small faces in 4-critical graphs, *Ars Combin.* 32 (1991), 203-207.

[2] O. V. Borodin, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, *J. Graph Theory* 21 (1996) 183-186.

[3] O. V. Borodin, A. N. Glebov, M. Montassier, and A. Raspaud, Planar graphs without 5- and 7-cycles and without adjacent triangles are 3-colorable, *J. Combin. Theory Ser. B* 99 (2009) 668-673.

[4] O. V. Borodin, A. N. Glebov, and A. Raspaud, Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable, *Discrete Math.* 310 (2010) 2584-2594.

[5] O. V. Borodin, A. N. Glebov, A. Raspaud, and M. R. Salavatipour, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, *J. Combin. Theory Ser. B* 93 (2005) 303-311.

[6] V. Cohen-Addad, M. Hebdige, D. Kral, Z. Li, and E. Salgado, Steinberg's conjecture is false, *J. Combin. Theory Ser. B* 122 (2017) 452-456.

[7] L. Jin, Y. Kang, M. Schubert and Y. Wang, Planar graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable, *SIAM J. Discrete Math.* 31 (2017) 1836-1847.

[8] D. P. Sanders and Y. Zhao, A note on the three color problem, *Graphs Combin.* 11 (1995) 91-94.

[9] B. Xu, On 3-colorable plane graphs without 5- and 7-cycles, *J. Combin. Theory Ser. B* 96 (2006) 958-963.