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Two Extremal Problems in Linear Hypergraphs

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Abstract. An r-uniform hypergraph is linear if any pair of edges of the hypergraph has at most one common vertex. In this paper, we focus on Turán type problems for linear hypergraphs. For a graph F, the r-expansion F^+ is the r-graph obtained from F by enlarging each edge of F with $r-2$ new vertices disjoint from $V(F)$ such that distinct edges of F are enlarged by distinct vertices. First, we prove that a $K_{s,t}^+$-free r-partite linear r-graph of order n has at most $\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}} n^{2-\frac{1}{s}} + O(n^{2-\frac{2}{s}})$ edges, which strengthens the results of Lazebnik, Verstraëte [Electron. J. Comb., 10(2003), #R25] and Timmons [Electron. J. Combin. 29 (3)(2022)#P3.46]. Second, we give a sharp upper bound for the number of edges of linear hypergraphs containing no expansion of double star, where the double star is a tree with two vertices of degree greater than one.

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1 Introduction

An r-uniform hypergraph (or r-graph for short) $H=(V(H),E(H))$ is a pair of a vertex set $V(H)$ and an edge set $E(H)$, where $E(H)$ is a collection of r-element

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subset of $V(H)$. By convention, a graph is defined as a 2-uniform hypergraph. Throughout this paper, we let $r\geq 2$.

For a family of r-graphs $\mathcal{F}$, the r-graph H is $\mathcal{F}$-free if H does not contain any member of $\mathcal{F}$ as a subgraph. The Turán number, denoted by $\mathrm{ex}_r(n,\mathcal{F})$, is the maximum possible number of edges of an n-vertex $\mathcal{F}$-free r-graph. Determining the Turán number of graphs and hypergraphs is one of the central problems in extremal combinatorics. For nonbipartite graphs, the Turán number was asymptotically solved by the celebrated Erdős-Stone-Simonovits Theorem [4, 5]. Nevertheless, there is little understanding of the Turán number of bipartite graphs and hypergraphs. We refer the reader to the surveys [1, 6, 7, 11, 13] for more details.

An r-uniform hypergraph H is linear if $|e_1\bigcap e_2|\leq 1$ for any $e_1,e_2\in E(H)$. Similar to the Turán number, for a family of r-graphs $\mathcal{F}$, the linear Turán number $ex_r^{lin}(n,\mathcal{F})$ of $\mathcal{F}$ is the maximum possible number of edges of an n-vertex $\mathcal{F}$-free linear r-graph. The linear Turán problem is closely related to the famous Brown-Erdős-Sós problem. Let (v,e)-configuration be the collection of 3-graphs with e edges and at most v vertices. Brown-Erdős-Sós [2] conjectured that the number of edges of an n-vertex 3-graph without (v,e)-configuration is $o(n^2)$. It was shown by Solymosi [14] that the Brown-Erdős-Sós conjecture is equivalent to proving the linear Turán number of (v,e)-configuration is $o(n^2)$.

Recently, the Turán type problems in linear hypergraphs have attracted considerable attention. For a graph F and a positive integer $r\geq 2$, the r-expansion of F, denoted by F^+, is the r-graph obtained from F by enlarging each edge of F with $r-2$ new vertices disjoint from $V(F)$ such that distinct edges of F are enlarged by distinct vertices. Let C_k be the cycle with k vertices and $K_{s,t}$ be the complete bipartite graph with one part of size s and the other part of size t. Lazebnik and Verstraëte [12] showed that $ex_3^{lin}(n,\{C_3^+,K_{2,2}^+\})=\frac{1}{6}n^{3/2}+O(n)$. For $t\geq 2$, Timmons [16] proved that $ex_r^{lin}(n,\{C_3^+,K_{2,t}^+\})\leq\frac{\sqrt{t-1}}{r(r-1)}n^{3/2}+O(n)$. Gerbner, Methuku and Vizer [8] further showed that $ex_r^{lin}(n,\text{Berge-}K_{2,t})\leq\frac{\sqrt{t-1}}{r(r-1)}n^{3/2}+O(n)$, where Berge-$K_{2,t}$ is a collection of hypergraphs whose member is obtained from enlarging each edge of $K_{2,t}$ to a unique hyperedge. Gao and Chang [9] enhanced these results by showing that for all integers $r\geq 2$ and $t\geq s\geq 2$, $ex_r^{lin}(n,K_{s,t}^+)\leq\frac{(t-1)^{\frac{1}{s}}}{r(r-1)}n^{2-\frac{1}{s}}+O(n)$. Meanwhile, Lazebnik and Verstraëte [12] also studied the Zarankiewicz type problems in linear hypergraph, i.e., Turán type problems with host graph is r-partite. Let $z_r^{lin}(n,\mathcal{F})$ be the maximum number of edges in a $\mathcal{F}$-free r-partite r-graph with n vertices in each part. Lazebnik and Verstraëte [12] showed that $\frac{1}{2}n^{3/2}-O(n)\leq z_3^{lin}(n,\{C_3^+,K_{2,2}^+\})\leq\frac{1}{\sqrt{2}}n^{3/2}+O(n)$. Timmons [16] proved that $z_3^{lin}(n,\{C_3^+,K_{2,3}^+\})=n^{3/2}+o(n^{3/2})$.

In this paper, we focus on Turán type problems for linear hypergraphs. Our first result strengthens the results of Lazebnik, Verstraëte [12] and Timmons [16].

Theorem 1.1. *For $t \geq s \geq 2$, we have*

$$z_r^{lin}(n, K_{s,t}^+) \leq \left(\frac{t-1}{r-1}\right)^{\frac{1}{s}} n^{2-\frac{1}{s}} + O(n^{2-\frac{2}{s}}).$$

Combining our Theorem 1.1 and the results of Timmons [16], Lazebnik and Verstraëte [12], we have the following corollaries.

Corollary 1.1.

$$z_3^{lin}(n, K_{2,3}^+) = n^{\frac{3}{2}} + O(n).$$

Corollary 1.2.

$$\frac{1}{2} n^{3/2} - O(n) \leq z_3^{lin}(n, K_{2,2}^+) \leq \frac{1}{\sqrt{2}} n^{3/2} + O(n).$$

A linear hypergraph is acyclic if it does not contain any C_k^+ for any $k \geq 3$. The linear Turán number of acyclic hypergraph was initiated by Carbonero, Fletchcher, Guo, Gyárfás, Wang and Yan [3], who studied the linear Turán number of "crown", where "crown" is a 3-graph with vertex set $\{1,2,\ldots,9\}$ and edge set $\{123,145,267,389\}$. Tang, Wu, Zhang and Zheng [15] gave the asymptotic bound for the linear Turán number of "crown". Gyárfás, Ruszinkó and Sárközy [10] proved upper bounds for linear Turán number of many acyclic 3-graphs. Zhang and Wang [17] extended these to 4-graphs. Zhou and Yuan [18] investigated the linear Turán number of star-path forests.

Our next result is a sharp bound for linear Turán number of the expansion of double star $S_{\alpha,\beta}$, where the double star $S_{\alpha,\beta}$ is a tree with two adjacent center vertices u and v, α leaves attached to u and β leaves attached to v.

A Steiner triple system $STS(n)$ is a linear 3-graph on n vertices whose edges cover all pairs of vertices exactly once. It is well known that the Steiner triple systems exist if and only if $n \equiv 1,3 \pmod 6$. The notion can be extended to other values of n, a maximal partial triple system, $MPTS(n)$, is a linear 3-graph on n vertices whose edges cover the maximum number of pairs of vertices. For convenience, we let $|MPTS(n)| = |E(MPTS(n))|$.

Theorem 1.2. *Let α and β be two positive integers with $\beta > \alpha \geq 1$, and let $l = 2(2\alpha+\beta)+1$. Then*

$$ex_3^{lin}(n, S_{\alpha,\beta}^+) \leq \frac{|MPTS(l)|}{l} n.$$

Moreover, the equality holds when $\alpha = 1$, $l|n$ and $l \equiv 1,3$ (mod 6).

We use standard notation throughout. In particular, $[n]$ denotes the set $\{1,2,...,n\}$. The rest of this paper is organized as follows. In the next section, we prove Theorem 1.1. Theorem 1.2 is proven in Section 3.

2 Proof of Theorem 1.1

To establish our main result, we employ the following lemma, whose proof can be found in [9]. For a vertex $v\in V(H)$, the neighborhood of v in H is defined as

$$N_H(v)=\{x\in V(H)\backslash\{v\}:\{v,x\}\subseteq h \text{ for some } h\in E(H)\}.$$

For convenience, we omit the subscript if it does not cause any confusion. We sometimes identify the pair $\{a,b\}$ and the triple $\{a,b,c\}$ with ab and abc.

Lemma 2.1. *Let H be an n-vertex $K^+_{s,t}$-free linear r-graph. For any $v\in V(H)$, the number of edges $h\in E(H)$ with $|h\cap N(v)|\geq 2$ and $v\notin h$ is $O(d_v^{2-\frac{1}{s-1}})$.*

Proof the Therorem 1.1. Let H be a $K^+_{s,t}$-free linear r-partite r-graph with each part n vertices. Denote the r parts of H by $V_1,V_2,\ldots,V_r$.

Let d be the average degree of H and δ be the minimum degree. Note that if $d\leq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{1-\frac{1}{s}}$, then the number of edges in H is at most $\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{2-\frac{1}{s}}$. Thus we assume that $d>\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{1-\frac{1}{s}}$. We may also assume that $\delta\geq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}\frac{n^{1-\frac{1}{s}}}{r}$. Otherwise, if there is a vertex whose degree less than $\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}\frac{n^{1-\frac{1}{s}}}{r}$, we delete this vertex and all these edges containing this vertex. This procedure does not decrease the average degree, since $\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}\frac{n^{1-\frac{1}{s}}}{r}<\frac{d}{r}$. Repeat this procedure as long as we can. Eventually, we obtain a hypergraph H' with at least $n_1\geq\sqrt{n}$ vertices, average degree $d'\geq d$ and minimum degree at least $\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}\frac{n^{1-\frac{1}{s}}}{r}$. If we have $d'\leq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n_1^{1-\frac{1}{s}}+O(n_1^{1-\frac{2}{s}})$, then $d\leq d'\leq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{1-\frac{1}{s}}+O(n^{1-\frac{2}{s}})$, which implies the Theorem. Thus we can assume that $\delta\geq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}\frac{n^{1-\frac{1}{s}}}{r}$.

Let $\mathcal{S}$ be the set of all pairs of the form $(v,\{x_1,x_2,\ldots,x_s\})$, where $v\in V(H)$, $x_1,x_2,\ldots,x_s$ are vertices in the same part with $v\notin\{x_1,x_2,\ldots,x_s\}$, and vx_i is contained in some edges for all $1\leq i\leq s$. Since H is a linear r-graph, each vx_i is contained in exactly one edge of H and we denote this edge by h_{vx_i}. As H is r-partite, we have $h_{vx_i}\neq h_{vx_j}$ for $i\neq j$.

Let d_v be the degree of v in H. Clearly, if we have already chosen a vertex v, then we have $(r-1)\binom{d_v}{s}$ to choose the vertex set $\{x_1,x_2,\ldots,x_s\}$. Thus

$$|S|=\sum_{v\in H}(r-1)\binom{d_v}{s}.$$

For a pair $(v,\{x_1,x_2,\ldots,x_s\})\in\mathcal{S}$, let $(v,\{x_1,x_2,\ldots,x_s\})$ be "bad" if there exist i,j such that h_{vx_i} and h_{vx_j}, together with another edge $h\in E(H)$, form a C_3^+, and we say $(v,\{x_1,x_2,\ldots,x_s\})$ is bad for h. If a pair is not a bad pair, then we say the pair is "good". Let $\mathcal{B}\subset\mathcal{S}$ be the set of bad pairs and $\mathcal{S}'=\mathcal{S}\backslash\mathcal{B}$ be the set of good pairs.

Claim 1. $|\mathcal{S}'|\geq \frac{r-1}{s!}rnd^s-O(nd^{s-\frac{1}{s-1}})$.

Choose a vertex $v\in V(H)$. If $(v,\{x_1,x_2,\ldots,x_s\})$ is a bad pair, then there is an edge $h\in E(H)$ with $|h\cap N(v)|\geq 2$ and $v\notin h$. Meanwhile, for an edge $h\in E(H)$ with $|h\cap N(v)|\geq 2$ and $v\notin h$, there are at most $(r-1)\binom{r}{2}\binom{d_v-2}{s-2}$ pairs $(v,\{x_1,x_2,\ldots,x_s\})$ such that these pairs are bad for h. By Lemma 2.1, the number of edges $h\in E(H)$ with $|h\cap N(v)|\geq 2$ and $v\notin h$ is $O(d_v^{2-\frac{1}{s-1}})$. Thus the number of bad pairs $(v,\{x_1,x_2,\ldots,x_s\})$ is at most $O(d_v^{s-\frac{1}{s-1}})$. Summing it for all vertices, we have

$$|\mathcal{B}|=\sum_{v\in V(H)}O(d_v^{s-\frac{1}{s-1}}).$$

As $\mathcal{S}'=\mathcal{S}\backslash\mathcal{B}$, we have

$$\begin{aligned}|\mathcal{S}'|&=\sum_{v\in H}(r-1)\binom{d_v}{s}-\sum_{v\in V(H)}O(d_v^{s-\frac{1}{s-1}})\\&=\frac{r-1}{s!}\sum_{v\in H}d_v^s-\sum_{v\in V(H)}O(d_v^{s-\frac{1}{s-1}}).\end{aligned}$$

Clearly, there exist an x_0 such that the function $f(x)=\frac{r-1}{s!}x^s-O(x^{s-\frac{1}{s-1}})$ is convex for $x\geq x_0$. Recall that $\delta\geq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}\frac{n^{1-\frac{1}{s}}}{r}$. Let n be sufficiently large with $\delta>x_0$. Then by Jensen¡¯s inequality, we have

$$|\mathcal{S}'|\geq\frac{r-1}{s!}rnd^s-O(nd^{s-\frac{1}{s-1}}),$$

as desired.

Claim 2. $|\mathcal{S}'|\leq r(t-1)\binom{n}{s}$.

Choose s vertices $\{x_1,x_2,\ldots,x_s\}$ from a part of H, say $\{x_1,x_2,\ldots,x_s\}\subset V_i$. If there are t vertices $v_1,v_2,\ldots,v_t$ such that $(v_k,\{x_1,x_2,\ldots,x_s\})$ is a good pair for each $1\leq k\leq t$, then the edge set $\{h_{v_kx_i}:\ 1\leq k\leq t,\ 1\leq i\leq s\}$ forms a copy of $K_{s,t}^+$, a contradiction. Thus there are at most $t-1$ choices of v such that $(v,\{x_1,x_2,\ldots,x_s\})$ is a good pair. Hence

$$|\mathcal{S}'|\leq(t-1)\sum_{1\leq i\leq r}\binom{|V_i|}{s}=r(t-1)\binom{n}{s}.$$

Combining Claim 1 and Claim 2, we have

$$\frac{r-1}{s!}rnd^s-O(nd^{s-\frac{1}{s-1}})\leq r(t-1)\binom{n}{s}<\frac{r(t-1)n^s}{s!}.$$

That is,

$$\left(d^s-O(d^{s-\frac{1}{s-1}})\right)^{1/s}\leq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{\frac{s-1}{s}}.$$

When x tends to 0, the function $g(x)=(1+x)^{\frac{1}{s}}$ is convergent to $1+\frac{x}{s}$. Thus,

$$\left(d^s-O(d^{s-\frac{1}{s-1}})\right)^{1/s}=d\left(1-O(d^{-\frac{1}{s-1}})\right)^{\frac{1}{s}}=d-O(d^{\frac{s-2}{s-1}}).$$

Clearly, $d=O(n^{\frac{s-1}{s}})$. So $O(d^{\frac{s-2}{s-1}})=O(n^{\frac{s-2}{s}})$. Hence,

$$d\leq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{\frac{s-1}{s}}+O(n^{\frac{s-2}{s}}).$$

As H has rn vertices,

$$E(H)\leq\frac{rnd}{r}=nd\leq\left(\frac{t-1}{r-1}\right)^{\frac{1}{s}}n^{2-\frac{1}{s}}+O(n^{2-\frac{2}{s}}).$$

This completes the proof. □

3 Proof of the Theorem 1.2

Proof the Therorem 1.2. According to the definition of $MPTS(l)$, we can obtain the following results.

Claim 3. $\frac{|MPTS(l)|}{l}\geq\frac{l-2}{6}>\alpha$.

Since $l=2(2\alpha+\beta)+1$, it is easy to see that $l\not\equiv 0,2,4 \pmod 6$. If $l\equiv 1,3 \pmod 6$, we know that $MPTS(l)\cong STS(l)$. Hence,

$$\frac{|MPTS(l)|}{l}=\frac{|STS(l)|}{l}=\frac{l-1}{6}>\frac{l-2}{6}=\frac{2(2\alpha+\beta)-1}{6}>\alpha,$$

where the last inequality follows from $\beta\geq\alpha+1$. Now, suppose $l\equiv 5 \pmod 6$. In this case, it is well known that $MPTS(l)$ is a linear 3-graph whose edges cover all the vertex pairs except four vertex pairs forming a four-cycle. So

$$|MPTS(l)|=\frac{1}{3}\left(\frac{l(l-1)}{2}-4\right)=\frac{l(l-1)}{6}-\frac{4}{3}.$$

Hence,

$$\frac{|MPTS(l)|}{l}=\frac{l(l-1)}{6l}-\frac{4}{3l}>\frac{l-1}{6}-\frac{1}{6}=\frac{l-2}{6}=\frac{2(2\alpha+\beta)-1}{6}>\alpha,$$

where the first inequality follows from $\frac{l}{2}>4$ and the second inequality follows from $\beta\geq\alpha+1$.

Let H be a minimal counterexample, i.e., an $S^+_{\alpha,\beta}$-free linear 3-graph with n vertices and more than $\frac{|MPTS(l)|}{l}n$ edges, where n is as small as possible. Let $\delta(H)$ and $\Delta(H)$ be the maximum and minimum degree of H, respectively. From the minimality, H has just one connected component. By Claim 3, we can deduce that $\delta(H)>\alpha$. Otherwise, suppose that there exists a vertex $v\in V(H)$ such that $d(v)\leq\alpha$. Let $H_1=H-\{v\}$. Then

$$\begin{aligned}|E(H_1)|=|E(H)|-d(v)&\geq\frac{|MPTS(l)|}{l}n-\alpha\\&>\frac{|MPTS(l)|}{l}n-\frac{|MPTS(l)|}{l}=\frac{|MPTS(l)|}{l}(n-1).\end{aligned}$$

Then H_1 is a smaller counterexample, which is a contradiction. Furthermore, by Claim 3 and the fact that H is $S^+_{\alpha,\beta}$-free, we can obtain the following two claims. We call the vertex of maximum degree of a star the center.

Claim 4. If there exists an $S^+_{2\alpha+\beta}$ with center w in H, then any edge that intersects it and does not contain the vertex w must be contained in $V(S^+_{2\alpha+\beta})$.

Suppose for a contradiction that there exists an edge $e_1\in E(H)$ such that $w\notin e_1$ and $1\leq|e_1\cap V(S^+_{2\alpha+\beta})|\leq 2$. Let $z\in e_1\cap V(S^+_{2\alpha+\beta})$ and e be the unique edge of $S^+_{2\alpha+\beta}$ such that $\{w,z\}\subseteq e$. Define $E_z(H):=\{e\in E(H)\mid z\in e \text{ and } w\notin e\}$. Clearly, $e_1\in E_z(H)$. As $\delta(H)>\alpha$, there exist at least $\alpha-1$ edges $e_2,e_2,\cdots e_\alpha\in E_z(H)\setminus\{e_1\}$. By $|e_1\cap V(S^+_{2\alpha+\beta})|\leq 2$, we have that there exist at least β edges $e'_1,e'_2,\cdots,e'_\beta\in E(S^+_{2\alpha+\beta})$ such that $e_i\cap e'_j=\emptyset$ for any $i\in[\alpha]$ and $j\in[\beta]$. Hence $e_1,e_2,\cdots,e_\alpha,e'_1,e'_2,\cdots,e'_\beta$ and e form an $S^+_{\alpha,\beta}$, a contradiction.

Claim 5. $\Delta(H)\leq 2\alpha+\beta$.

Suppose $\Delta(H)\geq 2\alpha+\beta+1$. Let $u\in V(H)$ be a vertex with maximum degree in H. Select the largest star S^+_k with center u. Clearly, $k\geq 2\alpha+\beta+1$. Select a vertex w in S^+_k different from u. By $\delta(H)>\alpha$, there exist at least α edges $e_1,e_2,\cdots,e_\alpha$ such that $w\in e_i$ and $u\notin e_i$ for each $i\in[\alpha]$. Meanwhile, since $k\geq 2\alpha+\beta+1$, there exist at least β edges $e'_1,e'_2,\cdots,e'_\beta\in E(S^+_k)$ such that $e_i\cap e'_j=\emptyset$ for any $i\in[\alpha]$ and $j\in[\beta]$. Let e be the unique edge of $S^+_{2\alpha+\beta}$ such that $\{u,w\}\subseteq e$. Then $e_1,e_2,\cdots,e_\alpha,e'_1,e'_2,\cdots,e'_\beta$ and e form an $S^+_{\alpha,\beta}$, a contradiction.

Next, we will show that $ex^{lin}_3(n,S^+_{\alpha,\beta})\leq\frac{|MPTS(l)|}{l}n$ by dividing it into the following two cases.

Case 1. $l\equiv 1,3 \pmod 6$.

In this case, it is clear that $MPTS(l)$ is $(2\alpha+\beta)$-regular and $\frac{|MPTS(l)|}{l}=\frac{l-1}{6}=\frac{2\alpha+\beta}{3}$. By Claim 5, we have that

$$E(H)\leq\frac{2\alpha+\beta}{3}n=\frac{|MPTS(l)|}{l}n,$$

which contradicts our assumption. Thus $ex_3^{lin}(n,S_{\alpha,\beta}^+)\leq\frac{|MPTS(l)|}{l}n$.

Case 2. $l\equiv 5 \pmod 6$.

Recall that $\frac{|MPTS(l)|}{l}=\frac{l(l-1)}{6l}-\frac{4}{3l}>\frac{l-2}{6}$ in this case. Suppose $\Delta(H)\leq 2\alpha+\beta-1$. Then,

$$E(H)\leq\frac{2\alpha+\beta-1}{3}n=\frac{2(2\alpha+\beta-1)}{6}n=\frac{l-3}{6}n<\frac{l-2}{6}n<\frac{|MPTS(l)|}{l}n,$$

which is a contradiction. Thus $\Delta(H)\geq 2\alpha+\beta$. Combining this with Claim 5, we have $\Delta(H)=2\alpha+\beta$.

Let $w\in V(H)$ be a vertex with maximum degree in H. Selecting an $S_{2\alpha+\beta}^+$ with center w, by Claim 4, we can observe that any edge intersecting it must be completely inside $V(S_{2\alpha+\beta}^+)$. Hence, $|E(H)|\leq|MPTS(l)|$, which is a contradiction. This finishes the proof of $ex_3^{lin}(n,S_{\alpha,\beta}^+)\leq\frac{|MPTS(l)|}{l}n$.

Let $\alpha=1$ and $l\equiv 1,3 \pmod 6$. In this case, $MPTS(l)$ is a Steiner triple system and every vertex pair is exactly covered by one edge. If $MPTS(l)$ contains a $S_{1,\beta}^+$ as subgraph, then we can easily find some vertex pairs that can not be covered by an edge, a contradiction. Thus $MPTS(l)$ is $S_{1,\beta}^+$-free. For $l|n$, the n-vertex linear hypergraph which consists of vertex disjoint $MPTS(l)$'s is $S_{1,\beta}^+$-free and has $\frac{|MPTS(l)|}{l}n$ edges. This completes the proof. □

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