

A Criterion for Andrásfai–Erdős–Sós Type Theorems and Applications

Jianfeng Hou^{1,*}, Xizhi Liu² and Hongbin Zhao¹

¹ *Center for Discrete Mathematics, Fuzhou University, Fuzhou 350003, China*

² *School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China*

Received 6 June 2025

Abstract. The classical Andrásfai–Erdős–Sós Theorem states that for $\ell \geq 2$, every n -vertex $K_{\ell+1}$ -free graph with minimum degree greater than $\frac{3\ell-4}{3\ell-1}n$ must be ℓ -partite. We establish a simple criterion for r -graphs, $r \geq 2$, to exhibit an Andrásfai–Erdős–Sós type property, also known as degree-stability. This leads to a classification of most previously studied hypergraph families with this property. An immediate application of this result, combined with a general theorem by Keevash–Lenz–Mubayi, solves the spectral Turán problems for a large class of hypergraphs.

We show an interesting application of the degree-stability in Complexity Theory: For every r -graph F with degree-stability, there is a simple algorithm to decide the F -freeness of an n -vertex r -graph with minimum degree greater than $(\pi(F) - \varepsilon_F) \binom{n}{r-1}$ in time $O(n^r)$, where $\varepsilon_F > 0$ is a constant. In particular, for the complete graph $K_{\ell+1}$, we can take $\varepsilon_{K_{\ell+1}} = (3\ell^2 - \ell)^{-1}$, and this bound is tight up to some multiplicative constant factor unless $\mathbf{W}[1] = \mathbf{FPT}$. Based on a result by Chen–Huang–Kanj–Xia, we further show that for every fixed $C > 0$, this problem cannot be solved in time $n^{o(\ell)}$ if we replace $\varepsilon_{K_{\ell+1}}$ with $(C\ell)^{-1}$ unless \mathbf{ETH} fails. Furthermore, we apply the degree-stability of $K_{\ell+1}$ to decide the $K_{\ell+1}$ -freeness of graphs whose size is close to the Turán bound in time $(\ell+1)n^2$, partially improving a recent result by Fomin–Golovach–Sagunov–Simonov.

As an intermediate step, we show that for a specific class of r -graphs F , the (surjective) F -coloring problem can be solved in time $O(n^r)$, provided the input r -graph has n vertices and a large minimum degree, refining several previous results.

* *Emails:* jfhou@fzu.edu.cn (J. Hou), xizhi.liu.ac@gmail.com (X. Liu), hbzhao2024@163.com (H. Zhao)

AMS: 05C35, 68R10, 68Q17

Key words: Andrásfai–Erdős–Sós Theorem, testing F -freeness, homomorphism, parameterized algorithms, Exponential Time Hypothesis (**ETH**).

1 Introduction

Fix an integer $r \geq 2$, an r -graph \mathcal{H} is a collection of r -subsets of some finite set V . We identify a hypergraph[†] \mathcal{H} with its edge set and use $V(\mathcal{H})$ to denote its vertex set. The size of $V(\mathcal{H})$ is denoted by $v(\mathcal{H})$. For a vertex $v \in V(\mathcal{H})$, the **degree** $d_{\mathcal{H}}(v)$ of v in \mathcal{H} is the number of edges in \mathcal{H} containing v . We use $\delta(\mathcal{H})$, $\Delta(\mathcal{H})$, and $d(\mathcal{H})$ to denote the **minimum degree**, the **maximum degree**, and the **average degree** of \mathcal{H} , respectively. We will omit the subscript \mathcal{H} if it is clear from the context.

Given a family \mathcal{F} of r -graphs, we say \mathcal{H} is **\mathcal{F} -free** if it does not contain any member of \mathcal{F} as a subgraph. The **Turán number** $\text{ex}(n, \mathcal{F})$ of \mathcal{F} is the maximum number of edges in an \mathcal{F} -free r -graph on n vertices. The **Turán density** of \mathcal{F} is defined as $\pi(\mathcal{F}) := \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{F}) / \binom{n}{r}$, the existence of the limit follows from a simple averaging argument of Katona–Nemetz–Simonovits [47]. We say a family \mathcal{F} is **nondegenerate** if $\pi(\mathcal{F}) > 0$. The study of $\text{ex}(n, \mathcal{F})$ has been a central topic in extremal graph and hypergraph theory since the seminal work of Turán [81] who proved that $\text{ex}(n, K_{\ell+1}) = \lfloor \frac{\ell-1}{2\ell} n^2 \rfloor$ for all $n \geq \ell \geq 2$ (with the case $\ell = 2$ proved even earlier by Mantel [62]). We refer the reader to surveys [29, 48, 75] for related results.

1.1 Degree-stability

In this subsection, we focus on the structure of dense F -free hypergraphs. Here, dense can refer to either a large number of edges or a large minimum degree. This important topic in Extremal Combinatorics traces its origins to the seminal work of Simonovits [78], who proved that for $\ell \geq 2$, every $K_{\ell+1}$ -free n -vertex graph with at least $\frac{\ell-1}{2\ell} n^2 - o(n^2)$ edges can be made ℓ -partite by removing $o(n^2)$ edges (see [30] for a concise proof). The classical Andrásfai–Erdős–Sós Theorem [4] demonstrates an even stronger stability for $K_{\ell+1}$ -free graphs. It states that for $\ell \geq 2$, every n -vertex $K_{\ell+1}$ -free graph with minimum degree greater than $\frac{3\ell-4}{3\ell-1} n$ must be ℓ -partite.

Let $r \geq 2$ be an integer, \mathcal{F} be a nondegenerate family of r -graphs, and \mathfrak{H} be a family of \mathcal{F} -free r -graphs. We say

[†]A graph is viewed as a 2-uniform hypergraph.

- \mathcal{F} is **edge-stable** with respect to \mathfrak{H} if for every $\delta > 0$, there exist $\varepsilon > 0$ and n_0 such that every \mathcal{F} -free r -graph \mathcal{H} on $n \geq n_0$ vertices with $|\mathcal{H}| \geq (\pi(F)/r! - \varepsilon)n^r$ becomes a member in \mathfrak{H} after removing at most δn^r edges,
- \mathcal{F} is **degree-stable** with respect to \mathfrak{H} if there exist $\varepsilon > 0$ and n_0 such that every \mathcal{F} -free r -graph \mathcal{H} on $n \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(F)/(r-1)! - \varepsilon)n^{r-1}$ is a member in \mathfrak{H} ,
- \mathcal{F} is **vertex-extendable** with respect to \mathfrak{H} if there exist $\varepsilon > 0$ and n_0 such that for every \mathcal{F} -free r -graph \mathcal{H} on $n \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(F)/(r-1)! - \varepsilon)n^{r-1}$ the following holds: if $\mathcal{H} - v$ is a member in \mathfrak{H} , then \mathcal{H} is a member in \mathfrak{H} as well.

Vertex-extendability was introduced in [60] to provide a unified framework for proving the degree-stability of certain classes of nondegenerate families of hypergraphs. However, one limitation of the main results in [60] is that they only apply to families \mathcal{F} that are either blowup-invariant (see [60, Theorem 1.7]) or have a strong stability called vertex-stability (see [60, Theorem 1.8]). In the following theorem, we extend the results of [60] to include the broader class of families with edge-stability, a property satisfied by almost all nondegenerate hypergraph families (with known Turán densities).

Theorem 1.1. *Let \mathcal{F} be a nondegenerate family of r -graph and \mathfrak{H} be a hereditary[‡] class of \mathcal{F} -free r -graphs. If \mathcal{F} is both edge-stable and vertex-extendable with respect to \mathfrak{H} , then \mathcal{F} is degree-stable with respect to \mathfrak{H} .*

Recall that an r -**multiset** is an unordered collection of r elements with repetitions allowed. The **multiplicity** $e(i)$ of i in a multiset e is the number of times that i appears. An r -**pattern** is a pair $P = (\ell, E)$ where ℓ is a positive integer and E is a collection of r -multisets on $[\ell]$. It is clear that pattern is a generalization of r -graph, since an r -graph is a pattern in which E consists of only simple r -sets. For convenience, we call E the edge set of P and omit the first coordinate ℓ if it is clear from the context. An r -graph \mathcal{H} is P -**colorable** if there exists a **homomorphism** ϕ from \mathcal{H} to P , where homomorphism means $\phi(e) \in P$ for every $e \in \mathcal{H}$.

For many r -graph families \mathcal{F} , extremal \mathcal{F} -free constructions are typically P -colorable for some specific pattern P . We refer to such a pair (\mathcal{F}, P) as a **Turán pair**. More specifically, given a pair (F, P) , where P is an r -uniform pattern and F is a family of r -graphs, we say (F, P) is a Turán pair if every P -colorable r -graph is F -free and every n -vertex F -free r -graph with $\text{ex}(n, F) - o(n^r)$ edges is P -colorable after removing at most $o(n^r)$ edges. In most cases, applying Theorem 1.1 involves

[‡]A family \mathfrak{H} is hereditary if all subgraphs of every member $H \in \mathfrak{H}$ are also contained in \mathfrak{H} .

choosing \mathfrak{H} as the collection of all P -colorable hypergraphs. Hence, when stating that \mathcal{F} is edge-stable/degree-stable/vertex-extendable, it is implied that this property is with respect to the family of P -colorable hypergraphs for simplicity.

Table 1: List of hypergraphs with and without degree-stability.

Hypergraph	Degree-stable?
Edge-critical graphs [4, 20]	Yes
Non-edge-critical graphs [60]	No
Expansion of edge-critical graphs [56, 60, 65, 72]	Yes
Expansion of non-edge-critical graphs	No
Expansion of extended Erdős–Sós tree [10, 69, 77]	Yes
Expansion of M_2^r for $r \geq 3$ [5, 39, 60]	Yes
Expansion of M_k^3 , L_k^3 , or L_k^4 for $k \geq 2$ [39, 45, 60]	Yes
Expansion of M_k^4 for $k \geq 2$ [84]	Yes
Expansion of $K_4^3 \sqcup K_3^3$ [83]	Yes
Generalized triangle \mathbb{T}_r for $r \in \{3, 4\}$ [9, 26, 50, 56, 60, 71, 76]	Yes
Generalized triangle \mathbb{T}_r for $r \in \{5, 6\}$ [27, 71]	No
Weak triangle Δ_r [57]	Yes
Expanded triangle C_3^{2r} [25, 51]	No
Fano Plane [15, 34, 52]	Yes
$\mathbb{F}_{3,2}$ (3-book with 3 pages) [32]	No
F_7 (4-book with 3 pages) [33]	Yes
$\mathbb{F}_{4,3}$ (4-book with 4 pages) [31]	No
C_5^{3-} (tight 5-cycle minus one edge) [7, 54]	No
$\{K_4^3, C_5^3\}$ (tetrahedron and the tight 5-cycle) [8]	No

In Table 1, we summarize (most of) the previously studied hypergraph families (their definitions are included in the Appendix.) with degree-stability. Since they are all edge-stable, according to Theorem 1.1, proving degree-stability is reduced to verifying their vertex-extendability. This verification is relatively straightforward, and we refer the reader to [42, 60] for systematic results on this property.

Proof for Theorem 1.1 is presented in Section 2.

1.2 Spectral Turán problems

In this subsection, we show a quick application of Theorem 1.1 in spectral Turán problems.

Given an r -graph \mathcal{H} on $[n]$, the **Lagrange polynomial** of \mathcal{H} is defined as

$$\Lambda_{\mathcal{H}}(X_1, \dots, X_n) := \sum_{E \in \mathcal{H}} \prod_{i \in E} X_i.$$

For every real number $\alpha \geq 1$, the α -spectral radius of \mathcal{H} is

$$\lambda_{\mathcal{H},\alpha} := \max \{ \Lambda_{\mathcal{H}}(x_1, \dots, x_n) : (x_1, \dots, x_n) \in \mathbb{S}_{\alpha}^{n-1} \},$$

where $\mathbb{S}_{\alpha}^{n-1} := \{ (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n : x_1^{\alpha} + \dots + x_n^{\alpha} = 1 \}$. The spectral Turán problem studies, for fixed family \mathcal{F} of r -graphs, the value

$$\text{specex}_{\alpha}(n, \mathcal{F}) := \max \{ \lambda_{\mathcal{G},\alpha} : \mathcal{G} \text{ is an } n\text{-vertex } \mathcal{F}\text{-free } r\text{-graph} \}.$$

This problem was the focus of long-term research by Nikiforov (see the survey [68]) and is relatively well-understood for graphs, with the general theorem by Wang–Kang–Xue [82] being a notable highlight. However, similar to the ordinary Turán problem, understanding of the hypergraph spectral Turán problem remains relatively limited, with only a few known examples. By combining Theorem 1.1 with a general theorem of Keevash–Lenz–Mubayi [49], we derive the following theorem, significantly enhancing our understanding of the hypergraph spectral Turán problem.

Theorem 1.2. *Suppose that F is an r -graph in Table 1 with degree-stability. Then for sufficiently large n , every F -free r -graph \mathcal{H} on n vertices satisfies*

$$\lambda_{\mathcal{H},\alpha} \leq \max \{ \lambda_{\mathcal{G},\alpha} : \mathcal{G} \text{ is } P\text{-colorable and } v(\mathcal{G}) = n \},$$

where P is the pattern such that (F, P) is a Turán pair. Moreover, equality holds only if \mathcal{H} is P -colorable.

Theorem 1.2 follows relatively straightforwardly from the following theorem by Keevash–Lenz–Mubayi [49]. For convenience, we use $x = y \pm \delta$ to represent that $y - \delta \leq x \leq y + \delta$. Given a family \mathfrak{H} of r -graphs, we let

$$\lambda_{\alpha}(\mathfrak{H}, n) := \max \{ \lambda_{\mathcal{G},\alpha} : \mathcal{G} \in \mathfrak{H} \text{ and } v(\mathcal{G}) = n \}.$$

Theorem 1.3 (Keevash–Lenz–Mubayi [49]). *Let $N \geq r \geq 2$, $\alpha > 1$, $\varepsilon > 0$, \mathcal{F} be a nondegenerate family of r -graphs, and \mathfrak{H} be a hereditary family of \mathcal{F} -free r -graphs. There exist $\delta > 0$ and $n_0 > N$ such that the following holds. Suppose that*

1. \mathcal{F} is degree-stable with respect to \mathfrak{H} ,
2. $\text{ex}(n, \mathcal{F}) - \text{ex}(n-1, \mathcal{F}) = \pi(\mathcal{F}) \binom{n}{r-1} \pm \delta n^{r-1}$ for all $n \geq N$, and
3. $\lambda_{\alpha}(\mathfrak{H}, n) = \frac{r! \cdot \text{ex}(n, \mathcal{F})}{n^{r/\alpha}} \pm \delta n^{r-r/\alpha-1}$ for all $n \geq N$.

Then $\text{specex}_{\alpha}(n, \mathcal{F}) = \lambda_{\alpha}(\mathfrak{H}, n)$ for every $n \geq n_0$.

In the special case where \mathfrak{H} is the collection of all P -colorable r -graphs for some pattern P , Assumption 2 in Theorem 1.3 is automatically true due to [42, Theorem 1.9]. Therefore, in this case, one only needs to verify Assumption 3, which involves calculating the maximum α -spectral radius of n -vertex P -colorable r -graphs. This task is generally non-trivial. However, for r -graphs in Table 1 with degree-stability, Assumption 3 can be verified either using theorems from [46] or through calculations similar to those in the proof of [49, Corollary 1.6]. Hence, we omit the proof for Theorem 1.2.

1.3 Deciding the F -freeness of dense hypergraphs

In this subsection, we present applications of the degree-stability in the decision problem of testing whether a hypergraph is F -free, a fundamental problem in Complexity Theory.

Given two r -graphs F and \mathcal{H} , a map $\psi: V(F) \rightarrow V(\mathcal{H})$ is an **embedding of F** if ψ is injective and $\psi(e) \in \mathcal{H}$ for all $e \in F$. For a fixed r -graph F , let **EMBED- F** denote the following decision problem:

F -EMBEDDING

Input: An r -graph \mathcal{H} on n vertices.

Question: Is there an embedding from F to \mathcal{H} ?

Notice that, for a fixed r -graph F , the brute-force search can solve **EMBED- F** in time $O(n^{v(F)})$. For $r=2$, faster algorithms improving the exponent $v(F)$ have been explored by many researchers [3, 19, 44, 53, 67]. In the case of $r \geq 3$, Yuster [85] demonstrated algorithms that improve the exponent $v(F)$ for certain classes of r -graphs. However, an open problem remains: determining whether there are faster algorithms that improve the exponent $v(F)$ when F is a complete r -graph with at least $r+1$ vertices. The current record for this problem is due to Nagle [66][§]. On the other hand, for the complete graph K_ℓ , a result by Chen–Huang–Kanj–Xia [12] shows that **Embed- K_ℓ** cannot be solved in time $n^{o(\ell)}$ unless the **ETH** (exponential time hypothesis [43]) fails. For complete bipartite graphs $K_{t,t}$, a result by Lin [55] shows that no algorithm can solve **EMBED- $K_{t,t}$** in time $n^{O(1)}$ if **W[1] \neq FPT** (see [17]), and no algorithm can solve **EMBED- $K_{t,t}$** in time $n^{o(\sqrt{t})}$ if randomized **ETH** holds (see [14]).

Motivated by the recent work of Fomin–Golovach–Sagunov–Simonov [24], we consider the following two embedding problems in dense hypergraphs. For a fixed r -graph F , let **EMBED_{avg}- (F, n, k)** denote the following decision problem:

[§]Nagle’s result was improved by a polylog(n) factor very recently [1].

F -EMBEDDING WITH AVERAGE DEGREE CONSTRAINT

Input: An r -graph \mathcal{H} on n vertices with $|\mathcal{H}| \geq \text{ex}(n, F) - k$.

Question: Is there an embedding from F to \mathcal{H} ?

In addition, define $\text{EMBED}_{\min}(F, \alpha)$ as the following decision problem:

F -EMBEDDING WITH MINIMUM DEGREE CONSTRAINT

Input: An r -graph \mathcal{H} on n vertices with $\delta(\mathcal{H}) \geq (\pi(F) - \alpha) \binom{n}{r-1}$.

Question: Is there an embedding from F to \mathcal{H} ?

Among other results, Fomin–Golovach–Sagunov–Simonov [24] proved that $\text{EMBED}_{\text{avg}}(K_\ell, n, k)$ can be solved in time $2.49^k n^{O(1)}$. A key ingredient in their proof involves reducing $\text{EMBED}_{\text{avg}}(K_\ell, n, k)$ to the task of verifying the K_ℓ -freeness of a graph with at most $5k$ vertices. This reduction is rooted in Erdős’ proof [22], which essentially employs the Zykov symmetrization [86] to establish the Turán theorem. Regrettably, Erdős’ proof does not appear to readily extend to general graphs and hypergraphs, and as a consequence, neither does the reduction employed by Fomin–Golovach–Sagunov–Simonov.

Using a different strategy based on the degree-stability, we show that $\text{EMBED}_{\min}(F, \alpha)$ (and hence, $\text{EMBED}_{\text{avg}}(F, n, \alpha \binom{n}{r-1})$) can be solved in time $O(n^r)$ when α is sufficiently small.

Theorem 1.4. *Suppose that F is an r -graph in Table 1 with degree-stability. Then there exist constants $\varepsilon_F > 0$ and $C_F > 0$ depending only on F such that the problem $\text{Embed}_{\min}(F, \alpha)$ can be solved in time $C_F n^r$ for all $\alpha \leq \varepsilon_F$, where n is the number of vertices of the input hypergraph.*

A natural and interesting problem arises in determining the optimal upper bound for ε_F in Theorem 1.4. For $F = K_{\ell+1}$, we can establish the lower bound $\frac{1}{3\ell^2 - \ell}$ for $\varepsilon_{K_{\ell+1}}$. Moreover, this bound is tight up to some multiplicative constant factor if $\mathbf{W}[1] \neq \mathbf{FPT}$. In addition, using the results of Chen–Huang–Kanj–Xia [12], we can also show that $\varepsilon_{K_{\ell+1}}$ must be of order $o(\frac{1}{\ell})$, even with a running time of $n^{o(\ell)}$.

Theorem 1.5. *There is an algorithm that solves $\text{EMBED}_{\min}(K_{\ell+1}, \alpha)$ in time $(\ell + 1)n^2$ for all $\ell \geq 2$ and $\alpha < \frac{1}{3\ell^2 - \ell}$, where n is the number of vertices of the input graph. On the other hand, for every $i \geq 2$ there exists $\delta_i > 0$ and n_i such that $\text{EMBED}_{\min}(K_{\ell+1}, (\delta_i \ell^2)^{-1})$ cannot be solved in time $O(n^i)$ for any $\ell \geq n_i$ if $\mathbf{W}[1] \neq \mathbf{FPT}$, and for every fixed $C > 0$ $\text{EMBED}_{\min}(K_{\ell+1}, (C\ell)^{-1})$ cannot be solved in time $n^{o(\ell)}$ if \mathbf{ETH} holds.*

Another interesting problem is to characterize the family of nondegenerate r -graphs F for which there exists a constant $\varepsilon_F > 0$ such that $\text{EMBED}_{\min}^-(K_{\ell+1}, \varepsilon_F)$ can be solved in time $O(n^r)$. The following result, whose proof relies on results by Lin [55], provides a natural class of graphs that do not lie in this family.

Given a graph F , the t -**blowup** $F[t]$ (of F) is obtained from F by replacing each vertex with a set of size t and each edge with a corresponding complete bipartite graph. It is well-known that $\pi(F) = \pi(F[t])$ for every $t \geq 1$ (see [21]).

Theorem 1.6. *For every fixed $\ell \geq 2$, there is no algorithm that solves $\text{EMBED}_{\min}^-(K_{\ell+1}[t], 0)$ in time $n^{O(1)}$ if $\mathbf{W}[1] \neq \mathbf{FPT}$, and there is no algorithm that solves $\text{EMBED}_{\min}^-(K_{\ell+1}[t], 0)$ in time $n^{o(\sqrt{t})}$ if randomized \mathbf{ETH} holds.*

We would like to remind the reader that in Theorem 1.4, one should view F as fixed and n as large (since for small n , we can just use the brute-force search). Similarly, in the second part of Theorem 1.5, the integer ℓ should also be considered fixed. Meanwhile, in the first part of Theorem 1.5, we do not require ℓ to be fixed.

For the problem $\text{EMBED}_{\text{avg}}^-(K_{\ell+1}, n, k)$, the following result improves upon the running time provided by Fomin–Golovach–Sagunov–Simonov within a specific range.

Theorem 1.7. *There is an algorithm that solves $\text{EMBED}_{\text{avg}}^-(K_{\ell+1}, n, k)$ in time $(\ell+4)n^2$ for integers $n \geq \ell \geq 2$ and $k \geq 0$ satisfying $\max\{6\ell^2, 30k\ell\} \leq n$.*

Note that if ℓ is fixed, then Theorem 1.7 implies that $\text{EMBED}_{\text{avg}}^-(K_{\ell+1}, n, k)$ can be solved in time $O(n^2)$ when $k \leq \frac{n}{30\ell}$. The following result shows that this linear bound cannot be improved to $n^{1+\delta}$ for any constant $\delta > 0$, thus leaving an open problem to determine the optimal bound for k in Theorem 1.7.

Theorem 1.8. *Unless \mathbf{ETH} fails, for every fixed $\delta > 0$ there is no algorithm that solves $\text{EMBED}_{\text{avg}}^-(K_{\ell+1}, n, n^{1+\delta})$ in time $n^{o(\ell)}$.*

Proofs for Theorems 1.4, 1.5, 1.6, 1.7, and 1.8 are presented in Section 4.

1.4 Homomorphism and surjective homomorphism

To establish the results in the previous subsection, we need to determine whether a given r -graph with a large minimum degree exhibits a specific structure. This is a particular instance of the homomorphism (or coloring) problem. To address this objective, we delve into the study of the homomorphism problem for dense hypergraphs in this subsection.

Given an r -pattern $P = (\ell, E)$, let the **Lagrange polynomial** of P be

$$\Lambda_P(X_1, \dots, X_\ell) := \sum_{e \in E} \prod_{i=1}^{\ell} \frac{X_i^{e(i)}}{e(i)!}.$$

The **Lagrangian** λ_P of P is defined as

$$\lambda_P := \max \{ \Lambda_P(x_1, \dots, x_\ell) : (x_1, \dots, x_\ell) \in \Delta^{\ell-1} \},$$

where $\Delta^{\ell-1} := \{ (x_1, \dots, x_\ell) \in \mathbb{R}_{\geq 0}^\ell : x_1 + \dots + x_\ell = 1 \}$ is the standard $(\ell-1)$ -dimensional simplex. It is worth mentioning that the definition of λ_P is equivalent to the definition of $\lambda_{P,1}$ from Section 1.2 when P is a hypergraph (see e.g. [13, 28]). Lagrangian is a crucial concept and has many applications in Extremal Combinatorics, and we refer the reader to [27, 64, 73, 76] for some examples.

We say a pattern $P = (\ell, E)$ is **minimal** if $\lambda_{P-i} < \lambda_P$ for all $i \in [\ell]$. Here $P-i$ denotes the pattern obtained from P by removing i and all edges containing i . Simple calculations show that a graph is minimal iff it is complete.

Given an r -graph \mathcal{H} and an r -pattern $P = (\ell, E)$, a map $\phi : V(\mathcal{H}) \rightarrow V(P)$ is a **homomorphism** (or a **P -coloring**) if $\phi(e) \in E$ for all $e \in \mathcal{H}$. We say \mathcal{H} is **P -colorable** if there is a homomorphism from \mathcal{H} to P . For a fixed r -pattern P , the **P -coloring problem** consists in deciding whether there exists a homomorphism of a given input r -graph \mathcal{H} to P . When P is a graph, the classical Hell–Nešetřil Theorem [40] states that the P -coloring problem is in **P** if P is bipartite and is **NP**-complete otherwise. For $r \geq 3$, the P -coloring problem is already **NP**-complete when $P = K_r^r$ (i.e. the r -graph with only one edge) [79].

For an r -pattern P and a real number $\alpha \in [0, 1]$, let $\text{HOM}(P, \alpha)$ denote the following problem:

HOMOMORPHISM WITH MINIMUM DEGREE CONSTRAINT

Input: An r -graph \mathcal{H} on n vertices with $\delta(\mathcal{H}) \geq \alpha n^{r-1}$.

Question: Is there a homomorphism from \mathcal{H} to P ?

Edwards [18] was the first to explore $\text{HOM}(K_\ell, \alpha)$, proving that for every $\ell \geq 3$, the problem $\text{Hom}(K_\ell, \alpha)$ is in **P** if $\alpha > \frac{\ell-3}{\ell-2}$ and is **NP**-complete otherwise. Subsequent extensions to general graphs and hypergraphs were considered in [16, 80], although many problems in this direction remain unresolved. Unfortunately, the exponents in the running time for algorithms provided in [16, 18, 80] for large α are excessively large and depend on F , rendering them impractical for our purposes (Theorem 1.4). Therefore, we present the following theorem, which efficiently solves $\text{HOM}(P, \alpha)$ in time $O(n^r)$ when P is a minimal r -pattern and α is close to $r\lambda_P$ (observe that, by Euler’s homogeneous function theorem, if $\alpha > r\lambda_P$, then there is not such a homomorphism).

Theorem 1.9. *Suppose that P is a minimal r -pattern on ℓ vertices. Then there exist $\varepsilon_P > 0$ and n_P depending only on P such that the problem $\text{HOM}(P, \alpha)$ can be*

solved in time $(\ell+1)n^r$ for all for $n \geq n_P$ and $\alpha \geq r\lambda_P - \varepsilon_P$, where n is the number of vertices of the input r -graph. Moreover, for every r -graph \mathcal{H} on $n \geq n_P$ vertices with $\delta(\mathcal{H}) \geq (r\lambda_P - \varepsilon_P)n^{r-1}$, the homomorphism from \mathcal{H} to P is unique (up to the automorphism[¶] of P) if it exists.

For complete graphs, the requirement that $n \geq n_P$ is not necessary.

Theorem 1.10. *For every $\ell \geq 2$, the problem $\text{HOM}(K_\ell, \alpha)$ can be solved in time $(\ell+1)n^2$ for all $\alpha > \frac{3\ell-4}{3\ell-1}$, where n is the number of vertices of the input graph. Moreover, for every graph G on n vertices with $\delta(G) > \frac{3\ell-4}{3\ell-1}n$, the homomorphism from G to K_ℓ is unique (up to the automorphism of K_ℓ) if it exists.*

Remark. It would be interesting to determine the infimum of α for which $\text{HOM}(K_\ell, \alpha)$ can be solved in time $O(n^2)$. It follows from Theorem 1.10 and Edwards' result that this infimum lies in the interval $[\frac{\ell-3}{\ell-2}, \frac{3\ell-4}{3\ell-1}]$ (assuming $\mathbf{P} \neq \mathbf{NP}$).

We take a step further by extending Theorem 1.9 to an important variant of the homomorphism problem, specifically the surjective homomorphism problem. For a fixed r -pattern P , the **surjective P -coloring problem** involves determining whether a surjective homomorphism exists for a given input r -graph \mathcal{H} to P . Unlike the P -coloring problem, a Hell–Nešetřil-type theorem for the graph surjective homomorphism problem is still elusive (see e.g. [6, 23, 36–38, 63] for some related results).

For an r -pattern P and a real number $\alpha \in [0, 1]$, let $\text{SHOM}(P, \alpha)$ denote the following decision problem^{||}:

SURJECTIVE HOMOMORPHISM WITH MINIMUM DEGREE CONSTRAINT

Input: An r -graph \mathcal{H} on n vertices with $\delta(\mathcal{H}) \geq \alpha n^{r-1}$.

Question: Is there a surjective homomorphism from \mathcal{H} to P ?

Let us introduce some technical definitions before stating the result. For an r -pattern $P = (\ell, E)$, let Φ_P denote the maximum value of the following optimization problem:

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & \exists (x_1, \dots, x_\ell) \in \Delta^{\ell-1} \quad \text{with} \quad \partial_i \Lambda_P(x_1, \dots, x_\ell) \geq z \quad \text{for all} \quad i \in [\ell]. \end{aligned} \tag{1.1}$$

Here $\partial_i \Lambda_P$ denotes the partial derivative of Λ_P with respect to the i -th variable. For convenience, let D_P be the collection of all optimal solutions to optimization

[¶]An automorphism of P is simply a bijective homomorphism from P to P .

^{||}A quick observation is that the results in [16, 18, 80] concerning $\text{HOM}(P, \alpha)$ can be easily extended to $\text{SHOM}(P, \alpha)$ with a minor modification to their original proofs.

problem (1.1), and let

$$\phi_P := \min_{(x_1, \dots, x_\ell) \in D_P} \min\{x_i : i \in [\ell]\}.$$

We say P is **rigid** if $\phi_P > 0$ and $\partial_i \Lambda_P(x_1, \dots, x_\ell) = \Phi_P$ holds for all $i \in [\ell]$ and for all $(x_1, \dots, x_\ell) \in D_P$. Otherwise, we say P is **non-rigid**. A quick observation, derived from the Lagrangian multiplier method, asserts that every minimal pattern P is rigid (but not vice versa, for example, C_k is rigid for $k \geq 5$ but not minimal). Furthermore, for a minimal r -pattern P we have $r\lambda_P = \Phi_P$.

Theorem 1.11. *Suppose P is a rigid r -pattern on ℓ vertices. Then there exist $\varepsilon_P > 0$ and n_P depending only on P such that the problem $\text{SHOM}(P, \alpha)$ can be solved in time $(\ell + 1)n^r$ for all for $n \geq n_P$ and $\alpha \geq \Phi_P - \varepsilon_P$, where n is the number of vertices of the input r -graph. Moreover, for every r -graph \mathcal{H} on $n \geq n_P$ vertices with $\delta(\mathcal{H}) \geq (\Phi_P - \varepsilon_P)n^{r-1}$, the surjective homomorphism from \mathcal{H} to P is unique (up to the automorphism of P) if it exists.*

Proofs for Theorems 1.9, 1.10, and 1.11 are presented in Section 3.

2 Proof of Theorem 1.1

Given an \mathcal{F} -free r -graph \mathcal{H} with large minimum degree, our objective is to show that \mathcal{H} is contained in \mathfrak{H} . The strategy in the following proof is to first use the edge-stability of \mathcal{F} to find a large minimum degree subgraph (i.e. $\mathcal{H}_1[U]$) of \mathcal{H} that is contained in \mathfrak{H} . Then we add the vertices in $V(\mathcal{H})$ back to $\mathcal{H}_1[U]$ one by one, where adding a vertex means adding all edges in $\mathcal{H} \setminus \mathcal{H}_1[U]$ containing this vertex. Using the vertex-extensibility, we will show that adding vertices preserves the containment in \mathfrak{H} , and hence, in the end, we obtain $\mathcal{H} \in \mathfrak{H}$.

Proof of Theorem 1.1. Let \mathcal{F} be a nondegenerate family of r -graphs and \mathfrak{H} be a hereditary family of \mathcal{F} -free r -graphs. Fix $\varepsilon_1 \geq \varepsilon_2 > 0$ sufficiently small and n_0 sufficiently large such that

1. every \mathcal{F} -free r -graph \mathcal{H} on $n \geq n_0$ vertices with $|\mathcal{H}| \geq (\pi(F)/r! - \varepsilon_2)n^r$ becomes a member in \mathfrak{H} after removing at most $\varepsilon_1 n^r/2$ edges, and
2. every \mathcal{F} -free r -graph \mathcal{H} on $n \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(F)/(r-1)! - 2\varepsilon_1^{1/3})n^{r-1}$ the following holds: if $\mathcal{H} - v$ is a member in \mathfrak{H} , then \mathcal{H} is a member in \mathfrak{H} as well.

Take $0 < \varepsilon_3 \leq \varepsilon_2/2$. Let \mathcal{H} be an \mathcal{F} -free r -graph on $n \geq 2n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(\mathcal{F})/(r-1)! - \varepsilon_3)n^{r-1}$. We aim to show that $\mathcal{H} \in \mathfrak{H}$.

First, notice that

$$|\mathcal{H}| \geq \frac{n}{r} \times \delta(\mathcal{H}) \geq \frac{n}{r} \times \left(\frac{\pi(\mathcal{F})}{(r-1)!} - \varepsilon_3 \right) n^{r-1} \geq \left(\frac{\pi(\mathcal{F})}{r!} - \varepsilon_3 \right) n^r.$$

So it follows from Assumption 1 that there exists a subgraph $\mathcal{H}_1 \subset \mathcal{H}$ such that $\mathcal{H}_1 \in \mathfrak{H}$ and

$$|\mathcal{H}_1| \geq |\mathcal{H}| - \frac{\varepsilon_1}{2} n^r \geq \left(\frac{\pi(\mathcal{F})}{r!} - \varepsilon_3 \right) n^r - \frac{\varepsilon_1}{2} n^r \geq \left(\frac{\pi(\mathcal{F})}{r!} - \varepsilon_1 \right) n^r.$$

Let

$$V := V(\mathcal{H}), Z := \left\{ v \in V : d_{\mathcal{H}_1}(v) < \left(\pi(\mathcal{F})/(r-1)! - \varepsilon_1^{1/3} \right) n^{r-1} \right\} \text{ and } U := V \setminus Z.$$

We claim that $|Z| \leq \varepsilon_1^{1/3} n$. Suppose, for contradiction, that $|Z| > \varepsilon_1^{1/3} n$. Then one could take a set $X \subseteq Z$ of size $|X| = \varepsilon_1^{1/3} n$. The definition of Z leads to

$$\begin{aligned} |\mathcal{H}_1 - X| &\geq \left(\frac{\pi(\mathcal{F})}{r!} - \varepsilon_1 \right) n^r - \left(\frac{\pi(\mathcal{F})}{(r-1)!} - \varepsilon_1^{1/3} \right) n^{r-1} |X| \\ &= \left(\frac{\pi(\mathcal{F})}{r!} - \frac{\pi(\mathcal{F})}{(r-1)!} \varepsilon_1^{1/3} \right) n^r + \left(\varepsilon_1^{2/3} - \varepsilon_1 \right) n^r \\ &> \frac{\pi(\mathcal{F})}{r!} (n - |X|)^r. \end{aligned}$$

However, this contradicts the fact that $\mathcal{H}_1 - X$ is \mathcal{F} -free. Hence,

$$\delta(\mathcal{H}_1[U]) \geq \left(\pi(\mathcal{F})/(r-1)! - \varepsilon_1^{1/3} \right) n^{r-1} - |Z| n^{r-2} \geq \left(\pi(\mathcal{F})/(r-1)! - 2\varepsilon_1^{1/3} \right) n^{r-1}.$$

Since \mathfrak{H} is hereditary and $\mathcal{H}_1 \in \mathfrak{H}$, we have $\mathcal{H}_1[U] \in \mathfrak{H}$. Let v_1, \dots, v_n be an ordering of vertices in V such that $\{v_1, \dots, v_{|U|}\} = U$. For convenience, let $V_i := \{v_1, \dots, v_i\}$ for $i \in [n]$. Let $\mathcal{G}_0 := \mathcal{H}_1[U]$, and for $i \in [n]$ let $\mathcal{G}_i := \mathcal{G}_{i-1} \cup \{e \in \mathcal{H}[V_i] : v_i \in e\}$. In particular, note that $V(\mathcal{G}_i) = U$ for $i \leq |U|$, and $V(\mathcal{G}_i) = V_i$ for $i \geq |U|$. We prove by induction on i that $\mathcal{G}_i \in \mathfrak{H}$. The base case $i=0$ is clear, so we may assume that $i \geq 1$. Assume that we have shown that $\mathcal{G}_{i-1} \in \mathfrak{H}$ for some $i \geq 1$, and we want to show that $\mathcal{G}_i := \mathcal{G}_{i-1} \cup \{e \in \mathcal{H}[V_i] : v_i \in e\}$ is also contained in \mathfrak{H} .

If $v_i \in U$, then it follows from $\mathcal{G}_i - v_i \subset \mathcal{G}_{i-1} \in \mathfrak{H}$,

$$\delta(\mathcal{G}_i) \geq \delta(\mathcal{H}_1[U]) \geq \left(\pi(\mathcal{F})/(r-1)! - 2\varepsilon_1^{1/3} \right) n^{r-1},$$

and Assumption 2 that $\mathcal{G}_i \in \mathfrak{H}$.

If $v_i \in Z$, then, similarly, it follows from $\mathcal{G}_i - v_i \subset \mathcal{G}_{i-1} \in \mathfrak{H}$,

$$\delta(\mathcal{G}_i) \geq \delta(\mathcal{H}) - |Z|n^{r-2} \geq \left(\frac{\pi(\mathcal{F})}{(r-1)!} - \varepsilon_3 \right) n^{r-1} - \varepsilon_1^{1/3} n^{r-1} \geq \left(\frac{\pi(\mathcal{F})}{(r-1)!} - 2\varepsilon_1^{1/3} \right) n^{r-1},$$

and Assumption 2 that $\mathcal{G}_i \in \mathfrak{H}$. This proves our claim. Therefore, $\mathcal{H} = \mathcal{G}_n \in \mathfrak{H}$. \square

3 Proofs for Theorems 1.9, 1.10, and 1.11

We prove Theorems 1.9, 1.10, and 1.11 in this section. The core of the proofs is a simple clustering algorithm based on the distance (defined below) between a pair of vertices.

Given an n -vertex r -graph \mathcal{H} the **link** $L_{\mathcal{H}}(v)$ of a vertex $v \in V(\mathcal{H})$ is

$$L_{\mathcal{H}}(v) := \left\{ A \in \binom{V(\mathcal{H})}{r-1} : A \cup \{v\} \in \mathcal{H} \right\}.$$

The (Hamming) **distance** between two vertices $u, v \in V(\mathcal{H})$ is $\text{dist}_{\mathcal{H}}(u, v) := |L_{\mathcal{H}}(u) \Delta L_{\mathcal{H}}(v)|$. It is clear that $\text{dist}_{\mathcal{H}}(u, v)$ can be calculated in time n^{r-1} . Observe that for a graph G , the value $\text{dist}_G(u, v)$ is simply the Hamming distance of the row vectors corresponding to u and v in the adjacency matrix of G . The following clustering algorithm based on the distance of vertices will be crucial for proofs in this section.

Proof of Theorem 1.10. Let $n \geq \ell \geq 2$ be integers. Suppose that G is an n -vertex graph with $\delta(G) = \alpha n$, where $\alpha > \frac{3\ell-4}{3\ell-1}$ is a real number. Let $\delta := \frac{2}{3\ell-1}$. Run Algorithm 1 with input (G, ℓ, δ) , and let $V_1 \cup \dots \cup V_{\ell} = V(G)$ denotes the output partition. It is easy to see that the running time for this step is at most ℓn^2 .

Claim 3.1. The graph G is ℓ -partite iff $\bigcup_{i \in [\ell]} G[V_i] = \emptyset$.

Proof. Suppose that G is ℓ -partite. Suppose that $U_1 \cup \dots \cup U_{\ell} = V(G)$ is a partition with $\bigcup_{i \in [\ell]} G[U_i] = \emptyset$ (this partition is used only for the proof and is not used for the algorithm). It suffices to show that $\{V_1, \dots, V_{\ell}\} = \{U_1, \dots, U_{\ell}\}$.

Let $x_i := |U_i|/n$ for $i \in [\ell]$. Since G is ℓ -partite, we obtain $1 - x_i = \sum_{j \in [\ell] \setminus \{i\}} x_j \geq \delta(G)/n > \alpha$ for all $i \in [\ell]$. Consequently, $x_i < 1 - \alpha$ for all $i \in [\ell]$, and hence, $1 - x_i = \sum_{j \in [\ell] \setminus \{i\}} x_j < (\ell - 1)(1 - \alpha)$ for all $i \in [\ell]$. In summary, we have

$$1 - (\ell - 1)(1 - \alpha) < x_i < 1 - \alpha \quad \text{for all } i \in [\ell]. \tag{3.1}$$

Algorithm 1 HAMMING CLUSTERING

Input: A triple $(\mathcal{H}, \ell, \delta)$, where \mathcal{H} is an n -vertex r -graph, $\ell \geq 2$ is an integer, and $\delta \in [0, 1]$ is a real number.

Output: A partition $V(\mathcal{H}) = V_1 \cup \dots \cup V_\ell$.

Operations:

1. Take a vertex $v_1 \in V(\mathcal{H})$, and let $W_1 := \{u \in V(\mathcal{H}) : \text{dist}_{\mathcal{H}}(u, v_1) \leq \delta n^{r-1}\}$. Suppose that we have defined W_1, \dots, W_i for some $i \in [\ell - 1]$. If $V(\mathcal{H}) \setminus (W_1 \cup \dots \cup W_i) \neq \emptyset$, then take an arbitrary vertex v_{i+1} from it, and let

$$W_{i+1} := \begin{cases} \{u \in V(\mathcal{H}) : \text{dist}_{\mathcal{H}}(u, v_{i+1}) \leq \delta n^{r-1}\} & \text{if } i \neq \ell - 1, \\ V(\mathcal{H}) \setminus (W_1 \cup \dots \cup W_i) & \text{if } i = \ell - 1. \end{cases}$$

Otherwise, let $W_{i+1} = \dots = W_\ell = \emptyset$.

2. Let $V_i := W_i \setminus (W_{i+1} \cup \dots \cup W_\ell)$ for $i \in [\ell - 1]$, and let $V_\ell := W_\ell$.

Fix $i, j \in [\ell]$ with $i \neq j$. Suppose that $u, u' \in U_i$ are two distinct vertices. Then it follows from the Inclusion-Exclusion Principle and (3.1) that

$$\begin{aligned} \text{dist}_G(u, u') &= d_G(u) + d_G(u') - 2|N_G(u) \cap N_G(u')| \\ &\leq d_G(u) + d_G(u') - 2(d_G(u) + d_G(u') - (1 - x_i)n) \\ &= 2(1 - x_i)n - (d_G(u) + d_G(u')) \\ &\leq 2(1 - x_i - \alpha)n \leq 2(\ell - 1 - \ell\alpha)n < \delta n. \end{aligned}$$

According to Algorithm 1, $u, u' \in V_{i^*}$ for some $i^* \in [\ell]$.

Suppose that $v \in U_i$ and $v' \in U_j$ are two distinct vertices. Then by (3.1),

$$\begin{aligned} \text{dist}_G(v, v') &\geq d_G(v) - \sum_{k \in [\ell] \setminus \{i, j\}} x_k n + d_G(v') - \sum_{k \in [\ell] \setminus \{i, j\}} x_k n \\ &\geq 2(\alpha - (1 - x_i - x_j))n \\ &\geq 2(\alpha + 2(1 - (\ell - 1)(1 - \alpha)) - 1)n \\ &= 2((2\ell - 1)\alpha - 2\ell + 3)n > \delta n. \end{aligned}$$

According to Algorithm 1, v and v' do not belong to the same part V_i . This proves that $\{V_1, \dots, V_\ell\} = \{U_1, \dots, U_\ell\}$, and hence, $\bigcup_{i \in [\ell]} G[V_i] = \bigcup_{i \in [\ell]} G[U_i] = \emptyset$. Additionally, observe that this proof shows the uniqueness of the partition $U_1 \cup \dots \cup U_\ell = V(G)$. \square

According to Claim 3.1, to check whether G is ℓ -partite we just need to check whether $\bigcup_{i \in [\ell]} G[V_i] = \emptyset$. This can be accomplished in at most n^2 time. Therefore, the overall time complexity is at most $(\ell + 1)n^2$. \square

Next, we present the proof for Theorem 1.11, while Theorem 1.9 follows from a similar argument and its proof is omitted.

Proof of Theorem 1.11. Fix a rigid r -pattern P on ℓ vertices. For simplicity, let us assume that the vertex set of P is $[\ell]$. For every $\beta > 0$, let

$$D_{P,\beta} := \left\{ (y_1, \dots, y_\ell) \in \Delta^{\ell-1} : \exists (x_1, \dots, x_\ell) \in D_P \text{ with } \max_{i \in [\ell]} |x_i - y_i| < \beta \right\},$$

and let $\Phi_{P,\beta}$ denote the optimal value of the following optimization problem:

$$\begin{aligned} & \max \quad y \\ & \text{s.t.} \quad \exists (x_1, \dots, x_\ell) \in \Delta^{\ell-1} \setminus D_{P,\beta} \quad \text{with} \quad \partial_i \Lambda_P(x_1, \dots, x_\ell) \geq y \quad \text{for all} \quad i \in [\ell]. \end{aligned}$$

It is easy to see from the definitions that $\Phi_{P,\beta} < \Phi_P$ for all $\beta > 0$. Take

$$\delta := \frac{(\phi_P/2)^{r-1}}{(r-1)!}, \quad \delta_* := \frac{\delta}{2^r \ell} < \frac{1}{\ell}, \quad \text{and} \quad \varepsilon_P := \min \left\{ \frac{\Phi_P - \Phi_{P,\delta_*}}{2}, \frac{\delta}{5} \right\}.$$

Let n be sufficiently large, and let \mathcal{H} be an n -vertex r -graph with $\delta(\mathcal{H}) = \alpha n^{r-1} \geq (\Phi_P - \varepsilon_P) n^{r-1}$. Run Algorithm 1 with input $(\mathcal{H}, \ell, \delta)$ and let $V_1 \cup \dots \cup V_\ell = V(\mathcal{H})$ denote the output partition. It is easy to see that the running time for this step is at most ℓn^r .

Claim 3.2. There is a surjective homomorphism from \mathcal{H} to P iff $\bigcup_{i \in [\ell]} \mathcal{H}[V_i] = \emptyset$ and $V_i \neq \emptyset$ for all $i \in [\ell]$.

Proof. Suppose that $\psi: V(\mathcal{H}) \rightarrow [\ell]$ is a surjective homomorphism from \mathcal{H} to P . Let $U_i := \psi^{-1}(i) \subset V(\mathcal{H})$ and $y_i := |U_i|/n$ for $i \in [\ell]$. Notice that $y_i > 0$ for $i \in [\ell]$ and $U_1 \cup \dots \cup U_\ell$ is a partition of $V(\mathcal{H})$. Observe from definitions that for every $i \in [\ell]$ we have

$$\Phi_{P,\delta_*} < \Phi_P - \varepsilon_P \leq \alpha = \frac{\delta(\mathcal{H})}{n^{r-1}} \leq \partial_i \Lambda_P(y_1, \dots, y_\ell).$$

So, it follows from the definitions of ε_P and Φ_{P,δ_*} that there exists $(x_1, \dots, x_\ell) \in D_P$ such that $\max_{i \in [\ell]} |x_i - y_i| < \delta_*$. Since for every vector $(z_1, \dots, z_\ell) \in \mathbb{R}^\ell$ with $\max_{k \in [\ell]} |z_k - x_k| \leq \delta_*$, the inequality

$$|\partial_{i,j}^2 \Lambda_P(z_1, \dots, z_\ell)| \leq (|z_1| + \dots + |z_\ell|)^{r-2} \leq (1 + \ell \delta_*)^{r-2} \leq 2^{r-2}.$$

holds for all $i, j \in [\ell]$, it follows Taylor's theorem that for every $i \in [\ell]$,

$$|\partial_i \Lambda_P(y_1, \dots, y_\ell) - \partial_i \Lambda_P(x_1, \dots, x_\ell)| \leq 2^{r-2} \cdot \sum_{k \in [\ell]} |y_k - x_k| \leq 2^{r-2} \ell \delta_*.$$

Therefore, $\partial_i \Lambda_P(y_1, \dots, y_\ell) \leq \partial_i \Lambda_P(x_1, \dots, x_\ell) + 2^{r-2} \ell \delta_* = \Phi_P + 2^{r-2} \ell \delta_*$.

Fix distinct $i, j \in [\ell]$. Suppose that $u, u' \in U_i$. Similar to the proof of Theorem 1.10, it follows from the Inclusion-Exclusion Principle and the inequality above that

$$\begin{aligned} \text{dist}_{\mathcal{H}}(u, u') &= d_{\mathcal{H}}(u) + d_{\mathcal{H}}(u') - 2|L_{\mathcal{H}}(u) \cap L_{\mathcal{H}}(u')| \\ &\leq d_{\mathcal{H}}(u) + d_{\mathcal{H}}(u') - 2(d_{\mathcal{H}}(u) + d_{\mathcal{H}}(u') - \partial_i \Lambda_P(y_1, \dots, y_\ell) \cdot n^{r-1}) \\ &= 2 \cdot \partial_i \Lambda_P(y_1, \dots, y_\ell) \cdot n^{r-1} - (d_{\mathcal{H}}(u) + d_{\mathcal{H}}(u')) \\ &\leq 2(\Phi_P + 2^{r-2} \ell \delta_* - \alpha) n^{r-1} \\ &= 2(2^{r-2} \ell \delta_* + \varepsilon_P) n^{r-1} < \delta n^{r-1}. \end{aligned}$$

Suppose that $v \in U_i$ and $v' \in U_j$ are two distinct vertices. A simple but crucial observation is that a rigid pattern does not contain twin vertices, i.e. vertices with the same link. Therefore, the two polynomials $\partial_i \Lambda_P(X_1, \dots, X_\ell)$ and $\partial_j \Lambda_P(X_1, \dots, X_\ell)$ are not identical. Hence,

$$\text{dist}_{\mathcal{H}}(v, v') \geq \frac{(\min_{i \in [\ell]} y_i)^{r-1}}{(r-1)!} \cdot n^{r-1} > \frac{(\phi_P - \delta_*)^{r-1}}{(r-1)!} \cdot n^{r-1} > \delta n^{r-1}.$$

It follows from the definition of Algorithm 1 that $\{U_1, \dots, U_\ell\} = \{V_1, \dots, V_\ell\}$, and hence, $\bigcup_{i \in [\ell]} \mathcal{H}[V_i] = \emptyset$ and $V_i \neq \emptyset$ for all $i \in [\ell]$. □

According to Claim 3.2, to check whether there is a surjective homomorphism from \mathcal{H} to P we just need to check whether $\bigcup_{i \in [\ell]} \mathcal{H}[V_i] = \emptyset$ and $V_i \neq \emptyset$ for all $i \in [\ell]$. This can be accomplished in at most n^r time. Therefore, the overall time complexity is at most $(\ell+1)n^r$. □

4 Proofs for Theorems 1.4, 1.5, 1.6, 1.7, and 1.8

We prove Theorems 1.4, 1.5, 1.6, 1.7, and 1.8 in this section.

Proof of Theorem 1.4. Let F be an r -graph in Table 1 with degree-stability. Let P be the minimal pattern such that (F, P) is a Turán pair. Simple calculations show that $\pi(F) = r! \lambda_P$. Let ℓ denote the number of vertices in P . Let $\varepsilon_P > 0$ and n_P be constants given by Theorem 1.9. Take $\varepsilon_F \in (0, \varepsilon_P)$ to be sufficiently small and $n_0 \geq n_P$ be sufficiently large such that: every F -free r -graph \mathcal{H} on $n \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(F) - \varepsilon_F) \binom{n}{r-1}$ is P -colorable (this is guaranteed by the degree-stability of F). Consider the following algorithm:

It is easy to see from Theorem 1.9 that the running time of Algorithm 2 is at most $\max\{n_0^{v(F)}, (\ell+1)n^r\} = O(n^r)$, proving Theorem 1.4. □

Algorithm 2 DECIDING F -FREEDOM IN LARGE MINIMUM DEGREE HYPERGRAPHS

Input: An n -vertex r -graph \mathcal{H} with $\delta(\mathcal{H}) \geq (\pi(F) - \varepsilon_F) \binom{n}{r-1}$.

Output: "Yes", if \mathcal{H} is F -free; "No", otherwise.

Operations:

1. If $n < n_0$, then use the brute-force search to check the F -freeness of \mathcal{H} and return the answer.
 2. If $n \geq n_0$, then run the Algorithm 1 with input $(\mathcal{H}, \ell, \varepsilon_F)$. Assume that $V_1 \cup \dots \cup V_\ell = V(\mathcal{H})$ is the output partition. Check whether $\mathcal{H}[V_i] = \emptyset$ holds for all $i \in [\ell]$. If it does hold for all $i \in [\ell]$, then return "Yes"; otherwise, return "No".
-

Next, we prove Theorem 1.5.

Proof of Theorem 1.5. The proof for the first part of Theorem 1.5 is similar to the proof of Theorem 1.4, so we omit it here and focus on the second part of Theorem 1.5.

First, we prove that $\text{EMBED}_{\min}(K_{\ell+1}, (C\ell)^{-1})$ cannot be solved in time $n^{o(\ell)}$ for any fixed C , unless **ETH** fails. Indeed, fix $C > 0$ (we may assume that C is an integer) and suppose to the contrary that there exists an algorithm \mathcal{A} that solves $\text{EMBED}_{\min}(K_{\ell+1}, \alpha)$ in time $n^{o(\ell)}$ for $\alpha = \frac{1}{C\ell}$. We claim that \mathcal{A} can also solve $\text{EMBED}_{\min}(K_{\ell+1}, \alpha)$ in time $n^{o(\ell)}$, which would contradict the result by Chen–Huang–Kanj–Xia [12]. Indeed, consider an arbitrary n -vertex graph G . Let $V_0 := V(G)$. Let \hat{G} be the graph obtained from G by adding $q := C\ell$ sets V_1, \dots, V_q , each of size n , and adding new edges $\{u, v\}$ for all $(u, v) \in V_i \times V_j$ whenever $0 \leq i < j \leq q$. Let $N := (q+1)n$ denote the number of vertices in \hat{G} and let $L := q + \ell = (C+1)\ell$. Observe that $K_{\ell+1} \subset G$ iff $K_{L+1} \subset \hat{G}$. Since

$$\delta(\hat{G}) \geq qn = \frac{q}{q+1}N = \left(\frac{L-1}{L} - \frac{\ell-1}{(q+1)L} \right)N > \left(\frac{L-1}{L} - \frac{1}{CL} \right)N, \tag{4.1}$$

by assumption, algorithm \mathcal{A} can decide in time $N^{o(L)} = ((C+1)\ell n)^{o((C+1)\ell)} = n^{o(\ell)}$ whether $K_{L+1} \subset \hat{G}$, and equivalently, whether $K_{\ell+1} \subset G$, proving our claim.

Now assume that **W[1] ≠ FPT**. For every $i \geq 2$ let ℓ_i be the smallest integer such that $\text{EMBED}_{\min}(K_{\ell_i+1})$ cannot be solved in time $O(n^i)$, let $\delta_i := \frac{1}{2(\ell_i-1)}$ and $n_i := 2\ell_i$. We claim that $\text{EMBED}_{\min}(K_{\ell_i+1}, (\delta_i \ell_i^2)^{-1})$ cannot be solved in time $O(n^i)$ for any $\ell \geq n_i$. Indeed, suppose to the contrary that there exist an $i_* \geq 2$ and an algorithm \mathcal{A}_{i_*} that solves $\text{EMBED}_{\min}(K_{\ell_*+1}, (\delta_{i_*} \ell_*^2)^{-1})$ in time $O(n^{i_*})$ for some $\ell_* \geq n_{i_*}$. Consider an arbitrary n -vertex graph G . Let \hat{G} be the same construction as defined above by replacing ℓ with ℓ_{i_*} and C with $\ell_*/\ell_{i_*} - 1 \geq 1$ (hence, $L = \ell_*$ and $L/2 \geq \ell_{i_*}$). It follows

from (4.1) that

$$\delta(\hat{G}) \geq \left(\frac{L-1}{L} - \frac{\ell_{i_*}-1}{(q+1)L} \right) N > \left(\frac{L-1}{L} - \frac{\ell_{i_*}-1}{L^2/2} \right) N = \left(\frac{L-1}{L} - \frac{1}{\delta_{i_*} L^2} \right) N.$$

By assumption, algorithm \mathcal{A}_{i_*} can decide in time $O(N^{i_*})=O(((C+1)\ell_{i_*}n)^{i_*})=O(n^{i_*})$ whether $K_{L+1} \subset \hat{G}$, and equivalently, whether $K_{\ell_{i_*}+1} \subset G$, contradicting the definition of ℓ_{i_*} . □

Next, we present the proof of Theorem 1.6.

Proof of Theorem 1.6. Fix $\ell \geq 0$, and suppose to the contrary that there is an algorithm \mathcal{A} that solves $\text{EMBED}_{\min}(K_{\ell+1}[t], 0)$ in time $n^{O(1)}$ (or $n^{o(\sqrt{t})}$). Consider an arbitrary n -vertex graph G . We may assume that $n \geq t$. Let $V_0 := V(G)$. Let \hat{G} be the graph obtained from G by adding $\ell-1$ sets $V_1, \dots, V_{\ell-1}$, each of size n , and adding new edges $\{u, v\}$ for all $(u, v) \in V_i \times V_j$ whenever $0 \leq i < j \leq \ell-1$. Let $N := \ell n$ denote the number of vertices in \hat{G} . Observe that $K_{t,t} \subset G$ iff $K_{\ell+1}[t] \subset \hat{G}$. Since

$$\delta(\hat{G}) \geq (\ell-1)n = \frac{\ell-1}{\ell} N = \pi(K_{\ell+1}[t])N,$$

by assumption, algorithm \mathcal{A} can decide in time $N^{O(1)} = (\ell n)^{O(1)} = n^{O(1)}$ (or $N^{o(\sqrt{t})} = n^{o(\sqrt{t})}$) whether $K_{\ell+1}[t] \subset \hat{G}$, and equivalently, whether $K_{t,t} \subset G$, contradicting the result by Lin [55]. □

Next, we prove Theorem 1.7. In the proof, we will use the Andrásfai–Erdős–Sós Theorem.

Theorem 4.1 (Andrásfai–Erdős–Sós [4]). *For all integers $n \geq \ell \geq 2$, every n -vertex $K_{\ell+1}$ -free graph G with $\delta(G) > \frac{3\ell-4}{3\ell-1}n$ is ℓ -partite.*

Proof of Theorem 1.7. Let $n \geq \ell \geq 2$ and $k \geq 1$ be integers satisfying $n \geq \max\{6\ell^2, 30k\ell\}$. Let G be an n -vertex graph with at least $\text{ex}(n, K_{\ell+1}) - k$ edges. Let $V_0 := V(G)$. For $0 \leq i \leq n-1$, we pick a vertex v_{i+1} of minimum degree in the induced subgraph $G[V_i]$, and let $V_{i+1} := V_i \setminus \{v_{i+1}\}$. Let z be the smallest positive integer i such that $d_{G[V_i]}(v_{i+1}) > \frac{3\ell-4}{3\ell-1}(n-i)$ (if there is no such i , then let $z=n$). Let $Z := \{v_i : i \in [z]\}$ and $U := V(G) \setminus Z$. We claim that the following Algorithm 3 can decide the $K_{\ell+1}$ -freeness of G in time $(\ell+4)n^2$.

The validity of Algorithm 3 will be established through the following claims. Suppose that G is $K_{\ell+1}$ -free. First, notice from the fact $\text{ex}(n, K_{\ell+1}) - \text{ex}(n-1, K_{\ell+1}) = n - \lceil n/\ell \rceil$ that $\delta(G) \geq n - \lceil n/\ell \rceil - k > \frac{\ell-1}{\ell}n - k - 1$. Since otherwise we would have $|G| < n - \lceil n/\ell \rceil - k + \text{ex}(n-1, K_{\ell+1}) \leq \text{ex}(n, K_{\ell+1}) - k$, a contradiction.

Algorithm 3 DECIDING $K_{\ell+1}$ -FREENESS IN DENSE GRAPHS

Input: An n -vertex graph G with $|G| \geq \text{ex}(n, K_{\ell+1}) - k$.

Output: "Yes", if G is $K_{\ell+1}$ -free; "No", otherwise.

Operations:

1. Let Z and U be as defined in the proof of Theorem 1.7. If $z > \frac{12\ell^2}{n} \left(k + \frac{\ell}{8}\right)$, then return "No"; otherwise, do the following operations.
 2. Run Algorithm 1 with input $(G[U], \ell, \delta)$, where $\delta := \frac{1}{3\ell+1}$, and assume that $U_1 \cup \dots \cup U_\ell = U$ is the output partition. If $G[U_i] \neq \emptyset$ for some $i \in [\ell]$, then return "No"; otherwise, do the following operations.
 3. For every $v \in Z$ find the smallest index $i_v \in [\ell]$ such that $N_G(v) \cap U_{i_v} = \emptyset$. If there is no such i_v for some $v \in Z$, then return "No"; otherwise, do the following operations.
 4. Let $V_j := U_j \cup \{v \in Z : i_v = j\}$ for $j \in [\ell]$. If $\bigcup_{j \in [\ell]} G[V_j] = \emptyset$, then return "Yes"; otherwise, return "No".
-

Claim 4.1. We have $z \leq \frac{12\ell^2}{n} \left(k + \frac{\ell}{8}\right)$. In particular, $z \leq \frac{13n}{120\ell}$.

Proof. Since $d_{G[V_i]}(v_{i+1}) \leq \frac{3\ell-4}{3\ell-1}(n-i)$ for $i \leq z$, we obtain

$$\begin{aligned}
 |G[U]| &\geq |G| - \sum_{i=0}^{z-1} \frac{3\ell-4}{3\ell-1}(n-i) \\
 &\geq \text{ex}(n, K_{\ell+1}) - k - \frac{3\ell-4}{2(3\ell-1)}(2n+1-z)z \\
 &= \text{ex}(n, K_{\ell+1}) - \frac{\ell-1}{2\ell}(2n-z)z - k + \frac{(2n-z)z}{2(3\ell^2-\ell)} - \frac{3\ell-4}{2(3\ell-1)}z \\
 &\geq \frac{\ell-1}{2\ell}(n-z)^2 - \left(k + \frac{\ell}{8} + \frac{3\ell-4}{2(3\ell-1)}z - \frac{(2n-z)z}{2(3\ell^2-\ell)}\right), \tag{4.2}
 \end{aligned}$$

where the last inequality follows from the fact that $\text{ex}(n, K_{\ell+1}) = \frac{\ell-1}{2\ell}n^2 - \frac{s}{2}\left(1 - \frac{s}{\ell}\right) \geq \frac{\ell-1}{2\ell}n^2 - \frac{\ell}{8}$, where $s := n - \ell \lfloor n/\ell \rfloor$. Let $k' := k + \frac{\ell}{8} + \frac{3\ell-4}{2(3\ell-1)}z - \frac{(2n-z)z}{2(3\ell^2-\ell)} \leq k + \frac{\ell}{8}$. Since $|G[U]| \leq \text{ex}(n-z, K_{\ell+1}) \leq \frac{\ell-1}{2\ell}(n-z)^2$, we have $k' \geq 0$, and hence,

$$z \leq \frac{4(3\ell^2-\ell)}{2n-z} \left(k + \frac{\ell}{8}\right) \leq \frac{12\ell^2}{n} \left(k + \frac{\ell}{8}\right).$$

Here we used fact that $\frac{3\ell-4}{2(3\ell-1)}z \leq \frac{(2n-z)z}{4(3\ell^2-\ell)}$, which follows from $n \geq 6\ell^2$. □

Since $\delta(G[U]) > \frac{3\ell-4}{3\ell-1}(n-z)$, it follows from Theorem 4.1 that there exists a partition $U_1 \cup \dots \cup U_\ell = U$ such that $G[U_i] = \emptyset$ for all $i \in [\ell]$. Moreover, it follows from uniqueness that this partition is identical to the one generated by Algorithm 3. Let $x_i := |U_i|/(n-z)$ for $i \in [\ell]$. It follows from (3.1) (by plugging in $\alpha = \frac{3\ell-4}{3\ell-1}$) that

$$\frac{2}{3\ell-1} < x_i < \frac{3}{3\ell-1} \quad \text{for all } i \in [\ell].$$

For every $i \in [\ell]$ let $B_i := \{v \in U_i : \exists u \in U \setminus U_i \text{ such that } \{v, u\} \notin G\}$ and $U'_i := U_i \setminus B_i$. Let $B := B_1 \cup \dots \cup B_\ell$, and notice that $|B| \leq 2k'$ and $|B_i| \leq k'$ for $i \in [\ell]$ (recall the definition of k' from Claim 4.1). Therefore, $|U'_i| \geq x_i(n-z) - k'$ for $i \in [\ell]$.

Observe that the induced subgraph of G on $U'_1 \cup \dots \cup U'_\ell$ is complete ℓ -partite. Therefore, for every $v \in Z$ there exists $i_v \in [\ell]$ such that v has no neighbor in U'_{i_v} . Suppose that v has a neighbor $u \in B_{i_v}$. Since $\delta(G[U]) > \frac{3\ell-4}{3\ell-1}(n-z)$, the vertex u has at least

$$\begin{aligned} \frac{3\ell-4}{3\ell-1}(n-z) - \sum_{k \in [\ell] \setminus \{i_v, j\}} x_k(n-z) - |B_j| &\geq \frac{3\ell-4}{3\ell-1}(n-z) - \frac{3(\ell-2)}{3\ell-1}(n-z) - k' \\ &\geq \frac{2(n-z)}{3\ell-1} - k' \end{aligned}$$

neighbors in U'_j for every $j \in [\ell] \setminus \{i_v\}$. Since G is $K_{\ell+1}$ -free, there exists $j_v \in [\ell] \setminus \{i_v\}$ such that v has no neighbor in $N_G(u) \cap U'_{j_v}$. This means that the number of non-neighbors of v in U is at least

$$\begin{aligned} x_{i_v}(n-z) - k' + \frac{2(n-z)}{3\ell-1} - k' &> \frac{4(n-z)}{3\ell-1} - 2k' \\ &> \frac{4(n-z)}{3\ell-1} - 2 \left(k + \frac{\ell}{8} + \frac{3\ell-4}{2(3\ell-1)}z - \frac{(2n-z)z}{2(3\ell^2-\ell)} \right) \\ &> \frac{4n}{3\ell-1} - 2 \left(k + \frac{\ell}{8} \right), \end{aligned}$$

where in the last inequality, we used $\frac{(2n-z)z}{2(3\ell^2-\ell)} \geq \frac{3\ell}{2(3\ell-1)}z$, which follows from $n \geq 6\ell^2$. Since $n \geq \max\{30k\ell, 6\ell^2\}$, we have $\frac{4n}{3\ell-1} - 2 \left(k + \frac{\ell}{8} \right) > \frac{n}{\ell} + k + 1$. Therefore, $d_G(v) < \frac{\ell-1}{\ell}n - k - 1$, a contradiction. This shows that v does not have any neighbor in B_{i_v} , and hence, v has no neighbor in U_{i_v} .

Let $Z_1 \cup \dots \cup Z_\ell = Z$ be a partition such that for every $i \in [\ell]$ and for every $v \in Z_i$ we have $N_G(v) \cap U_i = \emptyset$.

Claim 4.2. We have $G[Z_i] = \emptyset$ for $i \in [\ell]$.

Proof. First, we improve the lower bound $x_i > \frac{2}{3\ell-1}$ for $i \in [\ell]$. Notice that $|G[U]| \leq \sum_{1 \leq i < j \leq \ell} x_i x_j (n-z)^2$. So, it follows from (4.2) and the following inequality (which follows from the Maclaurin's inequality, see [60, Lemma 2.2])

$$\sum_{1 \leq i < j \leq \ell} x_i x_j \leq \frac{\ell-1}{2\ell} - \frac{1}{2} \sum_{i \in [\ell]} \left(x_i - \frac{1}{\ell}\right)^2$$

that $\frac{1}{2} \sum_{i \in [\ell]} \left(x_i - \frac{1}{\ell}\right)^2 \leq \frac{k'}{(n-z)^2} \leq \frac{k+\ell/8}{(n/2)^2}$. It follows from $n \geq \max\{6\ell^2, 30k\ell\}$ and $\ell \geq 2$ that

$$x_i \geq \frac{1}{\ell} - \left(\frac{8k+\ell}{n^2}\right)^{1/2} \geq \frac{1}{\ell} - \left(\frac{8}{30^2k} + \frac{1}{6^2\ell}\right)^{1/2} \cdot \frac{1}{\ell} \geq \frac{5}{6\ell} \quad \text{for all } i \in [\ell].$$

Suppose that this claim is not true, and by symmetry, we may assume that $\{v, v'\} \in G[Z_1]$ is an edge. It follows from the $K_{\ell+1}$ -freeness of G that there exists $2 \leq i_* \leq \ell$ such that $U'_{i_*} \cap N_G(v) \cap N_G(v') = \emptyset$. By the Pigeonhole Principle, we may assume that at least half vertices in U'_{i_*} are not adjacent to v . This implies that

$$\begin{aligned} d_G(v) &\leq n - |U_1| - \frac{1}{2}|U'_{i_*}| \leq n - \frac{5}{6\ell}(n-z) - \frac{1}{2} \left(\frac{5}{6\ell}(n-z) - 2k'\right) \\ &\leq \frac{\ell-1}{\ell}n - k - 1 - \left(\frac{n}{4\ell} - k' - k - \frac{5z}{4\ell} - 1\right). \end{aligned}$$

Since $k \leq \frac{n}{30\ell}$, $\ell \leq \frac{n}{6\ell}$, $k' \leq k + \ell/8 \leq \frac{13n}{240\ell}$ (all due to $n \geq \max\{6\ell^2, 30k\ell\}$), and $z \leq \frac{13n}{120\ell}$ (by Claim 4.1), we have $\frac{n}{4\ell} - k' - k - \frac{5z}{4\ell} - 1 \leq 0$. Therefore, $d_G(v) \leq \frac{\ell-1}{\ell}n - k - 1$, a contradiction. \square

Let $V_i := U_i \cup Z_i$ for $i \in [\ell]$. It follows from Claim 4.2 that $\bigcup_{i \in [\ell]} G[V_i] = \emptyset$, proving the correctness of Algorithm 3. \square

Now we present the proof of Theorem 1.8.

Proof of Theorem 1.8. Fix $\delta > 0$ (we may assume that $C := 1/\delta$ is an integer) and suppose to the contrary that there exists an algorithm \mathcal{A} that solves $\text{EMBED}_{\text{avg}}(K_{\ell+1}, n, n^{1+\delta})$ in time $n^{o(\ell)}$. We claim that \mathcal{A} can also solve $\text{EMBED}-K_{\ell+1}$ in time $n^{o(\ell)}$, which would contradict the result by Chen–Huang–Kanj–Xia [12].

The construction is very similar to that in the proof of Theorem 1.5. Consider an arbitrary n -vertex graph G . Let \hat{G} be the graph obtained from G by adding ℓ sets V_1, \dots, V_ℓ , each of size n^C , and adding new edges $\{u, v\}$ for all $(u, v) \in V_i \times V_j$

whenever $1 \leq i < j \leq \ell$. Let $N := \ell n^C + n$ denote the number of vertices in \hat{G} . Observe that $K_{\ell+1} \subset G$ iff $K_{\ell+1} \subset \hat{G}$. Since

$$|\hat{G}| = \binom{\ell}{2} n^{2C} + |G| \geq \frac{\ell-1}{\ell} \frac{(N-n)^2}{2} \geq \frac{\ell-1}{\ell} \frac{N^2}{2} - Nn > \text{ex}(N, K_{\ell+1}) - N^{1+\delta},$$

it follows from our assumption that algorithm \mathcal{A} can decide in time $N^{o(\ell)} = (\ell n^C + n)^{o(\ell)} = n^{o(\ell)}$ whether $K_{\ell+1} \subset \hat{G}$, and equivalently, whether $K_{\ell+1} \subset G$, proving our claim in the first paragraph. \square

5 Concluding remarks

Recall that the core of the algorithm for $\text{EMBED}_{\min}(K_{\ell+1}, \alpha)$ when $\alpha > \frac{3\ell-4}{3\ell-1}$ is the structural theorem by Andrásfai–Erdős–Sós [4]. It seems worth exploring whether refined structural theorems (see e.g. [2, 11, 35, 70]) can be used to push the lower bound for α further.

The proof of Theorem 1.4 can be easily modified to cover some hypergraph families with multiple extremal constructions (see e.g. [58, 59]). There are hypergraph Turán problems whose structure of extremal constructions can exhibit a nonminimal pattern (see e.g. [41]), a recursive pattern (see, e.g. [73]), or even a mixed recursive pattern (see, e.g. [61]). It is of interest to investigate whether Theorems 1.4 and 1.9 can be extended to cover nonminimal/recursive/mixed recursive patterns.

Recall the nice structural characterization of minimal graphs: a graph is minimal iff it is complete. It would be interesting to explore a characterization of rigid graphs. Simple linear algebra arguments show that non-singular (i.e. the adjacency matrix is full rank) regular graphs are rigid. We refer the reader to [74] for related results on singular graphs. In general, one could ask for a characterization of the families of all minimal/rigid r -graphs.

Acknowledgements

Theorem 1.1 was previously presented by X.L. in a 2023 summer school at Suzhou University, and we would like to thank the organizers for their warm hospitality. XL would also like to thank Dhruv Mubayi and Christian Reiher for very inspiring discussions in a previous related project, and to Dhruv Mubayi for pointing out the application of degree-stability in spectral Turán problem. We also thank Yixiao Zhang for verifying the vertex-extendability of some hypergraphs in Table 1. J.H. was supported by National Key R&D Program of China (Grant No. 2023YFA1010202),

and the Central Guidance on Local Science and Technology Development Fund of Fujian Province (Grant No. 2023L3003). X.L. was supported by the Excellent Young Talents Program (Overseas) of the National Natural Science Foundation of China. We would like to express our gratitude to referees for their valuable suggestions and comments.

References

- [1] A. Abboud, N. Fischer, and Y. Shechter. Faster combinatorial k -clique algorithms. In *LATIN 2024: Theoretical Informatics. Part I*, volume 14578 of *Lecture Notes in Comput. Sci.*, pages 193–206. Springer, Cham, [2024] ©2024.
- [2] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa, and R. Morris. The chromatic thresholds of graphs. *Adv. Math.*, 235:261–295, 2013.
- [3] N. Alon, R. Yuster, and U. Zwick. Finding and counting given length cycles. *Algorithmica*, 17(3):209–223, 1997.
- [4] B. Andrásfai, P. Erdős, and V. T. Sós. On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.*, 8:205–218, 1974.
- [5] A. Bene Watts, S. Norin, and L. Yepremyan. A Turán theorem for extensions via an Erdős-Ko-Rado theorem for Lagrangians. *Combinatorica*, 39(5):1149–1171, 2019.
- [6] M. Bodirsky, J. Kára, and B. Martin. The complexity of surjective homomorphism problems—a survey. *Discrete Appl. Math.*, 160(12):1680–1690, 2012.
- [7] L. Bodnár, J. León, X. Liu, and O. Pikhurko. The Turán density of the tight 5-cycle minus one edge. *arXiv preprint arXiv:2412.21011*, 2024.
- [8] L. Bodnár, J. León, X. Liu, and O. Pikhurko. The Turán density of short tight cycles. *arXiv preprint arXiv:2506.03223*, 2025.
- [9] B. Bollobás. Three-graphs without two triples whose symmetric difference is contained in a third. *Discrete Math.*, 8:21–24, 1974.
- [10] A. Brandt, D. Irwin, and T. Jiang. Stability and Turán numbers of a class of hypergraphs via Lagrangians. *Combin. Probab. Comput.*, 26(3):367–405, 2017.
- [11] C. C. Chen, G. P. Jin, and K. M. Koh. Triangle-free graphs with large degree. *Combin. Probab. Comput.*, 6(4):381–396, 1997.
- [12] J. Chen, X. Huang, I. A. Kanj, and G. Xia. Strong computational lower bounds via parameterized complexity. *J. Comput. System Sci.*, 72(8):1346–1367, 2006.
- [13] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra Appl.*, 436(9):3268–3292, 2012.
- [14] M. Cygan, F. V. Fomin, L. u. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized algorithms*. Springer, Cham, 2015.
- [15] D. De Caen and Z. Füredi. The maximum size of 3-uniform hypergraphs not containing a Fano plane. *J. Combin. Theory Ser. B*, 78(2):274–276, 2000.
- [16] J. Díaz, J. Nešetřil, M. Serna, and D. M. Thilikos. H -coloring of large degree graphs. In *Eurasian Conference on Information and Communication Technology*, pages 850–

857. Springer, 2002.
- [17] R. G. Downey and M. R. Fellows. *Parameterized complexity*. Monographs in Computer Science. Springer-Verlag, New York, 1999.
 - [18] K. Edwards. The complexity of colouring problems on dense graphs. *Theoret. Comput. Sci.*, 43(2-3):337–343, 1986.
 - [19] F. Eisenbrand and F. Grandoni. On the complexity of fixed parameter clique and dominating set. *Theoret. Comput. Sci.*, 326(1-3):57–67, 2004.
 - [20] P. Erdős and M. Simonovits. On a valence problem in extremal graph theory. *Discrete Math.*, 5:323–334, 1973.
 - [21] P. Erdős and M. Simonovits. Supersaturated graphs and hypergraphs. *Combinatorica*, 3(2):181–192, 1983.
 - [22] P. Erdős. On the graph theorem of Turán. *Mat. Lapok*, 21:249–251, 1970.
 - [23] J. Focke, L. A. Goldberg, and S. Živný. The complexity of counting surjective homomorphisms and compactions. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1772–1781. SIAM, Philadelphia, PA, 2018.
 - [24] F. V. Fomin, P. A. Golovach, D. Sagunov, and K. Simonov. Turán’s theorem through algorithmic lens. In *Graph-theoretic concepts in computer science*, volume 14093 of *Lecture Notes in Comput. Sci.*, pages 348–362. Springer, Cham, [2023] ©2023.
 - [25] P. Frankl. Asymptotic solution of a Turán-type problem. *Graphs Combin.*, 6(3):223–227, 1990.
 - [26] P. Frankl and Z. Füredi. A new generalization of the Erdős-Ko-Rado theorem. *Combinatorica*, 3(3-4):341–349, 1983.
 - [27] P. Frankl and Z. Füredi. Extremal problems whose solutions are the blowups of the small Witt-designs. *J. Combin. Theory Ser. A*, 52(1):129–147, 1989.
 - [28] S. Friedland, S. Gaubert, and L. Han. Perron-Frobenius theorem for nonnegative multilinear forms and extensions. *Linear Algebra Appl.*, 438(2):738–749, 2013.
 - [29] Z. Füredi. Turán type problems. In *Surveys in combinatorics, 1991 (Guildford, 1991)*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pages 253–300. Cambridge Univ. Press, Cambridge, 1991.
 - [30] Z. Füredi. A proof of the stability of extremal graphs, Simonovits’ stability from Szemerédi’s regularity. *J. Combin. Theory Ser. B*, 115:66–71, 2015.
 - [31] Z. Füredi, D. Mubayi, and O. Pikhurko. Quadruple systems with independent neighborhoods. *J. Combin. Theory Ser. A*, 115(8):1552–1560, 2008.
 - [32] Z. Füredi, O. Pikhurko, and M. Simonovits. On triple systems with independent neighbourhoods. *Combin. Probab. Comput.*, 14(5-6):795–813, 2005.
 - [33] Z. Füredi, O. Pikhurko, and M. Simonovits. 4-books of three pages. *J. Combin. Theory Ser. A*, 113(5):882–891, 2006.
 - [34] Z. Füredi and M. Simonovits. Triple systems not containing a Fano configuration. *Combin. Probab. Comput.*, 14(4):467–484, 2005.
 - [35] W. Goddard and J. Lyle. Dense graphs with small clique number. *J. Graph Theory*, 66(4):319–331, 2011.
 - [36] P. A. Golovach, M. Johnson, B. Martin, D. Paulusma, and A. Stewart. Surjective

- H -colouring: new hardness results. *Computability*, 8(1):27–42, 2019.
- [37] P. A. Golovach, B. Lidický, B. Martin, and D. Paulusma. Finding vertex-surjective graph homomorphisms. *Acta Inform.*, 49(6):381–394, 2012.
- [38] P. A. Golovach, D. Paulusma, and J. Song. Computing vertex-surjective homomorphisms to partially reflexive trees. *Theoret. Comput. Sci.*, 457:86–100, 2012.
- [39] D. Hefetz and P. Keevash. A hypergraph Turán theorem via Lagrangians of intersecting families. *J. Combin. Theory Ser. A*, 120(8):2020–2038, 2013.
- [40] P. Hell and J. Nešetřil. On the complexity of H -coloring. *J. Combin. Theory Ser. B*, 48(1):92–110, 1990.
- [41] J. Hou, H. Li, X. Liu, D. Mubayi, and Y. Zhang. Hypergraphs with infinitely many extremal constructions. *Discrete Anal.*, pages Paper No. 18, 34, 2023.
- [42] J. Hou, H. Li, X. Liu, L.-T. Yuan, and Y. Zhang. A step towards a general density Corrádi–Hajnal theorem. *arXiv preprint arXiv:2302.09849*, 2023.
- [43] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? volume 63, pages 512–530. 2001. Special issue on FOCS 98 (Palo Alto, CA).
- [44] A. Itai and M. Rodeh. Finding a minimum circuit in a graph. *SIAM J. Comput.*, 7(4):413–423, 1978.
- [45] T. Jiang, Y. Peng, and B. Wu. Lagrangian densities of some sparse hypergraphs and Turán numbers of their extensions. *European J. Combin.*, 73:20–36, 2018.
- [46] L. Kang, V. Nikiforov, and X. Yuan. The p -spectral radius of k -partite and k -chromatic uniform hypergraphs. *Linear Algebra Appl.*, 478:81–107, 2015.
- [47] G. Katona, T. Nemetz, and M. Simonovits. On a problem of Turán in the theory of graphs. *Mat. Lapok*, 15:228–238, 1964.
- [48] P. Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011.
- [49] P. Keevash, J. Lenz, and D. Mubayi. Spectral extremal problems for hypergraphs. *SIAM J. Discrete Math.*, 28(4):1838–1854, 2014.
- [50] P. Keevash and D. Mubayi. Stability theorems for cancellative hypergraphs. *J. Combin. Theory Ser. B*, 92(1):163–175, 2004.
- [51] P. Keevash and B. Sudakov. On a hypergraph Turán problem of Frankl. *Combinatorica*, 25(6):673–706, 2005.
- [52] P. Keevash and B. Sudakov. The Turán number of the Fano plane. *Combinatorica*, 25(5):561–574, 2005.
- [53] T. Kloks, D. Kratsch, and H. Müller. Finding and counting small induced subgraphs efficiently. *Inform. Process. Lett.*, 74(3-4):115–121, 2000.
- [54] B. Lidický, C. Mattes, and F. Pfender. The hypergraph Turán densities of tight cycles minus an edge. *arXiv preprint arXiv:2409.14257*, 2024.
- [55] B. Lin. The parameterized complexity of the k -biclique problem. *J. ACM*, 65(5):Art. 34, 21, 2018.
- [56] X. Liu. New short proofs to some stability theorems. *European J. Combin.*, 96:Paper

No. 103350, 8, 2021.

- [57] X. Liu. On a hypergraph Mantel theorem. *arXiv preprint arXiv:2501.19229*, 2025.
- [58] X. Liu and D. Mubayi. A hypergraph Turán problem with no stability. *Combinatorica*, 42(3):433–462, 2022.
- [59] X. Liu, D. Mubayi, and C. Reiher. Hypergraphs with many extremal configurations. *arXiv preprint arXiv:2102.02103*, 2021.
- [60] X. Liu, D. Mubayi, and C. Reiher. A unified approach to hypergraph stability. *J. Combin. Theory Ser. B*, 158(part 2):36–62, 2023.
- [61] X. Liu and O. Pikhurko. Finite hypergraph families with rich extremal Turán constructions via mixing patterns. *Forum Math. Sigma*, 13:Paper No. e53, 49, 2025.
- [62] W. Mantel. Vraagstuk XXVIII. *Wiskundige Opgaven*, 10(2):60–61, 1907.
- [63] B. Martin and D. Paulusma. The computational complexity of disconnected cut and $2K_2$ -partition. *J. Combin. Theory Ser. B*, 111:17–37, 2015.
- [64] T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turán. *Canadian J. Math.*, 17:533–540, 1965.
- [65] D. Mubayi. A hypergraph extension of Turán’s theorem. *J. Combin. Theory Ser. B*, 96(1):122–134, 2006.
- [66] B. Nagle. On computing the frequencies of induced subhypergraphs. *SIAM J. Discrete Math.*, 24(1):322–329, 2010.
- [67] J. Nešetřil and S. Poljak. On the complexity of the subgraph problem. *Comment. Math. Univ. Carolin.*, 26(2):415–419, 1985.
- [68] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra Appl.*, 457:455–535, 2014.
- [69] S. Norin and L. Yepremyan. Turán numbers of extensions. *J. Combin. Theory Ser. A*, 155:476–492, 2018.
- [70] H. Oberkampf and M. Schacht. On the structure of dense graphs with bounded clique number. *Combin. Probab. Comput.*, 29(5):641–649, 2020.
- [71] O. Pikhurko. An exact Turán result for the generalized triangle. *Combinatorica*, 28(2):187–208, 2008.
- [72] O. Pikhurko. Exact computation of the hypergraph Turán function for expanded complete 2-graphs. *J. Comb. Theory, Ser. B*, 103(2):220–225, 2013.
- [73] O. Pikhurko. On possible Turán densities. *Israel J. Math.*, 201(1):415–454, 2014.
- [74] I. Sciriha. A characterization of singular graphs. *Electron. J. Linear Algebra*, 16:451–462, 2007.
- [75] A. Sidorenko. What we know and what we do not know about Turán numbers. *Graphs Combin.*, 11(2):179–199, 1995.
- [76] A. F. Sidorenko. On the maximal number of edges in a homogeneous hypergraph that does not contain prohibited subgraphs. *Mat. Zametki*, 41(3):433–455, 459, 1987.
- [77] A. F. Sidorenko. Asymptotic solution for a new class of forbidden r -graphs. *Combinatorica*, 9(2):207–215, 1989.
- [78] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319.

Academic Press, New York, 1968.

- [79] E. Szymańska. The complexity of vertex coloring problems in uniform hypergraphs with high degree. In *Graph-theoretic concepts in computer science*, volume 6410 of *Lecture Notes in Comput. Sci.*, pages 304–314. Springer, Berlin, 2010.
- [80] E. Szymańska. H -colorings of dense hypergraphs. *Inform. Process. Lett.*, 112(23):899–902, 2012.
- [81] P. Turán. On an external problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.
- [82] J. Wang, L. Kang, and Y. Xue. On a conjecture of spectral extremal problems. *J. Combin. Theory Ser. B*, 159:20–41, 2023.
- [83] Z. Yan and Y. Peng. An irrational Lagrangian density of a single hypergraph. *SIAM J. Discrete Math.*, 36(1):786–822, 2022.
- [84] Z. Yan and Y. Peng. Lagrangian densities of 4-uniform matchings and degree stability of extremal hypergraphs. *Discrete Math.*, 348(1):Paper No. 114235, 19, 2025.
- [85] R. Yuster. Finding and counting cliques and independent sets in r -uniform hypergraphs. *Inform. Process. Lett.*, 99(4):130–134, 2006.
- [86] A. A. Zykov. On some properties of linear complexes. *Mat. Sbornik N.S.*, 24/66:163–188, 1949.

Definitions for hypergraphs in Table 1

- A graph F is **edge-critical** if there exists an edge $e \in F$ such that $\chi(F - e) < \chi(F)$.
- Fix a graph F , the **expansion** H_F^r of F is the r -graphs obtained from F by adding a set of $r - 2$ new vertices into each edge of F , and moreover, these new $(r - 2)$ -sets are pairwise disjoint.
- Given an r -graph F with $\ell + 1$ vertices, the **expansion** $H_{\ell+1}^F$ of F is the r -graph obtained from F by adding, for every pair $\{u, v\} \subset V(F)$ that is not contained in any edge of F , an $(r - 2)$ -set of new vertices, and moreover, these $(r - 2)$ -sets are pairwise disjoint.
- We say a tree T is an **Erdős–Sós tree** if it satisfies the famous Erdős–Sós conjecture on trees. The $(r - 2)$ -**extension** $\text{Ext}(T)$ of a tree T is

$$\text{Ext}(T) := \{e \cup A : e \in T\},$$

where A is a set of $r - 2$ new vertices that is disjoint from $V(T)$. An r -graph F is an **extended tree** if $F = \text{Ext}(T)$ for some tree.

- The $(r$ -uniform) **generalized triangle** \mathbb{T}_r is the r -graph with vertex set $[2r - 1]$ and edge set

$$\{\{1, \dots, r - 1, r\}, \{1, \dots, r - 1, r + 1\}, \{r, r + 1, \dots, 2r - 1\}\}.$$

- Given integers $r > i \geq 1$, let $\mathbb{T}_{r,i}$ denote the r -graph whose vertex set is $[2r+1]$ and whose edge set is

$$\{\{1, \dots, r\}, \{1, \dots, i, r+1, \dots, 2r-i\}, \{i+1, \dots, r, r+1, 2r-i+1, \dots, 2r-1\}\}.$$

The **weak triangle** Δ_r is the family defined by

$$\Delta_r := \{\mathbb{T}_{r,i} : 1 \leq i \leq \lceil r/2 \rceil\}.$$

- Let \mathcal{C}_3^{2r} (the **expanded triangle**) denote the $2r$ -graph with vertex set $[3r]$ and edge set

$$\{\{1, \dots, r, r+1, \dots, 2r\}, \{r+1, \dots, 2r, 2r+1, \dots, 3r\}, \{1, \dots, r, 2r+1, \dots, 3r\}\}.$$

- The **Fano plane** \mathbb{F} is the 3-graph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and edge set

$$\{123, 345, 561, 174, 275, 376, 246\}.$$

- Let F_7 (**4-book with 3-pages**) denote the 3-graph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and edge set

$$\{1234, 1235, 1236, 1237, 4567\}.$$

- Let $\mathbb{F}_{4,3}$ denote the 4-graph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and edge set

$$\{1234, 1235, 1236, 1237, 4567\}.$$

- Let $\mathbb{F}_{3,2}$ denote the 3-graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set

$$\{123, 124, 125, 345\}.$$

- The 3-graph $K_4^3 \sqcup K_3^3$ has vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and edge set

$$\{123, 124, 234, 567\}.$$

- The r -graph M_k^r (r -uniform k -**matching**) is the r -graph consisting of k pairwise disjoint edges.

- The r -graph L_k^r (r -uniform k -**sunflower**) is the r -graph consisting of k edges e_1, \dots, e_k such that for all $1 \leq i < j \leq k$, it holds that $e_i \cap e_j = \{v\}$ for some fixed vertex v .

- The 3-uniform **tight 5-cycle** C_5^3 is the 3-graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set

$$\{123, 234, 345, 451, 512\}.$$

- The 3-uniform **tight 5-cycle minus one edge** C_5^{3-} is the 3-graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set

$$\{123, 234, 345, 451\}.$$