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Minimum Degrees of Minimally $\frac{1}{k}$-tough Graphs

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Abstract. Let t be a non-negative real number. If a graph G has toughness t, and deleting any edge of G decreases its toughness, then G is a minimally t-tough graph. Katona et al. conjectured that the minimum degree of every minimally t-tough graph is $\lceil 2t \rceil$. Although the conjecture is disproved in general, authors attempt to confirm it for some classes of graphs. In this paper, for each positive integer $k \geq 2$, we prove that every minimally $\frac{1}{k}$-tough graph whose matching number is at most 3 has a vertex of degree one.

AMS subject classifications: 05C07; 05C40

Key words: minimally t-tough; toughness; matching number; minimum degree.

1 Introduction

All graphs that we consider are simple, undirected and finite. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a non-empty subset $T \subseteq V(G)$, let $G[T]$ be the subgraph induced by T, and let $G-T = G[V(G)-T]$. If $T = \{v\}$, then we use $G-v$ to replace $G-T$. A vertex subset $T \subseteq V(G)$ is a cutset if $G-T$ is disconnected. The empty set is considered a cutset of a disconnected graph. We use $\omega(G)$ to denote the number of components and $\kappa(G)$ the connectivity of G. Given a graph H, G is called H-free if it does not contain H as an induced subgraph.

Let t be a non-negative real number. A graph G is *t-tough* if $\frac{|S|}{\omega(G-S)} \geq t$ for any cutset S of G. The *toughness* of G, denoted by $\tau(G)$, is defined as

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$\tau(G)=\min\{\frac{|S|}{\omega(G-S)}\,|\,S$ is a cutset of $G\}$ if G is a non-complete graph, and $\tau(G)=\infty$ if G is a complete graph, which was introduced in [5]. The toughness of a non-complete tree is the reciprocal of its maximum degree. Chvátal [5] proposed a conjecture that there exists a constant t' such that all t'-tough graphs are hamiltonian. Bauer, Broersma and Veldman [1] showed that $t'\geq\frac{9}{4}$ if the conjecture is true. The conjecture is still open. Broersma, Engbers and Trommel [2] introduced the concept of minimally t-tough graphs to investigate the conjecture.

Definition 1.1. *Let t be a positive real number. A connected graph G is minimally t-tough if $\tau(G)=t$ and $\tau(G-e)<t$ for every edge e in G.*

A graph G is *minimal* if there exists a positive real number t such that G is minimal t-tough. Clearly, for a non-complete graph, we have $\kappa(G)\geq 2\tau(G)$. Kriesell [6] conjectured that each minimally 1-tough graph has a vertex of degree two. Katona et al. confirmed this conjecture for minimally 1-tough claw-free graphs in [7] and proposed a generalized version in [8].

Conjecture 1.1 ([8])**.** The minimum degree of each minimally t-tough graph is $\lceil 2t\rceil$.

Conjecture 1.1 was disproved in [12] by constructing a family of counterexamples. But authors attempt to prove that Conjecture 1.1 holds for some classes of graphs, see [3-5, 7-11]. Let k be a positive integer. Conjecture 1.1 was confirmed for minimally $\frac{1}{2}$-tough claw free graphs [8], minimally $\frac{1}{k}$-tough $2K_2$-free graphs with $k\geq 1$ [10], and minimally $\frac{1}{k}$-tough $K_{1,k+1}$-free graphs with $k\geq 2$ [11]. In this paper, for each positive integer $k\geq 2$, we confirm Conjecture 1.1 for minimally $\frac{1}{k}$-tough graphs with matching number at most three by characterizing their structures.

Theorem 1.2. *Let $k\geq 2$ be a positive integer and G be a minimally $\frac{1}{k}$-tough graph. If $\mu(G)\leq 3$, then G belongs to one of the graph families $\mathcal{H}_i$ for $1\leq i\leq 21$ as shown in Figures 2 and 3.*

Theorem 1.3. *Let $k\geq 2$ be a positive integer and G be a minimally $\frac{1}{k}$-tough graph. If $\mu(G)\leq 3$, then the minimum degree of G is one.*

2 Preliminaries

Let G be a simiple graph. We say that a component of G is trivial if it has only one vertex, otherwise, it is non-trivial. For a vertex $v\in V(G)$, we let $N_G(v)$ be the neighbor set and $d_G(v)$ the degree of v. We define $V_i(G)$ as the set of all vertices of degree i in G, and $V_{\geq 3}(G)$ be the set of vertices of degree at least three, that is $V_{\geq 3}(G)=\{v\,|\,d_G(v)\geq 3, v\in V(G)\}$. In a graph, we call a vertex of degree one as a leaf vertex. The vertex of degree r in a star $K_{1,r}$ is called its center vertex or center. For two non-empty vertex subsets T_1 and T_2 of G, let $G[T_1,T_2]$ denote the set of edges with one end in T_1 and another end in T_2, and we denote the set obtained from T_1 by deleting all the vertices in T_2 by T_1-T_2. We replace T_1-T_2 by $T_1-\{x\}$ if $T_2=\{x\}$. We say v is adjacent to T_1 if $N_G(v)\cap T_1\neq\emptyset$. The matching number $\mu(G)$ is defined as the size of a largest matching in G. For an edge $e=xy$ of G, we define $g(xy)=|N_G(x)\cap N_G(y)|$.

Lemma 2.1 ([8])**.** *Let $t\leq 1$ be a positive rational number, G be a graph with toughness t, and let S be a non-empty proper subset of $V(G)$. Then $|S|\geq t\cdot\omega(G-S)$.*

Lemma 2.2 ([8])**.** *Let t be a positive rational number, and let G be a minimally t-tough graph. For each edge e of G, then one of the following results holds:*
(i) e is a bridge of G.
(ii) there exists a vertex set $S=S(e)\subseteq V(G)$ with

$$t\cdot\omega(G-S)\leq|S|<t\cdot\omega((G-e)-S).$$

If e is a bridge of G, we define $S=S(e)=\emptyset$.

The following lemma is obtained in [10] based on Lemma 2.2.

Lemma 2.3 ([10])**.** *Let k be a positive integer, and let G be a minimally $\frac{1}{k}$-tough graph. For any non-bridge edge $e\in E(G)$, there exists a non-empty vertex set $S\subseteq V(G)$ such that*

$$k|S|=\omega(G-S)=\omega((G-e)-S)-1.$$

Lemma 2.4. *Let $k\geq 2$ be a positive integer, G be a minimally $\frac{1}{k}$-tough graph, and let S be a vertex set which satisfies Lemma 2.2 for a non-bridge edge e of G. If $|S|\geq 2$, then each vertex of S is adjacent to at least three components of $G-S$ and $S\subseteq V_{\geq 3}(G)$.*

Proof. Since S is a vertex set satisfying Lemma 2.2 for the edge e of G, we have

$$\frac{1}{k}\cdot\omega(G-S)\leq|S|<\frac{1}{k}\cdot\omega((G-e)-S)$$

and e is a bridge of $G-S$. Then $\omega(G-S)>k|S|-1\geq 3$. Suppose to the contrary that there exists a vertex $s\in S$ which is adjacent to less than three components of $G-S$. Then $\omega(G-(S-\{s\}))\geq\omega(G-S)-1\geq 3$. Thus, $S-\{s\}$ is a cutset of G. However,

$$\frac{|S-\{s\}|}{\omega(G-(S-\{s\}))}\leq\frac{|S|-1}{\omega(G-S)-1}<\frac{\frac{1}{k}\cdot\omega(G-S)+\frac{1}{k}-1}{\omega(G-S)-1}=\frac{1}{k}+\frac{\frac{2}{k}-1}{\omega(G-S)-1}\leq\frac{1}{k},$$

which contradicts $\tau(G)=\frac{1}{k}$. Thus, each vertex of S is adjacent to at least three components of $G-S$. This implies that each vertex of S is of degree at least three in G. Therefore, $S\subseteq V_{\geq 3}(G)$. □

Lemma 2.5. *Let* $k\geq 2$ *be a positive integer,* G *be a minimally* $\frac{1}{k}$*-tough graph, and let* S *be a vertex set which satisfies Lemma 2.2 for a non-bridge edge* $e=uv$ *of* G*. Then the following results hold:*
(1) *If* $V_{\geq 3}(G)-\{u,v\}$ *is of size at most one, then* $|S|=1$*.*
(2) *If* $V_{\geq 3}(G)-\{u,v\}=\{u',v'\}$ *and* $\omega(G-\{u',v'\})=|A|+|B|+1$*, then* $|S|=1$*, where* A *and* B *are sets of leaf vertices which are adjacent to* u' *and* v'*, respectively.*

Proof. If $|S|\geq 2$, then by Lemma 2.4, we have $S\subseteq V_{\geq 3}(G)-\{u,v\}$. Then the result (1) holds obviously. For the second part, suppose to the contrary that $|S|\geq 2$. Then $S=\{u',v'\}$ and $\omega(G-S)=k|S|\geq 2k$. It follows from $\tau(G)=\frac{1}{k}$ that $0\leq |A|\leq k-1$ and $0\leq |B|\leq k-1$. Then $\omega(G-S)=|A|+|B|+1\leq 2k-1$, which contradicts $\omega(G-S)\geq 2k$. $\square$

Lemma 2.6. *Let* $t\leq\frac{1}{2}$ *be a positive rational number,* G *be a minimally* t*-tough graph, and let* S *be a vertex set which satisfies Lemma 2.2 for an edge* e*. If* e *is a part of a cycle in which every vertex, except for the endpoints of* e*, is not a cut vertex of* G*, then* $|S|\geq 2$*.*

Proof. Note that S is a vertex set satisfying Lemma 2.2 for the edge e of G, that is,

$$t\cdot\omega(G-S)\leq |S|<t\cdot\omega((G-e)-S)$$

and e is a bridge of $G-S$. Suppose to the contrary that $|S|=1$. Let u and v be the endpoints of the edge e. Since e is contained in a cycle, say C, the unique member of S must be a vertex of $V(C)-\{u,v\}$. Since each vertex of $V(C)-\{u,v\}$ is not a cut vertex of G, we have $\omega(G-S)=1$. Then $t>\frac{|S|}{\omega((G-e)-S)}=\frac{|S|}{\omega(G-S)+1}=\frac{1}{2}$, which contradicts $t\leq\frac{1}{2}$. $\square$

Lemma 2.7. *Let* $k\geq 2$ *be a positive integer,* G *be a minimally* $\frac{1}{k}$*-tough graph, and let* w *be a cut vertex of* G*. If* W *is a component of* $G-w$*, then the induced subgraph* $G[V(W)\cup\{w\}]$ *is not isomorphic to* G_0 *as shown in Figure 1.*

Proof. Suppose to the contrary that $G[V(W)\cup\{w\}]$ is isomorphic to G_0. Let $v_1,v_2,\ldots,v_l$ be vertices of W which is adjacent to w, where $l\geq 2$. Let $e=wv_l$, and let S be a vertex set satisfying Lemma 2.2 for e. Clearly, e is a part of a cycle in which every vertex, excepting the endpoints of e, is not a cut vertex of G. By Lemma 2.6, we have $|S|\geq 2$. Since vv_1wv_lv forms a cycle, it follows that $v\in S$ or $v_1\in S$. However, since $N_G(v_1)=\{v,w\}\subseteq N_G(v_l)$ and $N_G(v)=\{v_1,v_2,\ldots,v_l\}\subseteq N_G(w)$, if $v\in S$ (or $v_1\in S$), then v (or v_1) is adjacent to exactly one component of $G-S$, which contradicts Lemma 2.4. $\square$

Lemma 2.8 ([4])**.** *Let* t *be a positive rational number, and let* G *be a minimally* t*-tough graph. Then each vertex of any triangle in* G *has degree at least three.*

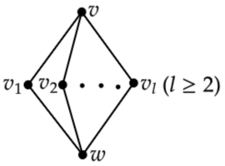

Figure 1. G_0.

Observation 2.1. *If H is a connected simple graph with $\mu(H)=1$, then H is a triangle or a star.*

3 Minimally $\frac{1}{k}$-tough graphs

In this section, we first construct a family of minimally $\frac{1}{k}$-tough graphs.

Lemma 3.1. *Let C be a cycle of length n, and let $\mathcal{T}$ be a set of disjoint trees whose maximum degree is at most k, where $k\geq 2$. Let $G(n,k)$ be the graph obtained from C by joining a leaf vertex of each tree in $\mathcal{T}$ and a vertex on C by an edge such that the maximum degree of $G(n,k)$ is $k+1$. Then the following results hold:*
(i) If $n\geq 4$, then $G(n,k)$ is a minimally $\frac{1}{k}$-tough graph if and only if $G(n,k)$ has two non-adjacent vertices with degree $k+1$;
(ii) If $n=3$, then $G(n,k)$ is a minimally $\frac{1}{k}$-tough graph if and only if each vertex on C has degree $k+1$.

Proof. Obviously, each vertex of degree $k+1$ in $G(n,k)$ is on the cycle C. We first show $\tau(G(n,k))=\frac{1}{k}$. Since each vertex of degree $k+1$ is a cut vertex of $G(n,k)$ such that the deletion of which produces exactly k components, we have $\tau(G(n,k))\leq\frac{1}{k}$. We now show $\tau(G(n,k))\geq\frac{1}{k}$. Let S be an arbitrary cutset of $G(n,k)$. Then S must contain at least one non-leaf vertex of $G(n,k)$. Without loss of generality, we assume that $u\in S$. Then $\omega(G(n,k)-u)\leq k$ and let $m=|S-\{u\}|$. Since the removal of each vertex in $S-\{u\}$ creates at most k new components, we have

$$\frac{|S|}{\omega(G(n,k)-S)}\geq\frac{1+m}{k+km}=\frac{1}{k}.$$

Therefore, $\tau(G(n,k))=\frac{1}{k}$.

Suppose that $G(n,k)$ has two non-adjacent vertices if $n\geq 4$ and each vertex on C has degree $k+1$ if $n=3$. Let e be an arbitrary edge of $G(n,k)$. If $e\notin E(C)$, then e is a bridge of $G(n,k)$, and so $\tau(G(n,k)-e)=0<\frac{1}{k}$. If $e\in E(C)$, then $G(n,k)-e$ is a tree with maximum degree $k+1$, and so $\tau(G(n,k)-e)=\frac{1}{k+1}<\frac{1}{k}$. Therefore, $G(n,k)$ is a minimally $\frac{1}{k}$-tough graph.

Assume now that $G(n,k)$ is a minimally $\frac{1}{k}$-tough graph. For $n\geq 4$, suppose to the contrary that there does not exist two non-adjacent vertices of degree $k+1$

in $G(n,k)$. Then $G(n,k)$ has either exactly one vertex of degree $k+1$, or exactly two adjacent vertices of degree $k+1$. For the former case, let v be the unique vertex with degree $k+1$ in $G(n,k)$, and let e_1 be an edge on C that is incident to v. For the later case, let v_1 and v_2 denote the vertices of degree $k+1$, and take the edge v_1v_2 as e_1. Then $G(n,k)-e_1$ is a tree with maximum degree k, and thus, $\tau(G(n,k)-e_1)=\frac{1}{k}=\tau(G(n,k))$, a contradiction. Thus, $G(n,k)$ has two non-adjacent vertices of degree $k+1$.

For $n=3$, let z_1, z_2 and z_3 denote the three vertices on C. Note that at least one vertex on C has degree $k+1$. Suppose to the contrary that at least one vertex on C has degree at most k. Without loss of generality, we assume that z_1 has degree at most k. Then $G(n,k)-z_2z_3$ is a tree with maximum degree k, and thus $\tau(G(n,k)-z_2z_3)=\frac{1}{k}=\tau(G(n,k))$, which contradicts the definition of minimally t-tough graphs. Therefore, each vertex on C has degree $k+1$. $\square$

By Lemma 3.1, we have the following result.

Proposition 3.1. *If G is a graph that belongs to the graph classes in Figures 2 and 3, then G is minimally $\frac{1}{k}$-tough for some positive integer $k\geq 2$.*

Proof. If G belongs to classes in Figure 2, then G is a tree with maximum degree greater than two. Thus, G is a minimally $\frac{1}{\triangle(G)}$-tough graph.

By Lemma 3.1, all graphs in Figure 3, except for $\mathcal{H}_{19}$, are minimally $\frac{1}{k}$-tough graphs. Now we will prove that $\mathcal{H}_{19}$ is a minimally $\frac{1}{k}$-tough graph. Since the deletion of any cut vertex of $\mathcal{H}_{19}$ produces exactly k components, we have $\tau(\mathcal{H}_{19})\leq \frac{1}{k}$. We now show $\tau(\mathcal{H}_{19})\geq\frac{1}{k}$. Let S be an arbitrary cutset of $\mathcal{H}_{19}$. Then S must contain at least one non-leaf vertex of $\mathcal{H}_{19}$. Without loss of generality, we assume that $u\in S$. Then $\omega(\mathcal{H}_{19}-u)\leq k$ and let $m=|S-\{u\}|$. Since the removal of each vertex in $S-\{u\}$ creates at most k new components, we have

$$\frac{|S|}{\omega(\mathcal{H}_{19}-S)}\geq\frac{1+m}{k+km}=\frac{1}{k}.$$

Therefore, $\tau(\mathcal{H}_{19})=\frac{1}{k}$.

Let e be an arbitrary edge of $\mathcal{H}_{19}$. If e is a bridge of $\mathcal{H}_{19}$, then $\tau(\mathcal{H}_{19}-e)=0<\frac{1}{k}$. If e is not a bridge of G, then since $\mathcal{H}_{19}-e$ contains a vertex whose deletion yields $k+1$ components, we have $\tau(\mathcal{H}_{19}-e)\leq\frac{1}{k+1}<\frac{1}{k}$. Therefore, $\mathcal{H}_{19}$ is a minimally $\frac{1}{k}$-tough graph. $\square$

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by examining two cases depending on whether G is a tree.

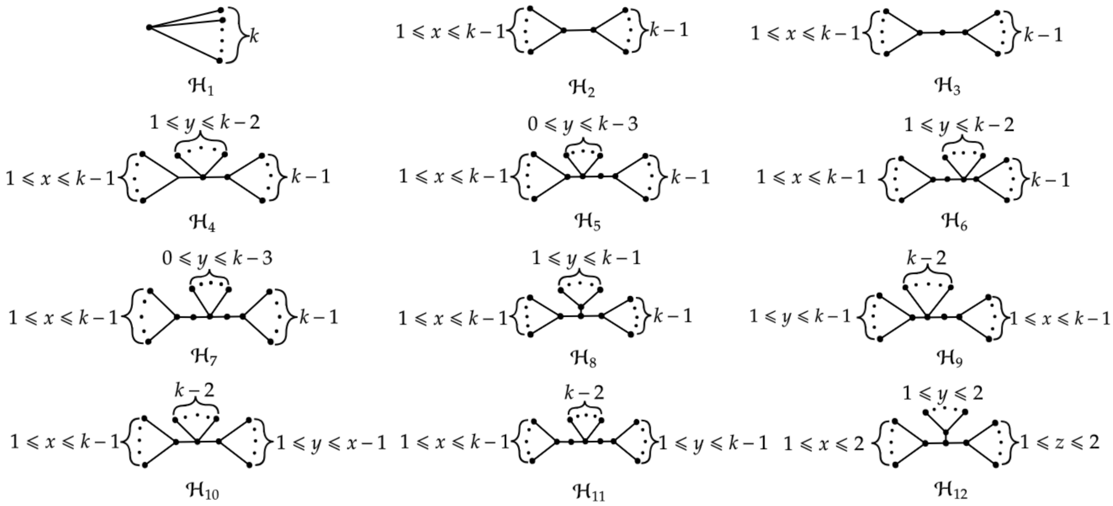

Figure 2. Minimally $\frac{1}{k}$-tough trees with $k\geq 2$.

Theorem 4.1. *let $k\geq 2$ be a positive integer and G be a minimally $\frac{1}{k}$-tough tree. If $\mu(G)\leq 3$, then G belongs to one of the graph families $\mathcal{H}_i$ for $1\leq i\leq 12$ as shown in Figure 2.*

Proof. The maximum degree of G is k. Let v be a vertex with maximum degree in G. If $\mu(G)=1$, then G belongs to class $\mathcal{H}_1$. If $\mu(G)=2$, the graph $G-v$ must have a non-trivial component; otherwise, G would be a star with center v, contradicting $\mu(G)=2$. Let us denote one such non-trivial component of $G-v$ by G_1, and let $G_2=G-V(G_1)$. Then $\mu(G_1)=\mu(G_2)=1$, meaning both graphs G_1 and G_2 are stars. Let the vertex u be the center of graph G_1. If v is the center of G_2, then: G belongs to class $\mathcal{H}_2$ if v is adjacent to u and G belongs to class $\mathcal{H}_3$ if v is adjacent to any leaf vertex within G_1. If v is not the center of G_2, then it can be concluded that $k=2$ and G_2 is a path of length two with one endpoint being vertex v. Thus, G is a path of length four that belongs to class $\mathcal{H}_3$. Finally, we encounter the case where $\mu(G)=3$, we will proceed by analyzing two subcases.

Case 1. $G-v$ has trivial components.

Let N be the union of the non-trivial components of $G-v$. Since $\mu(G)=3$, we have $1\leq |N|\leq 2$.

If $|N|=1$, we denote the unique non-trivial component by G_1. Since $\mu(G)=3$, it can be inferred that $\mu(G_1)=2$. Based on previous discussions, it follows that G_1 resembles to either graph class $\mathcal{H}_2$ or $\mathcal{H}_3$ and $\triangle(G_1)\leq k$. If G_1 belongs to class $\mathcal{H}_2$, then: G belongs to class $\mathcal{H}_5$ if v is adjacent to a leaf vertex in G_1, and G belongs to class $\mathcal{H}_4$ if v is adjacent to a non-leaf vertex in G_1. By similar discussion, if G_1 belongs to class $\mathcal{H}_3$, then: G belongs to class $\mathcal{H}_7$ if v is adjacent to a leaf vertex

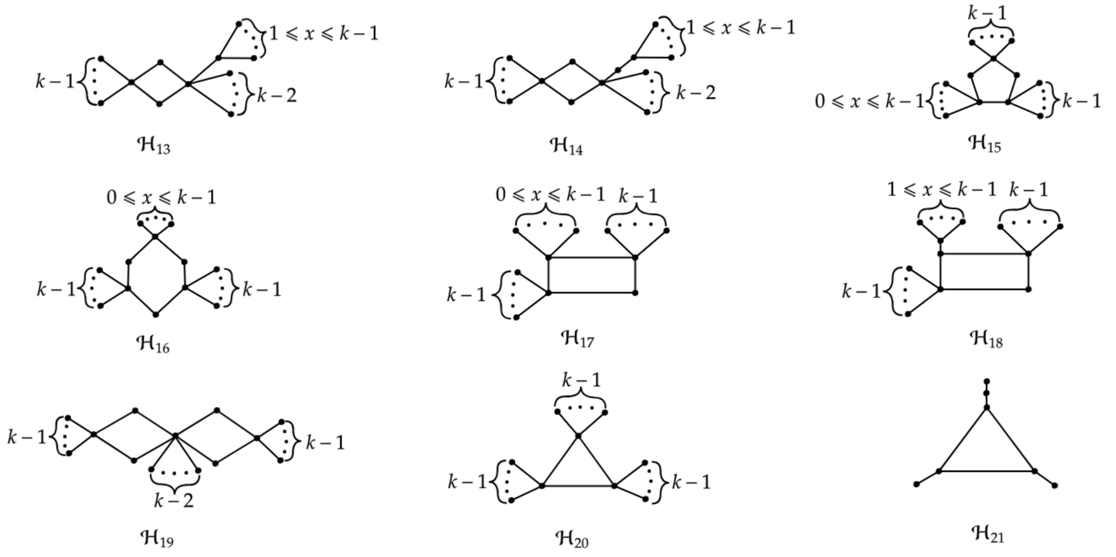

Figure 3. Minimally $\frac{1}{k}$-tough graphs with $k \geq 2$.

in G_1, and G belongs to class either $\mathcal{H}_6$ or $\mathcal{H}_8$ if v is adjacent to a non-leaf vertex in G_1.

If $|N| = 2$, we denote the two non-trivial components by G_2 and G_3. Since $\mu(G)=3$, we have $\mu(G_2)=\mu(G_3)=1$. So both G_2 and G_3 are stars. If v is adjacent to the centers of both G_2 and G_3, then G belongs to class $\mathcal{H}_{10}$. If v is adjacent to the center of G_2 and a leaf vertex of G_3(or the center of G_3 and a leaf vertex of G_2), then G belongs to class $\mathcal{H}_9$. If v is adjacent to leaf vertices of both G_2 and G_3, then G belongs to class $\mathcal{H}_{11}$.

Case 2. $G-v$ does not have trivial components.

Clearly, $d_G(v)=\omega(G-v)$. Since $\mu(G)=3$, we have $2 \leq \omega(G-v) \leq 3$. If $\omega(G-v)=2$, then $d_G(v)=2$, implying that G is a path of length at least five. Consequently, G must be either P_6 or P_7, which belongs to the classes $\mathcal{H}_5$ or $\mathcal{H}_7$, respectively. If $\omega(G-v)=3$, then $d_G(v)=3$. Let us denote the components of $G-v$ by G_1, G_2 and G_3. Since $\mu(G)=3$, each G_i satisfies $\mu(G_i)=1$ and is a star, and v must be adjacent to the centers of each G_i. Given that v has maximum degree and considering that each center of these stars is of degree at most three. We conclude that graph G belongs to class $\mathcal{H}_{12}$. □

Theorem 4.2. *Let $k \geq 2$ be a positive integer and G be minimally $\frac{1}{k}$-tough. If G is not a tree and $\mu(G) \leq 3$, then G belongs to one of the graph families $\mathcal{H}_i$ for $13 \leq i \leq 21$ as shown in Figure 3.*

Proof. Let e be a non-bridge edge of G with endpoints u and v such that

$$g(uv) = \max\{g(xy) \mid xy \in E(G)\}.$$

By Lemma 2.3, there exists a vertex set $S\subseteq V(G)$ with

$$\omega(G-S)=k|S|,$$

and e is a bridge of $G-S$. Let $C(e)$ be the component of $G-S$ containing the edge e, and let $D(e)$ be the union of the components of $G-S-C(e)$. We let $|D(e)|$ denote the numbers of components of $D(e)$. Let G_u and G_v denote the components of $(G-e)-S$ containing u and v, respectively. Denote their sets by L_u and L_v, where

$$L_u=L_0\cup L_1\cup\cdots\cup L_a,$$

$$L_v=R_0\cup R_1\cup\cdots\cup R_b,$$

with

$$L_i=\{x\,|\,d_{G_u}(x,u)=i,x\in L_u\},\quad 0\le i\le a,$$

$$R_j=\{y\,|\,d_{G_v}(y,v)=j,y\in L_v\},\quad 0\le j\le b.$$

Here, a is the longest distance from any vertex in G_u to u, and b is the longest distance from any vertex in G_v to v. Since e is a bridge of $G-S$, we have $N_G(u)\cap N_G(v)\subseteq S$. Thus, $g(uv)\le|S|$.

Case 1. All components of $D(e)$ are non-trivial.

Since $\mu(G)\le 3$, we have $\omega(G-S)\le 3$. Because $k\ge 2$, the following inequality holds:

$$2|S|\le k|S|=\omega(G-S)\le 3.$$

Thus, we conclude that $|S|=1$, and $k=2$ or 3. Hence, $g(uv)=0$ or 1. Let $S=\{s\}$ and $G_1=G[V(C(e))\cup S]$. Then the following claim holds.

Claim 1. $\mu(G_1)=2$.

Proof. Suppose to the contrary that $\mu(G_1)=1$. By Observation 2.1, G_1 is either a triangle or a star. Since e is not a bridge of G, G_1 must be a triangle. Then, $d_G(u)=d_G(v)=2$, which contradicts Lemma 2.8. Thus, $\mu(G_1)=2$. $\square$

If $k=3$, then $\omega(G-S)=3$, implying $\mu(G_1)=1$, which contradicts Claim 1. Therefore, we assume $k=2$ in the following discussion. Since $D(e)$ is non-trivial, it follows that $\mu(D(e))\ge 1$. Note that $\mu(G_1)+\mu(D(e))\le\mu(G)\le 3$. By Claim 1, we have $\mu(G_1)=2$ and $\mu(D(e))=1$.

Claim 2. The following results hold for a and b:

(1) $a\le 2$, $b\le 2$ and $a+b\le 3$;

(2) If $a=2$ and $b=1$, then s is not adjacent to $L_v-\{v\}$; and if $b=2$ and $a=1$, then s is not adjacent to $L_u-\{u\}$.

Proof. Since $\mu(C(e))\leq\mu(G_1)=2$, the longest path in $C(e)$ has length at most four. By the definition of L_i and R_j, there exists a path with one endpoint in L_a and the other in R_b. Thus, $a+b+1\leq 4$, which implies $a+b\leq 3$. If $a=3$, then $b=0$. Since e is not a bridge of G, vertex s must be adjacent to v. This implies $\mu(G_1)=3$, contradicting $\mu(G_1)=2$. Hence, $a\leq 2$. A similar argument shows that $b\leq 2$.

For the second part, we assume that $a=2$ and $b=1$. Since each path from L_2 to R_1 in $C(e)$ has length four and $\mu(G_1)=2$, it follows that s cannot be adjacent to $R_1=L_v-\{v\}$. By symmetry, if $b=2$ and $a=1$, then s is not adjacent to $L_1=L_u-\{u\}$. □

Claim 3. If $\mu(C(e))=2$, then the induced subgraph $G[V(D(e))\cup S]$ is a path of length two.

Proof. Given $\mu(G)\leq 3$ and $\mu(C(e))=2$, we have $\mu(G[V(D(e))\cup S])=1$. By Observation 2.1, the subgraph $G[V(D(e))\cup S]$ is a triangle or a star. If it were a triangle, then each vertex in $D(e)$ would have degree two in G, contradicting Lemma 2.8. Thus, the subgraph $G[V(D(e))\cup S]$ is a star with its center vertex in $D(e)$, which is a cut vertex of G. Since $\tau(G)=\frac{1}{2}$ and $D(e)$ is non-trivial, the subgraph $G[V(D(e))\cup S]$ must be a path of length two. □

Claim 4. If $|S|=1$ and $\mu(C(e))=1$, then $g(uv)=0$ and $C(e)$ is a star whose center vertex is a cut vertex.

Proof. Let $S=\{s\}$. Since $\mu(C(e))=1$, then $C(e)$ is a star by Observation 2.1. Without loss of generality, we assume that v is the center of $C(e)$. Then $L_u=\{u\}$ and $L_v=\{v\}\cup R_1$. Clearly, $us\in E(G)$. By Lemma 2.8, we get $vs\notin E(G)$. Thus, $g(uv)=0$. It follows that R_1 is an independent set. Noting that e is not a bridge of G, it follows that s is adjacent to R_1. By Lemma 2.7, the vertex v is a cut vertex of G. □

Case 1.1. $g(uv)=1$.

In this case, $us,vs\in E(G)$. By Lemma 2.8, we have $L_1\neq\emptyset$ and $R_1\neq\emptyset$. Then there exists a path of length four from L_1 to R_1 with internal vertices u, s and v. Since $\mu(G_1)=2$, we deduce that $L_2\cup R_2=\emptyset$, and L_1 and R_1 are independent sets. Given $g(uv)$=1, s is not adjacent to $L_1\cup R_1$. Consequently, both u and v are cut vertices of G. Since $\tau(G)=\frac{1}{2}$, we have $|L_1|=|R_1|=1$. Thus, $\mu(C(e))=2$. By Claim 3, the subgraph $G[V(D(e))\cup S]$ is a path of length two. Therefore, G is isomorphic to the graph $\mathcal{H}_{21}$.

Case 1.2. $g(uv)=0$.

In this case, s is adjacent to at most one of u and v. By the symmetry of u and v, we only discuss the cases $us\notin E(G)$, $vs\notin E(G)$ and $us\in E(G)$, $vs\notin E(G)$.

Case 1.2.1. $us\notin E(G)$ and $vs\notin E(G)$.

Since e is not a bridge of G, it follows that $L_1 \neq \emptyset$ and $R_1 \neq \emptyset$. By Claim 2, we have $a=b=1$. Then s is adjacent to both L_1 and R_1, and both L_1 and R_1 are independent sets. Thus, $\mu(C(e))=2$. Without loss of generality, let $x_i \in N_{L_1}(s)$ and $y_j \in N_{R_1}(s)$. Then ux_isy_jv is a path in G_1, which implies $|L_1|=|R_1|=1$. Consequently, G_1 is C_5. By Claim 3, the subgraph $G[V(D(e)) \cup S]$ is a path of length two. By Lemma 3.1(i), G is not minimal, a contradiction.

Case 1.2.2. $us \in E(G)$ and $vs \notin E(G)$.

Since e is not a bridge of G, we have $R_1 \neq \emptyset$ and s is adjacent to $L_v - \{v\}$. By Claim 2(2), $a \leq 1$. It follows from $g(uv)=0$ that R_1 is an independent set. If $b=2$, then since us and an edge of $G[R_1,R_2]$ consist of a matching, we have $|R_1|=1$.

Case 1.2.2.1. $a=1$.

By Claim 3, the subgraph $G[V(D(e)) \cup S]$ is a path of length two. Since $g(uv)=0$ and $us \in E(G)$, we have that L_1 is an independent set and s is not adjacent to L_1. Thus, the vertex u is a cut vertex of G. It follows from $\tau(G)=\frac{1}{2}$ that $|L_1|=1$. Let x denote the unique vertex in L_1.

If $R_2=\emptyset$, then s is adjacent to R_1. Let $y_i \in N_{R_1}(s)$. Then $xusy_iv$ is a path in G. Combining this with $\mu(G_1)=2$, we have $|R_1|=1$. By Lemma 3.1(i), G is not minimal, a contradiction.

If $R_2 \neq \emptyset$, then $|R_1|=1$. Let $R_1=\{y\}$. Since $g(uv)=0$, R_2 is an independent set. Assume that s is adjacent to R_2. Let $y' \in N_{R_2}(s)$. Then there exists a path $vyy'sux$ in G_1, which contradicts $\mu(G_1)=2$. Thus, s is not adjacent to R_2. So s is adjacent to R_1. This implies that y is a cut vertex of G. As $\tau(G)=\frac{1}{2}$, we conclude that $|R_2|=1$. Therefore, G belongs to class $\mathcal{H}_{18}$ with $k=2$.

Case 1.2.2.2. $a=0$.

By Claim 2(1), $b \leq 2$. If $b=2$, then $|R_1|=1$. By Claim 3, the subgraph $G[V(D(e)) \cup S]$ is a path of length two. It follows from $g(uv)=0$ that R_2 is an independent set and s is adjacent to exactly one of R_1 and R_2. If s is adjacent to R_1, then G is a graph as constructed in Lemma 3.1. By Lemma 3.1(i), G is not minimal, a contradiction. If s is adjacent to R_2, let $y' \in N_{R_2}(s)$. Then $\{uv, sy'\}$ is a matching. So $|R_2|=1$. By Lemma 3.1(i), G is not minimal, a contradiction.

If $b=1$, then R_1 is an independent set and s is adjacent to R_1. By Lemma 2.7, the vertex v is a cut vertex of G. So $|R_1| \geq 2$. Since $\{v,s\}$ is a cutset of G, by the definition of toughness, we have $\omega(G-\{v,s\})=2+|R_1| \leq k \cdot |\{v,s\}|=4$, which implies $|R_1|=2$. Then s is adjacent to exactly one vertex of R_1. Since $\mu(D(e))=1$ and $g(uv)=0$, Observation 2.1 implies that $D(e)$ is a star. Let w denote the center vertex of $D(e)$. If s is adjacent to w, then since $g(uv)=0$, it follows that s is not adjacent to any leaf vertex of $D(e)$. Thus, w is a cut vertex of G. Noting that $\tau(G)=\frac{1}{2}$, we have that $D(e)$ is path of order two. Therefore, G belongs to class $\mathcal{H}_{13}$ with $k=2$. If s is not adjacent to w, then s is adjacent to some leaf vertices of $D(e)$. If s is adjacent to exactly one leaf vertex of $D(e)$, then $D(e)$ is a path of order two or three, and so G belongs to class $\mathcal{H}_{13}$ or $\mathcal{H}_{14}$ with $k=2$. If s is adjacent to at least two leaf vertices of $D(e)$, then Lemma 2.7 ensures s can not be adjacent to all leaf vertices of $D(e)$. Then w is a cut vertex of G. Because $\tau(G)=\frac{1}{2}$ and $\{w,s\}$ is a cutset of G, we get $D(e)$ is a star $K_{1,3}$ and G belongs to class $\mathcal{H}_{19}$ with $k=2$.

Case 2. $D(e)$ has both trivial and non-trivial components.

In this case, $\omega(G-S)\geq 3$. Since $\mu(G)\leq 3$, we have that $D(e)$ contains exactly one non-trivial component. Let $D_1(e)$ be the non-trivial component in $D(e)$, and let $D_2(e)=D(e)-D_1(e)$. Then $D_2(e)$ is an independent set. As $\mu(G)\leq 3$, we have $\mu(C(e))=\mu(D_1(e))=\mu(G[V(D_2(e))\cup S])=1$.

We claim that $|S|=1$ and $k\geq 3$. Suppose to the contrary that $|S|\geq 2$. By Lemma 2.3, $\omega(G-S)=k|S|\geq 4$, which implies $|D_2(e)|\geq 2$. Since $\mu(G[V(D_2(e))\cup S])=1$, the subgraph $G[V(D_2(e))\cup S]$ is a star with center vertex in S. By Lemma 2.4, each vertex in S has at least one neighbor in $D_2(e)$, which contradicts $G[V(D_2(e))\cup S]$ is a star. Therefore, $|S|=1$. Since $\omega(G-S)=k|S|\geq 3$, we have $k\geq 3$.

Let $S=\{s\}$. By Claim 4, $g(uv)=0$ and $C(e)$ is a star such that the center vertex of which is a cut vertex. Without loss of generality, we assume that v is the center of $C(e)$. Then $L_u=\{u\}$ and $L_v=\{v\}\cup R_1$. It follows that $us\in E(G)$ and $vs\notin E(G)$. Since e is not a bridge of G, s must be adjacent to R_1. Since $\mu(D_1(e))=1$, then by Observation 2.1, the subgraph $D_1(e)$ is a star. Let w denote the center vertex of $D_1(e)$. If s is adjacent to w, then s is not adjacent to any leaf vertex of $D_1(e)$; and if s is not adjacent to w, then by Lemma 2.7, s is not adjacent to all leaf vertices of $D_1(e)$. In both cases, w must be a cut vertex of G. Let m and n be the number of the vertices of degree one in $C(e)$ and $D_1(e)$, and let m' and n' be the number of the vertices of degree two in $C(e)$ and $D_1(e)$, respectively. Since $\tau(G)=\frac{1}{k}$ and v is a cut vertex of G, by the definition of toughness, we have $\omega(G-v)=m+1\leq k\cdot|\{v\}|=k$, which implies $1\leq m\leq k-1$. Using the cut vertex w, we can get $1\leq n\leq k-1$. Noting that $d_G(u)=2$ and $N_{R_1}(s)\neq\emptyset$, we have $m'\geq 2$. Since $\{v,s\}$ is a cutset of G, we deduce that $\omega(G-\{v,s\})\leq k\cdot|\{v,s\}|=2k$. Clearly, the components of $G-\{v,s\}$ are $D(e)$, R_1 and an isolated vertex u. Thus, $\omega(G-\{v,s\})=m+m'+k-1$, and so $m+m'\leq k+1$. Let $y_1\in N_{R_1}(s)$. Let $e_0=y_1s$ and let S_0 be a vertex set satisfying Lemma 2.3 for e_0. We divide the following discussion into two cases.

Case 2.1. s is adjacent to exactly one vertex in $D_1(e)$.

If $|S_0|\geq 2$, then $\omega(G-S_0)=k|S_0|\geq 2k$. By Lemma 2.4, $S_0\subseteq V_{\geq 3}(G)\subseteq\{v,s,w\}$. Then $S_0=\{v,w\}$, and $D_1(e)$ has at least three leaf vertices or $D_1(e)$ is $K_{1,2}$ and s is adjacent to w. For the former case, $\omega(G-S_0)=m+n+1\leq k-1+k-1+1=2k-1$. For the later case, $n=2$ and it follows from $k\geq 3$ that $\omega(G-S_0)=m+n+1\leq k-1+2+1=k+2\leq 2k-1$. In all these cases, we conclude that $\omega(G-S_0)\leq 2k-1$, which contradicts $\omega(G-S_0)\geq 2k$. Hence, $|S_0|=1$, $\omega(G-S_0)=k\geq 2$ and the vertex in S_0 is a cut vertex. The cycle usy_1vu implies $S_0=\{v\}$. Then $\omega(G-S_0)=m+1$. Thus, $m=k-1$. Combining this with $m'\geq 2$ and $m+m'\leq k+1$, we have $m'=2$. According to the above analysis, if s is adjacent to w, then G belongs to class $\mathcal{H}_{13}$; and if s is adjacent to exactly one leaf vertex of $D_1(e)$, then G belongs to class $\mathcal{H}_{14}$.

Case 2.2. s is adjacent to at least two vertices in $D_1(e)$.

As $g(uv)=0$, s is not adjacent to w. Thus, $n'\geq 2$. Without loss of generality, let $w_1,w_2\in N_{D_1(e)}(s)$. Lemma 2.7 implies that w is a cut vertex of G and $n\geq 1$. Since $\{w,s\}$ is a cutset of G, we have $\omega(G-\{w,s\})=n+n'+k-1\leq k\cdot|\{w,s\}|=2k$. Thus, $n+n'\leq k+1$.

For the edge $e_0=sy_1$, since $V_{\geq 3}(G)-\{s,y_1\}\subseteq\{v,w\}$ and $\omega(G-\{v,w\})=m+n+1$, then by lemma 2.5(2), we have $|S_0|=1$. Since vy_1suv forms a cycle and u is not a cut vertex, we have $S_0=\{v\}$. Then $\omega(G-S_0)=m+1=k$, forcing $m=k-1$. It follows from $m'\geq 2$ and $m+m'\leq k+1$ that $m'=2$.

Let $e_1=sw_1$ and let S_1 be a vertex set satisfying Lemma 2.3 for e_1. Since $V_{\geq 3}(G)-\{s,w_1\}\subseteq\{v,w\}$ and $\omega(G-\{v,w\})=m+n+1$, then by Lemma 2.5(2), we have $|S_1|=1$. The cycle ww_2sw_1w implies $S_1=\{w\}$. Then $\omega(G-S_1)=n+1=k$, and so $n=k-1$. Combining this with $n'\geq 2$ and $n+n'\leq k+1$, we have $n'=2$. Then G belongs to class $\mathcal{H}_{19}$.

Case 3. All components of $D(e)$ are trivial.

We first show $|S|\leq 2$. Suppose to contrary that $|S|\geq 3$. Then $\omega(G-S)=k|S|\geq 6$. Thus, $|D(e)|\geq 5$. By Lemma 2.4, every vertex of S is adjacent to at least two components of $D(e)$. Since every vertex of $D(e)$ is adjacent to S and $|S|\geq 3$, we have $\mu(G[S\cup V(D(e))])\geq 3$. Thus, we get $\mu(G)\geq\mu(C(e))+\mu(G[S\cup V(D(e))])\geq 4$, a contradiction. Therefore, $|S|\leq 2$.

Case 3.1. $|S|=1$.

Let $S=\{s\}$. In this case, $|D(e)|=\omega(G-S)-1=k-1\geq 1$ and the subgraph $G[V(D(e))\cup S]$ is a star with center s. Since $\mu(G)\leq 3$, we have $\mu(C(e))\leq 2$.

Case 3.1.1. $\mu(C(e))=1$.

By Claim 4, we have that $g(uv)=0$ and $C(e)$ is a star such that the center vertex of which is a cut vertex. Without loss of generality, assume that v is the center of $C(e)$. Then $L_u=\{u\}$ and $L_v=\{v\}\cup R_1$. Since e is not a bridge of G, we have that $su\in E(G)$ and s is adjacent to R_1. Let m and m' be the number of vertices of degree one and two in $C(e)$, respectively. Noting that $d_G(u)=2$, it follows that $m'\geq 2$. Let $y\in N_{R_1}(s)$. Let $e_2=sy$ and let S_2 be a vertex set satisfying Lemma 2.3 for e_2. Since $V_{\geq 3}(G)\subseteq\{v,s\}$, Lemma 2.5(1) implies $|S_2|=1$. Thus, $\omega(G-S_2)=k\geq 2$. The cycle $uvysu$ implies $S_2=\{v\}$. So $\omega(G-S_2)=m+1=k$. Thus, $m=k-1$. Since $\{v,s\}$ is a cutset of G, we have

$$\omega(G-\{v,s\})=m+m'+|D(e)|=m+m'+k-1=2(k-1)+m'\leq 2k.$$

Hence, $m' \leq 2$. Combining this with $m' \geq 2$, it follows that $m'=2$. Therefore, G is belongs to class $\mathcal{H}_{17}$.

Case 3.1.2. $\mu(C(e))=2$.

Since there exists a path of length $a+b+1$ from L_a to R_b in $C(e)$, we have $a+b \leq 3$.

Case 3.1.2.1. $g(uv)=1$.

Then $us,vs \in E(G)$ and $uvsu$ is a triangle. By Lemma 2.8, we have $L_1 \neq \emptyset$ and $R_1 \neq \emptyset$. As $g(uv)=1$, s is adjacent to neither L_1 nor R_1. Without loss of generality, we assume that $R_2=\emptyset$. Since $g(uv)=1$, each vertex in R_1 has degree one or two in G. By Lemma 2.8, the subgraph $G[L_v]$ contains no triangle. Therefore, R_1 is an independent set, and thus v is a cut vertex of G. It follows from $\tau(G)=\frac{1}{k}$ that $1 \leq |R_1| \leq k-1$.

If $L_2=\emptyset$, then by similar discussion, we have L_1 is an independent set and $1 \leq |L_1| \leq k-1$. By Lemma 3.1(ii), G is a minimally $\frac{1}{k}$-tough graph if and only if $|L_1|=|R_1|=k-1$. Thus, G belongs to class $\mathcal{H}_{20}$.

If $L_2 \neq \emptyset$, then since $R_1 \neq \emptyset$, we have $\mu(G[L_u])=1$. Thus, $G[L_u]$ is a star whose center is in L_1, which implies $|L_1|=1$ and L_2 is an independent set. Let m be the number of vertex in L_2 that has degree one in G. Let $e_3=us$ and let S_3 be a vertex set satisfying Lemma 2.3 for e_3. Since $V_{\geq 3}(G)-\{u,s\} \subseteq \{x,v\}$ and $\omega(G-\{x,v\})=m+|R_1|+1$, then by Lemma 2.5(2), we have $|S_3|=1$. Thus, s is not adjacent to L_2. By Lemma 3.1(ii), G is not minimal, a contradiction.

Case 3.1.2.2. $g(uv)=0$.

In this case, s is adjacent to at most one of u and v. By the symmetric property, we only need to consider the cases $us,vs \notin E(G)$ and $us \notin E(G)$, $vs \in E(G)$.

Case 3.1.2.2.1. $us \notin E(G)$ and $vs \notin E(G)$.

Since e is not a bridge of G, we have $L_1 \neq \emptyset$ and $R_1 \neq \emptyset$. It follows from $g(uv)=0$ that L_1 and R_1 are independent sets. Note that $a+b \leq 3$. Without loss of generality, we assume that $R_2=\emptyset$. Then s is adjacent to R_1. Let m' and n' be the number of vertices in R_1 that have degree one and two in G, respectively. Since $\tau(G)=\frac{1}{k}$, we have $0 \leq m' \leq k-1$. Since $\{v,s\}$ is a cutset of G, by the definition of toughness, we have

$$\omega(G-\{v,s\})=|R_1|+|D(e)|+1=|R_1|+k \leq 2k.$$

Thus, $|R_1|=m'+n' \leq k$. Let $y_1 \in N_{R_1}(s)$. Let $e_4=sy_1$ and let S_4 be a vertex set satisfying Lemma 2.3 for e_4.

Subcase 1. $a=1$.

Then s is adjacent to L_1. Let m and n denote the number of vertices in L_1 which have degree one and two in G, respectively. By symmetric property of u and v, we have $0 \leq m \leq k-1$ and $|L_1|=m+n \leq k$. Since $V_{\geq 3}(G)-\{s,y_1\} \subseteq \{u,v\}$ and $\omega(G-\{u,v\})=m+m'+1$, we have $|S_4|=1$ by Lemma 2.5(2). Let $x_1 \in N_{L_1}(s)$.

The cycle uvy_1sx_1u forces $S_4=\{u\}$ or $S_4=\{v\}$.

If $S_4=\{u\}$, then $\omega(G-S_4)=m+1=k$, implying $m=k-1$. Combining this with $n\geq 1$ and $|L_1|=m+n\leq k$, we have $n=1$. Since e_4 is a bridge of $G-S_4$, we have $n'=1$. Therefore, G belongs to class $\mathcal{H}_{15}$.

If $S_4=\{v\}$, then $\omega(G-S_4)=m'+1=k$. Thus, $m'=k-1$. Combining this with $n'\geq 1$ and $|R_1|=m'+n'\leq k$, we have $n'=1$. Let $e_5=sx_1$ and let S_5 be a vertex set satisfying Lemma 2.3 for e_5. Similar to the discussion that is for e_4, we have $S_5=\{u\}$ or $S_5=\{v\}$. If $S_5=\{v\}$, then since e_5 is the bridge of $G-S_5$, we have $n=1$. If $S_5=\{u\}$, then $\omega(G-S_5)=m+1=k$. Therefore, $m=k-1$, and thus $n=1$. We conclude that if $S_4=\{v\}$, then $n=n'=1$. Thus, G also belongs to class $\mathcal{H}_{15}$.

Subcase 2. $a=2$.

Since $R_1\neq\emptyset$, we have $G[L_u]$ is a star. Thus, $|L_1|=1$, L_2 is an independent set and s is adjacent to exactly one of L_1 and L_2. Let x denote the unique vertex in L_1.

If s is adjacent to L_1, then x is a cut vertex. So $1\leq|L_2|\leq k-1$. For the edge $e_5=sx$, since $V_{\geq 3}(G)\subseteq\{s,x,v\}$, by Lemma 2.5(1), we have $|S_5|=1$. The cycle uvy_1sxu implies $S_5=\{v\}$. Then $\omega(G-S_5)=m'+1=k$, which forces $m'=k-1$. Thus, $n'=1$. Therefore, G belongs to class $\mathcal{H}_{15}$.

Assume now that s is adjacent to L_2. Let T_1 and T_2 be the sets of vertex in L_2 that has degree one and two in G, respectively. Clearly, $T_1\subseteq V_1(G)$, $T_2\subseteq V_2(G)$, $|T_1|\leq k-1$ and $|T_2|\geq 1$. Since $\{s,x\}$ is a cutset of G, we have $\omega(G-\{s,x\})=|L_2|+k\leq k\cdot|\{s,x\}|=2k$. Thus, $|L_2|=|T_1|+|T_2|\leq k$.

If $|S_4|\geq 2$, then by Lemma 2.4, we have $S_4\subseteq V_{\geq 3}(G)\subseteq\{v,s,x\}$. Thus, $S_4=\{x,v\}$. Then $\omega(G-S_4)=|T_1|+m'+2=2k$. Note that $|T_1|\leq k-1$ and $m'\leq k-1$. Then $|T_1|=m'=k-1$. Since $|T_1|+|T_2|\leq k$ and $n'+m'\leq k$, we have $|T_2|=n'=1$. Hence, G belongs to class $\mathcal{H}_{16}$.

If $|S_4|=1$, then $S_4=\{x\}$ or $S_4=\{v\}$. If $S_4=\{x\}$, then $\omega(G-S_4)=|T_1|+1$, forcing $|T_1|=k-1$, and thus $|T_2|=1$. Since e_4 is a bridge of $G-S_4$, we get $n'=1$. Therefore, G also belongs to class $\mathcal{H}_{16}$.

If $S_4=\{v\}$, then $\omega(G-S_4)=m'+1=k$. Thus, $m'=k-1$. Combining this with $m'+n'\leq k$, we have $n'=1$. Let x_i be an arbitrary vertex in T_2. Let $e_6=sx_i$ and let S_6 be a vertex set satisfying Lemma 2.3 for e_6. If $|S_6|\geq 2$, then since Lemma 2.4 implies $S_6\subseteq V_{\geq 3}(G)\subseteq\{v,s,x\}$, we have $S_6=\{x,v\}$. Thus, $\omega(G-S_6)=|T_1|+m'+2=|T_1|+k+1=2k$. It follows that $|T_1|=k-1$. So $|T_2|=1$. If $|S_6|=1$, then the cycle uvy_1sx_ixu implies $S_6=\{x\}$ or $S_6=\{v\}$. If $S_6=\{x\}$, then $\omega(G-S_6)=|T_1|+1=k$. Thus, $|T_1|=k-1$ and $|T_2|=1$. If $S_6=\{v\}$, since e_6 is the bridge of $G-S_6$, then $|T_2|=1$. We conclude that if $S_4=\{v\}$, then $m'=k-1$ and $|T_2|=n'=1$. Therefore, G also belongs to class $\mathcal{H}_{16}$.

Case 3.1.2.2.2. $us\notin E(G)$ and $vs\in E(G)$.

Since e is not a bridge of G, we have $L_1\neq\emptyset$. It follows from $g(uv)=0$ that L_1

is an independent set.

Subcase 1. $R_1=\emptyset$.

Since $\mu(C(e))=2$, we have $2\le a\le 3$ and $\mu(G[L_u-u])=1$.

Subcase 1.1. $a=2$ and $G[L_u-u]$ is disconnected.

Since $\mu(G[L_u-u])=1$, then $G[L_u-u]$ has exactly one non-trivial component, denoted by G_1. Since $g(uv)=0$, we have that G_1 is a star by Observation 2.1. Denote the center of G_1 by x_1. We first characterize the structure of G_1. If $x_1\in L_1$, then since L_1 is an independent set, we have $L_1\cap V(G_1)=\{x_1\}$ and $V(G_1)=L_2\cup\{x_1\}$. If $x_1\in L_2$, then $L_2=\{x_1\}$ and $V(G_1)=\{x_1\}\cup L_1'$, where $L_1'\subseteq L_1$. Let $\{A_1,A_2\}$ be a partition of $L_1-V(G_1)$ where

$$A_1=\{x\in L_1-V(G_1)\,|\,sx\in E(G)\},$$

$$A_2=\{x\in L_1-V(G_1)\,|\,sx\notin E(G)\}.$$

Clearly, $A_1\subseteq V_2(G)$ and $A_2\subseteq V_1(G)$. It follows from $\tau(G)=\frac{1}{k}$ that $0\le|A_2|\le k-1$.

Subcase 1.1.1. $x_1\in L_1$.

Then $V(G_1)=L_2\cup\{x_1\}$ and L_2 is an independent set. Let $\{A_3,A_4\}$ be a partition of L_2 where

$$A_3=\{w\in L_2\,|\,sw\in E(G)\},$$
$$A_4=\{w\in L_2\,|\,sw\notin E(G)\}.$$

Clearly, $A_3\subseteq V_2(G)$, $A_4\subseteq V_1(G)$ and $0\le|A_4|\le k-1$. Then $V_{\ge3}(G)\subseteq\{u,s,x_1\}$. Since $\{u,s\}$ is a cutset of G such that the components obtained from G by deleting $\{u,s\}$ are G_1 and all isolated vertices in $V(D(e))\cup A_1\cup A_2\cup\{v\}$, by the definition of toughness, we have that

$$\omega(G-\{u,s\})=|A_1|+|A_2|+2+|D(e)|=|L_1|+k\le 2k.$$

Thus, $|L_1|\le k$. The following discussions are based on whether s is adjacent to L_1.

Assume first that s is adjacent to L_1. Let $e_7=vs$ and let S_7 be a vertex set satisfying Lemma 2.3 for e_7. Since $V_{\ge3}(G)-\{v,s\}\subseteq\{u,x_1\}$ and $\omega(G-\{u,x_1\})=|A_2|+|A_4|+1$, then by Lemma 2.5(2), we have $|S_7|=1$.

If s is adjacent to x_1, then since $g(uv)=0$, we have $|A_3|=0$. Thus, $1\le|A_4|\le k-1$. The cycle $uvsx_1u$ implies $S_7=\{u\}$ or $S_7=\{x_1\}$. If $S_7=\{u\}$, then $\omega(G-S_7)=|A_2|+1=k$. Thus, $|A_2|=k-1$. Combining this with $|L_1|=|A_1|+|A_2|+1\le k$, we have $|A_1|=0$. Therefore, G belongs to class $\mathcal{H}_{17}$. If $S_7=\{x_1\}$, then $\omega(G-S_7)=|A_4|+1=k$, forcing $|A_4|=k-1$. Since e_7 is a bridge of $G-S_7$, we have $|A_1|=0$. By Lemma 3.1, G is a minimally $\frac{1}{k}$-tough graph if and only if $|A_2|=k-1$. Therefore, G also belongs to class $\mathcal{H}_{17}$.

If s is not adjacent to x_1, then $A_1\neq\emptyset$. Without loss of generality, let x_2 be a vertex in A_1. Since $uvsx_2u$ forms a cycle, we can obtain that $S_7=\{u\}$. If $A_3\neq\emptyset$, then $\omega(G-S_7)=|A_2|+1=k$, implying $|L_1|=|A_1|+|A_2|+1\geq k+1$, which contradicts $|L_1|\leq k$. Therefore, $A_3=\emptyset$. Then $|A_4|\geq 1$ and $\omega(G-S_7)=|A_2|+2=k$. Hence, $|A_2|=k-2$. Noting that $|L_1|=|A_1|+|A_2|+1\leq k$, we have $|A_1|=1$. Thus, G belongs to class $\mathcal{H}_{13}$.

Suppose now that s is not adjacent to L_1. Then $sx_1\notin E(G)$ and $A_1=\emptyset$. Since e is not a bridge of G, s is adjacent to L_2. Thus, $A_3\neq\emptyset$. Without loss of generality, let $x_3\in A_3$. Let $e_8=sx_3$ and let S_8 be a vertex set satisfying Lemma 2.3 for e_8. Since $V_{\geq 3}(G)-\{s,x_3\}\subseteq\{u,x_1\}$ and $\omega(G-\{u,x_1\})=|A_2|+|A_4|+1$, then by Lemma 2.5(2), we have $|S_8|=1$. Noting that $uvsx_3x_1u$ is a cycle, it follows that $S_8=\{u\}$ or $S_8=\{x_1\}$. If $S_8=\{u\}$, then $\omega(G-S_8)=|A_2|+1=k$. Thus, $|A_2|=k-1$. Since e_8 is a bridge of $G-S_8$, we have $|A_3|=1$. Therefore, G belongs to class $\mathcal{H}_{15}$. If $S_8=\{x_1\}$, then $\omega(G-S_8)=|A_4|+1=k$. Thus, $|A_4|=k-1$. Since $\{x_1,s\}$ is a cutset of G, we have

$$\omega(G-\{x_1,s\})=|A_3|+|A_4|+|D(e)|+1=|A_3|+|A_4|+k\leq 2k.$$

Thus, $|A_3|+|A_4|\leq k$. It follows from $A_3\neq\emptyset$ that $|A_3|=1$. In this case, G also belongs to class $\mathcal{H}_{15}$.

Subcase 1.1.2. $x_1\in L_2$.

Then $L_2=\{x_1\}$. If G_1 is of order two, we have already discussed in the case that $x_1\in L_1$. Thus, assume $|V(G_1)|\geq 3$. That is, $|L_1'|\geq 2$. Clearly, each vertex of L_1' has degree two or three in G. Let $x_i\in L_1'$ such that $d_G(x_i)=3$ if L_1' has vertex of degree three. Take $e_9=ux_i$. Let S_9 be a vertex set satisfying Lemma 2.3 for e_9. Without loss of generality, let x_j be an arbitrary vertex in $L_1'-x_i$. By the choice of x_i, we have $N_G(x_j)\subseteq N_G(x_i)\cup N_G(u)$. If $|S_9|\geq 2$, then $x_j\notin S_9$ by Lemma 2.4. Thus, $S_9=\{s,x_1\}$ if $|S_9|\geq 2$. Then $\omega(G-S_9)=|D(e)|+1=k$, which contradicts $\omega(G-S_9)\geq 2k$. Hence, $|S_9|=1$. Since $ux_ix_1x_ju$ forms a cycle, we have $x_j\in S_9$ or $x_1\in S_9$. However, neither of x_1 and x_j is a cut vertex of G, a contradiction.

Subcase 1.2. $a=2$ and $G[L_u-u]$ is connected.

Since $g(uv)=0$, then by Observation 2.1, we can obtain that $G[L_u-u]$ is a star. Let us denote the center vertex of $G[L_u-u]$ by x_1. It follows from e is not a bridge of G and $g(uv)=0$ that s is adjacent to exactly one of L_1 and L_2.

Assume first that $x_1\in L_1$. Since L_1 is an independent set, we have $|L_1|=1$. If s is adjacent to L_1, then $G-sx_1$ is a tree in which the vertex s has the maximum degree k, which contradicts G is a minimally $\frac{1}{k}$-tough graph. Hence, s is adjacent to L_2. Since $\{x_1,s\}$ is a cutset of G, we have $\omega(G-\{x_1,s\})=|L_2|+1+|D(e)|\leq 2k$. Thus, $|L_2|\leq k$. For the edge $e_7=vs$ and its corresponding vertex S_7, since $V_{\geq 3}(G)\subseteq\{x_1,s\}$, then by Lemma 2.5(1), we deduce that $|S_7|=1$. Since x_1 is the only possible cut vertex in G, we have $S_7=\{x_1\}$. So $\omega(G-S_7)=|T|+1=k$, where T is the set of vertex in L_2 which is of degree one in G. Thus, $|T|=k-1$, and s is adjacent to exactly one vertex in L_2. Therefore, G is a graph which belongs to class $\mathcal{H}_{15}$.

Suppose now that $x_1 \in L_2$. Since each vertex of L_2 has a neighbor in L_1, we have $L_2 = \{x_1\}$. Since the case $|L_1| = 1$ has been already discussed in the case $x_1 \in L_1$, we assume that $|L_1| \geq 2$. Without loss of generality, let $x_i \in L_1$ such that $d_G(x_i) = 3$ if s is adjacent to L_1. Suppose that s is adjacent to L_1. Let $e_{10} = ux_i$ and let S_{10} be a vertex set satisfying Lemma 2.3 for e_{10}. Since $uvsx_i$ and $ux_jx_1x_i$ are two u-x_i paths in G, we have $|S_{10}| \geq 2$. Without loss of generality, let x_j be an arbitrary vertex in $L_1 - \{x_i\}$. By the choice of x_i, we have $N_G(x_j) \subseteq N_G(x_i) \cup N_G(x_u)$. By Lemma 2.4, we have $S_{10} \subseteq V_{\geq 3}(G) \subseteq \{u,s,x_1\}$ and $L_1 \cap S_{10} = \emptyset$. Thus, $S_{10} = \{s,x_1\}$. Then $\omega(G - S_{10}) = |D(e)| + 1 = k$, which contradicts $\omega(G - S_{10}) \geq 2k$. Thus, s is adjacent to L_2. Let $e_{11} = sx_1$ and let S_{11} be a vertex set satisfying Lemma 2.3 for e_{11}. Since $V_{\geq 3}(G) \subseteq \{x_1,s,u\}$, then by Lemma 2.5(1), we have $|S_{11}| = 1$. However, s is the only cut vertex of G, a contradiction.

Subcase 1.3. $a = 3$ and $G[L_u - u]$ is disconnected.

As $\mu(G[L_u - u]) = 1$, the subgraph $G[L_u - u]$ has exactly one non-trivial component, denoted by G_2. By the definition of L_2 and L_3, we have $V(G_2) \cup L_1 \neq \emptyset$ and $L_2 \cup L_3 \subseteq V(G_2)$. By Observation 2.1, we have that G_2 is a star and the center vertex of which is in L_2. Let us denote the center vertex of G_2 by x_1. It follows from $g(uv) = 0$ that $L_2 = \{x_1\}$. Let $\{T_1,T_2,T_3\}$ be a partition of L_1 where

$$\begin{aligned} &T_1 = \{x \in L_1 - V(G_2) \,|\, sx \notin E(G)\}, \\ &T_2 = \{x \in L_1 - V(G_2) \,|\, sx \in E(G)\}, \\ &T_3 = L_1 \cap V(G_2), \end{aligned}$$

and let $\{T_1',T_2'\}$ be a partition of L_3 where

$$\begin{aligned} &T_1' = \{x \in L_3 \,|\, sx \notin E(G)\}, \\ &T_2' = \{x \in L_3 \,|\, sx \in E(G)\}. \end{aligned}$$

Clearly, $T_1 \cup T_1' \subseteq V_1(G)$ and $T_2 \cup T_2' \subseteq V_2(G)$. For each vertex $x_i \in T_3$, we have $N_G(x_i) = \{u,x_1\}$ or $N_G(x_i) = \{u,x_1,s\}$. Since $\tau(G) = \frac{1}{k}$, we conclude that $0 \leq |T_1| \leq k-1$ and $0 \leq |T_1'| \leq k-1$. As $\{u,x_1,s\}$ is a cutset of G, we can obtain that

$$\omega(G - \{u,x_1,s\}) = |L_1| + |L_3| + |D(e)| + 1 = |L_1| + |L_3| + k \leq 3k.$$

Thus, $|L_1| + |L_3| \leq 2k$. Since $\{u,s\}$ is a cutset of G, we have

$$\omega(G - \{u,s\}) = |T_1| + |T_2| + \omega(G_2) + |D(e)| + 1 = |T_1| + |T_2| + k + 1 \leq 2k.$$

Hence, $|T_1| + |T_2| \leq k-1$.

Subcase 1.3.1. s is adjacent to T_3.

Without loss of generality, let $x_i \in N_{T_3}(s)$. Let $e_{12} = ux_i$ and let S_{12} be a vertex set satisfying Lemma 2.3 for e_{12}. We first prove that $|T_3| = 1$. Suppose to the contrary that $|T_3|\geq 2$. Let x_j be an arbitrary vertex in T_3-x_i. Since $uvsx_i$ and $ux_jx_1x_i$ are two u-x_i paths in G, we have $|S_{12}|\geq 2$. Note that $V_{\geq 3}(G)\subseteq\{x_1,s,u\}\cup T_3$. By the choice of x_i, we have $N_G(x_j)\subseteq N_G(x_i)\cup N_G(u)$. By Lemma 2.4, $x_j\notin S_{12}$ and $S_{12}\cap T_3=\emptyset$. Thus, $S_{12}=\{s,x_1\}$ and $\omega(G-S_{12})=|T_1'|+|T_2'|+|D(e)|+1=|T_1'|+|T_2'|+k=2k$. Hence, $|T_1'|+|T_2'|=k$. Let $e_{13}=x_1x_i$ and let S_{13} be a vertex set satisfying Lemma 2.3 for e_{13}. Suppose that $|S_{13}|\geq 2$. Lemma 2.4 implies $S_{13}\subseteq V_{\geq 3}(G)$. By the choice of x_i, we have $S_{13}=\{s,u\}$ and $\omega(G-S_{13})=|T_1|+|T_2|+|D(e)|+2=|T_1|+|T_2|+k+1=2k$. Thus, $|T_1|+|T_2|=k-1$. Then $|L_1|+|L_3|=|T_1|+|T_2|+|T_3|+|T_1'|+|T_2'|=2k-1+|T_3|\geq 2k+1$, which contradicts $|L_1|+|L_3|\leq 2k$. Therefore, $|S_{13}|=1$. Since there exists a cycle $ux_ix_1x_ju$ where x_j is not a cut vertex, we have $S_{13}=\{u\}$. Thus, $T_2'=\emptyset$. It follows from $|T_1'|+|T_2'|=k$ that $|T_1'|=k$, which contradicts $|T_1'|\leq k-1$. Therefore, $|T_3|=1$.

Let $e_{14}=sx_i$ and let S_{14} be a vertex set satisfying Lemma 2.3 for e_{14}. Since $V_{\geq 3}(G)-\{s,x_i\}\subseteq\{u,x_1\}$ and $\omega(G-\{u,x_1\})=|T_1|+|T_1'|+1$, then by Lemma 2.5(2), we have $|S_{14}|=1$. Since $svux_i$ is a s-x_i path and v is not a cut vertex in G, we have $S_{14}=\{u\}$. Then $T_2'=\emptyset$ and $\omega(G-S_{14})=|T_1|+1=k$. Thus, $|T_1'|\geq 1$ and $|T_1|=k-1$. It follows from $|T_1|+|T_2|\leq k-1$ that $|T_2|=0$. Then, G belongs to class $\mathcal{H}_{18}$.

Subcase 1.3.2. s is adjacent to L_1 but not to T_3.

In this case, $T_2\neq\emptyset$. Since $|T_1|+|T_2|\leq k-1$, we have $|T_1|\leq k-2$. Without loss of generality, let $x_2\in T_2$ and $x_3\in T_3$.

Subcase 1.3.2.1. $T_2'=\emptyset$.

In this case, vertex x_1 becomes a cut vertex. Thus, $1\leq|T_1'|\leq k-1$. Suppose that $sx_1\in E(G)$. Let $e_{15}=sx_1$ and let S_{15} be a vertex set satisfying Lemma 2.3 for e_{15}. Since $V_{\geq 3}(G)\subseteq\{u,s,x_1\}$, then by Lemma 2.5(1), we have $|S_{15}|=1$. The cycle $ux_2sx_1x_3u$ implies $S_{15}=\{u\}$. Then, $\omega(G-S_{15})=|T_1|+1=k$. Thus, $|T_1|=k-1$, which contradicts $|T_1|\leq k-2$. Therefore, $sx_1\notin E(G)$.

Assume first that $|T_3|=1$. Let $e_{16}=sx_2$ and let S_{16} be a vertex set satisfying Lemma 2.3 for e_{16}. Assume that $|S_{16}|\geq 2$. Since $V_{\geq 3}(G)\subseteq\{u,x_1,s\}$, then by Lemma 2.4, we have $S_{16}=\{u,x_1\}$. Then

$$\omega(G-S_{16})=|T_1|+|T_1'|+|T_3|+1=|T_1|+|T_1'|+2\leq k-2+k-1+2=2k-1,$$

which contradicts $\omega(G-S_{16})\geq 2k$. Therefore, $|S_{16}|=1$. The cycle $uvsx_2u$ implies $S_{16}=\{u\}$, leading to $\omega(G-S_{16})=|T_1|+\omega(G_2)+1=|T_1|+2=k$. Hence, $|T_1|=k-2$. Combining this with $|T_1|+|T_2|\leq k-1$ and $T_2\neq\emptyset$, we get $|T_2|=1$. Thus, G belongs to the family $\mathcal{H}_{14}$.

Assume now that $|T_3|\geq 2$. Select an arbitrary vertex $x_i\in T_3-x_3$. Let $e_{17}=ux_3$ and let S_{17} be a vertex set satisfying Lemma 2.3 for e_{17}. Since $V_{\geq 3}(G)-\{u,x_3\}\subseteq\{s,x_1\}$ and $\omega(G-\{s,x_1\})=|D(e)|+|T_1'|+1$, then by Lemma 2.5(2), we have $|S_{17}|=1$. Given that x_1 is the only possible cut vertex in the cycle $ux_3x_1x_iu$, we have $S_{17}=\{x_1\}$. Then $\omega(G-S_{17})=|T_1'|+1=k$, implying $|T_1'|=k-1$.

Let $e_{18}=x_3x_1$ and let S_{18} be a vertex set satisfying Lemma 2.3 for e_{18}. If $|S_{18}|=1$, then $\omega(G-S_{18})=k$. The cycle $ux_3x_1x_iu$ implies $S_{18}=\{u\}$. Then $\omega(G-S_{18})=|T_1|+1+\omega(G_2)=|T_1|+2=k$, implying $|T_1|=k-2$. Since $|T_1|+|T_2|\leq k-1$, we have $|T_2|=1$. Noting that $|L_1|+|L_3|=|T_1|+|T_2|+|T_3|+|T_1'|\leq 2k$, it follows that $|T_3|\leq 2$. Since $|T_3|\geq 2$, we have $|T_3|=2$. Therefore, G belongs to class $\mathcal{H}_{19}$.

If $|S_{18}|\geq 2$, then by Lemma 2.4, we have $S_{18}\subseteq V_{\geq 3}(G)\subseteq\{u,s,x_1\}$. So $S_{18}=\{u,s\}$. Then

$$\omega(G-S_{18})=|T_1|+|T_2|+\omega(G_2)+|D(e)|+1=|T_1|+|T_2|+k+1=2k.$$

Thus, $|T_1|+|T_2|=k-1$. Noting that $|L_1|+|L_3|=|T_1|+|T_2|+|T_3|+|T_1'|\leq 2k$, we get $|T_3|\leq 2$. As $|T_3|\geq 2$, it follows that $|T_3|=2$. For the edge $e_{16}=sx_2$ and its corresponding vertex set S_{16}. If $|S_{16}|\geq 2$, then Lemma 2.4 implies $S_{16}\subseteq V_{\geq 3}(G)\subseteq\{u,s,x_1\}$. So $S_{16}=\{u,x_1\}$. Then $\omega(G-S_{16})=|T_1|+|T_1'|+|T_3|+1=|T_1|+k+2=2k$. If $|S_{16}|=1$, the cycle $uvsx_2u$ forces $S_{16}=\{u\}$ and $\omega(G-S_{16})=|T_1|+\omega(G_2)+1=|T_1|+2=k$. Therefore, we conclude that $|T_1|=k-2$. So $|T_2|=1$. Therefore, G also belongs to class $\mathcal{H}_{19}$.

Subcase 1.3.2.2. $T_2'\neq\emptyset$.

Let $e_{19}=vs$ and let S_{19} be a vertex set satisfying Lemma 2.3 for e_{19}. Suppose that $|S_{19}|\geq 2$. Lemma 2.4 implies $S_{19}\subseteq V_{\geq 3}(G)\subseteq\{u,s,x_1\}$. So $S_{19}=\{u,x_1\}$ and $\omega(G-S_{19})=|T_1|+|T_1'|+|T_3|+1$. Note that $|L_1|+|L_3|=|T_1|+|T_2|+|T_3|+|T_1'|+|T_2'|\leq 2k$. Since $T_2\neq\emptyset$ and $T_2'\neq\emptyset$, we have $|T_1|+|T_1'|+|T_3|\leq 2k-2$. Thus, $\omega(G-S_{19})\leq 2k-1$, which contradicts $\omega(G-S_{19})\geq 2k$. Therefore, $|S_{19}|=1$. The cycle $uvsx_2u$ implies $S_{19}=\{u\}$. Then $\omega(G-S_{19})=|T_1|+1=k$, forcing $|T_1|=k-1$. Since $T_2\neq\emptyset$, it follows that $|T_1|+|T_2|\geq k$, which contradicts the conclusion $|T_1|+|T_2|\leq k-1$.

Subcase 1.3.3. s is not adjacent to L_1.

Then $|T_2|=0$ and $T_3\subseteq V_2(G)$. Since e is not a bridge and $g(uv)=0$, s is adjacent to exactly one of L_2 and L_3. Let x_i be an arbitrary vertex in T_3.

If s is adjacent to L_2, then $T_2'=\emptyset$ and $1\leq|T_1'|\leq k-1$. Let $e_{20}=sx_1$ and let S_{20} be a vertex set satisfying Lemma 2.3 for e_{20}. Since $V_{\geq 3}(G)\subseteq\{u,x_1,s\}$, then by Lemma 2.5(1), we have $|S_{20}|=1$. The cycle $uvsx_1x_iu$ implies $S_{20}=\{u\}$. Thus, $\omega(G-S_{20})=|T_1|+1=k$, forcing $|T_1|=k-1$. If $|T_3|=1$, then G belongs to class $\mathcal{H}_{15}$. If $|T_3|\geq 2$, since $\{u,x_1\}$ is a cutset of G, we have $\omega(G-\{u,x_1\})=|T_1|+|T_3|+|T_1'|+1=|T_3|+|T_1'|+k\leq 2k$. Thus, $|T_3|+|T_1'|\leq k$, which implies $|T_1'|\leq k-2$. Let $e_{21}=ux_i$ and let S_{21} be a vertex set satisfying Lemma 2.3 for e_{21}. Since $V_{\geq 3}(G)-\{u,x_i\}\subseteq\{x_1,s\}$ and $\omega(G-\{x_1,s\})=|D(e)|+|T_1'|+1$, we have $|S_{21}|=1$ by Lemma 2.5(2). Let x_j be an arbitrary vertex in T_3-x_i. Since $ux_ix_1x_ju$ forms a cycle in which x_1 is the only possible cut vertex, we have $S_{21}=\{x_1\}$. Then $\omega(G-S_{21})=|T_1'|+1\leq k-2+1=k-1$, contradicting $\omega(G-S_{21})=k$.

If s is adjacent to L_3, we claim that $|T_3|=1$. Suppose to the contrary that $|T_3|\geq 2$. Let $x_j\in T_3-x_i$. Let $e_{22}=x_1x_i$ and let S_{22} be a vertex set satisfying

Lemma 2.3 for e_{22}. If $|S_{22}|\geq 2$, Lemma 2.4 implies $S_{22}\subseteq V_{\geq 3}(G)\subseteq\{u,x_1,s\}$, then $S_{22}=\{s,u\}$ and $\omega(G-S_{22})=|T_1|+|D(e)|+2=|T_1|+k+1=2k$. If $|S_{22}|=1$, the cycle $ux_ix_1x_ju$ implies $S_{22}=\{u\}$, and so $\omega(G-S_{22})=|T_1|+1=k$. In both cases, we conclude that $|T_1|=k-1$. Then $|L_1|=|T_1|+|T_3|\geq k-1+2=k+1$. Noting that $|L_1|+|L_3|\leq 2k$, we have $|L_3|\leq k-1$. We use the edge $e_{21}=ux_i$ and its corresponding set S_{21} again. Since $V_{\geq 3}(G)-\{u,x_i\}\subseteq\{x_1,s\}$ and $\omega(G-\{x_1,s\})=|D(e)|+|T_1'|+1$, we have $|S_{21}|=1$ by Lemma 2.5(2). The cycle $ux_ix_1x_ju$ forces $S_{21}=\{x_1\}$. Thus, $\omega(G-S_{21})=|T_1'|+1<|L_3|+1\leq k$, which contradicts $\omega(G-S_{21})=k$. Therefore, $|T_3|=1$.

Let x_l be a vertex in T_2'. Let $e_{23}=x_ls$ and let S_{23} be a vertex set satisfying Lemma 2.3 for e_{23}. If $|S_{23}|\geq 2$, Lemma 2.4 implies $S_{23}\subseteq V_{\geq 3}(G)\subseteq\{u,x_1,s\}$. Thus, $S_{23}=\{u,x_1\}$. It follows that $\omega(G-S_{23})=|T_1'|+|T_1|+|T_3|+1=2k$. So $|T_1'|+|T_1|=2k-2$. Given that $|T_1'|\leq k-1$ and $|T_1|\leq k-1$, we deduce that $|T_1'|=|T_1|=k-1$. Noting that $|L_1|+|L_3|\leq 2k$, which implies $|T_2'|=1$. Therefore, G belongs to class $\mathcal{H}_{16}$.

If $|S_{23}|=1$, the cycle $uvsx_lx_1x_iu$ implies $S_{23}=\{u\}$ or $S_{23}=\{x_1\}$. If $S_{23}=\{u\}$, then $\omega(G-S_{23})=|T_1|+1=k$, forcing $|T_1|=k-1$. Since e_{23} is a bridge of $G-S_{23}$, we have $|T_2'|=1$. Thus, G belongs to class $\mathcal{H}_{16}$. If $S_{23}=\{x_1\}$, then $\omega(G-S_{23})=|T_1'|+1=k$, which implies $|T_1'|=k-1$. Since $\{s,x_1\}$ is a cutset of G, we have

$$\omega(G-\{s,x_1\})=|T_1'|+|T_2'|+1+|D(e)|=|T_2'|+2k-1\leq 2k.$$

Then $|T_2'|=1$. Thus, G belongs to class $\mathcal{H}_{16}$.

Subcase 1.4. $a=3$ and $G[L_u-u]$ is connected.

Since $g(uv)=0$, then by Observation 2.1, we conclude that $G[L_u-u]$ is a star with its center vertex in L_2. Thus, $|L_2|=1$. Let $L_2=\{x_1\}$. For each vertex $x_i\in L_1$, the neighbor set $N_G(x_i)$ is either $\{u,x_1\}$ or $\{u,x_1,s\}$.

We first prove that $|L_1|=1$. Suppose to the contrary that $|L_1|\geq 2$. Let x_i be a vertex of L_1 such that $d_G(x_i)=3$ if s is adjacent to L_1, and let $x_j\in L_1-x_i$. Let $e_{24}=x_1x_i$ and let S_{24} be a vertex set satisfying Lemma 2.3 for e_{24}. If $|S_{24}|=1$, since $ux_ix_1x_ju$ forms a cycle, we have that S_{24} must be $\{u\}$ or $\{x_j\}$. However, neither u nor x_j is a cut vertex of G, which leads to a contradiction. Thus, $|S_{24}|\geq 2$. By Lemma 2.4, $S_{24}\subseteq V_{\geq 3}(G)\subseteq\{u,x_1,s\}\cup L_1$. By the choice of x_i, we have $S_{24}=\{s,u\}$. Since $G[L_u-u]$ is connected, it follows that $\omega(G-S_{24})=|D(e)|+2=k+1$, which contradicts $\omega(G-S_{24})\geq 2k$ as $k\geq 2$. Therefore, $|L_1|=1$.

Let $L_1=\{x_2\}$. Suppose that s is adjacent to L_1. Then s is not adjacent to L_2. Suppose s is not adjacent to L_3. Then x_1 is a cut vertex of G, and so $|L_3|\leq k-1$. By Lemma 3.1(i), G is not minimal, a contradiction. Hence, s is adjacent to L_3. Let $x_3\in N_{L_3}(s)$. Let $e_{25}=sx_2$ and let S_{25} be a vertex set satisfying Lemma 2.3 for e_{25}. Using the cycles $uvsx_2u$ and $x_2sx_3x_1x_2$, we conclude that $|S_{25}|\geq 2$, and at least one of u and v must belong to S_{25}. According to Lemma 2.4, we have

$S_{25} \subseteq V_{\geq 3}(G)$. However, both u and v have degree two, a contradiction. Hence, s is not adjacent to L_1.

Since e is not a bridge of G and $g(uv)=0$, it follows that s is adjacent to exactly one of L_2 and L_3. Assume that s is adjacent to L_2, then by Lemma 3.1(i), G is not minimal, a contradiction. Thus, s is adjacent to L_3. Let $x_l \in N_{L_3}(s)$. Let $e_{26}=sx_l$ and let S_{26} be a vertex set satisfying Lemma 2.3 for e_{26}. Since $V_{\geq 3}(G) \subseteq \{x_1,s\}$, Lemma 2.5(1) yields that $|S_{26}|=1$. Let T be the set of leaf vertices in L_3. Clearly, $0 \leq |T| \leq k-1$. The cycle $uvsx_lx_1x_2u$ implies $S_{26}=\{x_1\}$. Thus, $\omega(G-S_{26})=|T|+1=k$, forcing $|T|=k-1$. Since $\{s,x_1\}$ is a cutset of G, we have $\omega(G-\{s,x_1\})=|L_3|+|D(e)|+1=|L_3|+k \leq 2k$. Hence, $|L_3| \leq k$ and s is adjacent to exactly one vertex in L_3. Thus, G belongs to class $\mathcal{H}_{16}$.

Subcase 2. $R_1 \neq \emptyset$.

Since $g(uv)=0$, vertex s is not adjacent to R_1, and both R_1 and L_1 are independent sets. Noting that $2 \leq a+b \leq 3$ and $L_1 \neq \emptyset$, we have $1 \leq b \leq 2$. If $b=1$, then v is a cut vertex. Since $\omega(G-\{v\})=|R_1|+1$, we have $1 \leq |R_1| \leq k-1$.

Claim 5. If $a=2$, then $|L_1|=1$ and L_2 is an independent set, and if $b=2$, then $|R_1|=1$ and R_2 is an independent set.

Proof. Suppose that $a=2$. It follows from $\mu(C(e))=2$ that $\mu(G[L_u])=1$. Since $g(uv)=0$, then by Observation 2.1, $G[L_u]$ is a star with center vertex in L_1. As L_1 is an independent set, we have $|L_1|=1$, and thus L_2 is an independent set. Similarly, if $b=2$, then $|R_1|=1$ and R_2 is an independent set. □

Subcase 2.1. $a=1$.

Let $T_1=N_{L_1}(s)$ and $T_2=L_1-T_1$. Clearly, $T_1 \subseteq V_2(G)$ and $T_2 \subseteq V_1(G)$. Since e is not a bridge of G, we have $T_1 \neq \emptyset$. Let x_i be an arbitrary vertex in T_1. Let $e_{27}=sv$ and let S_{27} be a vertex set satisfying Lemma 2.3 for e_{27}.

If $b=1$, then $V_{\geq 3} \subseteq \{v,s,u\}$. By Lemma 2.5(1), we have $|S_{27}|=1$. The cycle $uvsx_iu$ implies $S_{27}=\{u\}$. Consequently, $\omega(G-S_{27})=|T_2|+1=k$, which forces $|T_2|=k-1$. Since $\{s,u\}$ is a cutset of G, we have $\omega(G-\{s,u\})=|L_1|+|D(e)|+1=|L_1|+k \leq 2k$. Hence, $|L_1| \leq k$. Combining this with $|L_1|=|T_1|+|T_2|$, we deduce $|T_1|=1$. Thus, G belongs to class $\mathcal{H}_{17}$.

If b=2, then by Claim 5, $|R_1|$=1 and R_2 is an independent set. Let us denote the unique vertex in R_1 by y_1. Let R_2' be a set of vertex of R_2 which has degree one in G. Since $\tau(G)=\frac{1}{k}$, we have $0 \leq |R_2'| \leq k-1$. For the edge e_{27}, since $V_{\geq 3}-\{v,s\} \subseteq \{u,y_1\}$ and $\omega(G-\{u,y_1\})=|T_2|+|R_2'|+1$, then by Lemma 2.5(2), we have $|S_{27}|$=1. Suppose that s is adjacent to R_2. Let $y_i \in N_{R_2}(s)$. Since vux_is and vy_1y_is are two v-s paths in G, we have $|S_{27}| \geq 2$, a contradiction. Thus, s is not adjacent to R_2. Then y_1 is a cut vertex. Since $\tau(G)=\frac{1}{k}$, we have $1 \leq |R_2| \leq k-1$. As $uvsx_iu$ forms a cycle, we have $S_{27}=\{u\}$. Then $\omega(G-S_{27})=|T_2|+1=k$, forcing $|T_2|=k-1$. Since $\{u,s\}$ is a cutset of G, we have $\omega(G-\{u,s\})=|L_1|+|D(e)|+1=|L_1|+k \leq 2k$. Thus, $|L_1| \leq k$.

Combining this with $|T_1|+|T_2|=|L_1|$, we get $|T_1|=1$. Therefore, G belongs to class $\mathcal{H}_{18}$.

Subcase 2.2. $a=2$.

In this case, $b=1$. By Claim 5, $|L_1|=1$ and L_2 is an independent set. Denote the unique vertex of L_1 by x_1. Obviously, s is adjacent to exactly one of L_1 and L_2, and $V_{\geq 3}\subseteq\{v,s,x_1\}$.

If s is adjacent to L_1, then by Lemma 2.5(1), we have $|S_{27}|=1$. The cycle $uvsx_1u$ implies $S_{27}=\{x_1\}$. Then $\omega(G-S_{27})=|L_2|+1=k$. Thus, $|L_2|=k-1$. Choose the edge sx_1, we can get $|R_1|=k-1$ by similar discussion. Therefore, G belongs to class $\mathcal{H}_{17}$.

If s is adjacent to L_2, let $T_3=N_{L_2}(s)$ and $T_4=L_2-T_3$. Then $T_3\subseteq V_2(G)$ and $T_4\subseteq V_1(G)$. By Lemma 2.5(1), we have $|S_{27}|=1$. Let $x_i\in T_3$. The cycle $uvsx_ix_1u$ implies $S_{27}=\{x_1\}$. Then $\omega(G-S_{27})=|T_4|+1=k$, which implies $|T_4|=k-1$. Since $\{s,x_1\}$ is a cutset of G, we have $\omega(G-\{s,x_1\})=|L_2|+|D(e)|+1=|L_2|+k\leq 2k$. Hence, $|L_2|\leq k$, and so $|T_3|=1$. Therefore, G belongs to class $\mathcal{H}_{15}$.

Case 3.2. $|S|=2$.

Let $S=\{s_1,s_2\}$. By Lemma 2.3, $\omega(G-S)=k|S|=2k\geq 4$. By Lemma 2.4, every vertex of S must be adjacent to at least three components of $D(e)$. Since $\mu(G)\leq 3$ and $\mu(C(e))\geq 1$, we have $\mu(G[S\cup V(D(e))])=2$, and so $\mu(C(e))=1$. Since e is a bridge of $C(e)$, $C(e)$ is a star by Observation 2.1. Without loss of generality, let u be the center vertex of $C(e)$. Then $b=0$ and $0\leq a\leq 1$. Let $\{N_{s_1},N_{s_2},N_{s_{1,2}}\}$ be a partition of $V(D(e))$ where

$$\begin{aligned}
N_{s_1}&=\{w\in V(D(e))\,|\,s_1w\in E(G),s_2w\notin E(G)\},\\
N_{s_2}&=\{w\in V(D(e))\,|\,s_1w\notin E(G),s_2w\in E(G)\},\\
N_{s_{1,2}}&=\{w\in V(D(e))\,|\,s_1w\in E(G),s_2w\in E(G)\}.
\end{aligned}$$

Observe that $N_{s_1}\cup N_{s_2}\subseteq V_1(G)$ and $N_{s_{1,2}}\subseteq V_2(G)$. Since $\tau(G)=\frac{1}{k}$, we have $0\leq |N_{s_1}|\leq k-1$ and $0\leq|N_{s_2}|\leq k-1$. As $|D(e)|=\omega(G-S)-1=2k-1$, we conclude that $|N_{s_{1,2}}|\geq 1$. Let w_1 be a vertex in $N_{s_{1,2}}$. It follows from $g(uv)\leq|S|$ that $g(uv)\leq 2$.

Case 3.2.1. $g(uv)=2$.

Then every vertex of S is adjacent to both u and v. Let $e_{28}=us_2$ and let S_{28} be a vertex set satisfying Lemma 2.3 for e_{28}. Since uvs_2 and $us_1w_1s_2$ are two u-s_2 paths in G, we have $|S_{28}|\geq 2$ and $v\in S_{28}$. However, $N_G(v)=\{u,s_1,s_2\}\subseteq N_G(u)\cup N_G(s_2)$, which contradicts Lemma 2.4.

Case 3.2.2. $g(uv)=1$.

Without loss of generality, we assume that $us_1,vs_1\in E(G)$. By Lemma 2.8, we have $L_1\neq\emptyset$ and $vs_2\in E(G)$. Given that $g(uv)=1$, it follows that s_1 is not adjacent to L_1, and $us_2,s_1s_2\notin E(G)$. Let $T_1=N_{L_1}(s_2)$ and $T_2=L_1-T_1$. Clearly, $T_1\subseteq V_2(G)$, $T_2\subseteq V_1(G)$ and $|T_2|\leq k-1$. Suppose that $T_1\neq\emptyset$ and let $x_i\in T_1$. Let $e_{29}=vs_2$ and let S_{29}

be a vertex set satisfying Lemma 2.3 for e_{29}. Since vux_is_2 and $vs_1w_1s_2$ are two v-s_2 paths in G, we have $|S_{29}|\geq 2$. By Lemma 2.4, we have $S_{29}\subseteq V_{\geq 3}\subseteq\{u,v,s_1,s_2\}$, which implies $S_{29}=\{u,s_1\}$. Then $\omega(G-S_{29})=|T_2|+|N_{s_1}|+1\leq k-1+k-1+1=2k-1$, which contradicts $\omega(G-S_{29})\geq 2k$. Hence, $T_1=\emptyset$, and so $|L_1|=|T_2|\leq k-1$. Let $e_{30}=vs_1$ and let S_{30} be a vertex set satisfying Lemma 2.3 for e_{30}. Since vus_1 and $vs_2w_1s_1$ are two v-s_1 paths in G, we have $|S_{30}|\geq 2$. By Lemma 2.4, we have $S_{30}\subseteq V_{\geq 3}\subseteq\{u,v,s_1,s_2\}$, which implies $S_{30}=\{u,s_2\}$. Then $\omega(G-S_{30})=|L_1|+|N_{s_2}|+1\leq k-1+k-1+1=2k-1$, which contradicts $\omega(G-S_{30})\geq 2k$.

Case 3.2.3. $g(uv)=0$.

Without loss of generality, suppose that $vs_2\in E(G)$. Since $g(uv)=0$ and $|N_{s_{1,2}}|\geq 1$, we can obtain that $s_1s_2\notin E(G)$.

Case 3.2.3.1. $L_1=\emptyset$.

Since $g(uv)=0$, we have $us_1\in E(G)$ and $us_2,vs_1\notin E(G)$. Let $e_{31}=us_1$ and let S_{31} be a vertex set satisfying Lemma 2.3 for e_{31}. Since $V_{\geq 3}(G)\subseteq\{s_1,s_2\}$, then by Lemma 2.5(1), we have $|S_{31}|=1$. The cycle $uvs_2w_1s_1$ implies $S_{31}=\{s_2\}$. Consequently, $\omega(G-S_{31})=|N_{s_2}|+1=k$, forcing $|N_{s_2}|=k-1$. By a similar argument for the edge vs_2, we deduce that $|N_{s_1}|=k-1$. Noting that $|D(e)|=|N_{s_1}|+|N_{s_2}|+|N_{s_{1,2}}|=2k-1$, it follows that $|N_{s_{1,2}}|=1$. Therefore, G belongs to the family $\mathcal{H}_{15}$.

Case 3.2.3.2. $L_1\neq\emptyset$.

Since $g(uv)=0$, this implies that L_1 is an independent set. Since $\{u,s_1,s_2\}$ is a cutset of G, we have:

$$\omega(G-\{u,s_1,s_1\})=|L_1|+1+|D(e)|=|L_1|+2k\leq 3k.$$

Thus, $|L_1|\leq k$. Let $\{M_1,M_2,M_3,M_4\}$ be a partition of L_1 where

$$\begin{aligned}
M_1&=\{x\in L_1\,|\,s_1x\in E(G),s_2x\notin E(G)\},\\
M_2&=\{x\in L_1\,|\,s_2x\in E(G),s_1x\notin E(G)\},\\
M_3&=\{x\in L_1\,|\,s_1x\notin E(G),s_2x\notin E(G)\},\\
M_4&=\{x\in L_1\,|\,s_1x\in E(G),s_2x\in E(G)\}.
\end{aligned}$$

Clearly, $M_3\subseteq V_1(G)$, $M_1\cup M_2\subseteq V_2(G)$ and $M_4\subseteq V_3(G)$. Since $\tau(G)=\frac{1}{k}$, we have $0\leq|M_3|\leq k-1$.

Case 3.2.3.2.1. s_1 is not adjacent to L_u.

Then $M_1=M_4=\emptyset$ and $s_1u\notin E(G)$. Since e is not a bridge of G and $g(uv)=0$, we have $s_2u\notin E(G)$ and s_2 is adjacent to L_u. Therefore, $M_2\neq\emptyset$. Without loss of generality, let x_1 be a vertex in M_2.

Let $e_{32}=vs_2$ and let S_{32} be a vertex set satisfying Lemma 2.3 for e_{32}. Since $V_{\geq 3}(G)-\{v,s_2\}\subseteq\{u,s_1\}$ and $\omega(G-\{u,s_1\})=|N_{s_1}|+|M_3|+1$, then by Lemma 2.5(2), we have $|S_{32}|=1$. Consequently, $vs_1\notin E(G)$. Otherwise, vux_1s_2 and $vs_1w_1s_2$ are two v-s_2 paths in G, which implies $|S_{32}|\geq 2$, a contradiction. The cycle uvs_2x_1u

implies $S_{32}=\{u\}$. Then $\omega(G-S_{32})=|M_3|+1=k$, forcing $|M_3|=k-1$. Since $|L_1|=|M_3|+|M_2|\le k$ and $M_2\neq\emptyset$, we have $|M_2|=1$. As $\omega(G-\{s_2\})=|N_{s_2}|+2$, it follows that $|N_{s_2}|\le k-2$. Noting that $|D(e)|=|N_{s_1}|+|N_{s_2}|+|N_{s_{1,2}}|=2k-1$ and $|N_{s_1}|\le k-1$, we have $|N_{s_{1,2}}|\ge 2$.

Let $w_2\in N_{s_{1,2}}-w_1$. Let $e_{33}=s_2w_1$ and let S_{33} be a vertex set satisfying Lemma 2.3 for e_{33}. Since $V_{\ge 3}(G)-\{s_2,w_1\}\subseteq\{u,s_1\}$ and $\omega(G-\{u,s_1\})=|N_{s_1}|+|M_3|+1$, we have $|S_{33}|=1$ by Lemma 2.5(2). The cycle $s_1w_1s_2w_2s_1$ implies $S_{33}=\{s_1\}$. Thus, $\omega(G-S_{33})=|N_{s_1}|+1=k$, forcing $|N_{s_1}|=k-1$. Let $e_{34}=s_1w_1$ and let S_{34} be a vertex set satisfying Lemma 2.3 for e_{34}. If $|S_{34}|\ge 2$, then by Lemma 2.4, we have $S_{34}=\{u,s_2\}$ and $\omega(G-S_{34})=|M_1|+|M_2|+|N_{s_2}|+2=|N_{s_2}|+k+2=2k$. If $|S_{34}|=1$, the cycle $s_1w_1s_2w_2s_1$ implies $S_{34}=\{s_2\}$ and $\omega(G-S_{34})=|N_{s_2}|+2=k$. By the above equalities, we have $|N_{s_2}|=k-2$. Noting that $|D(e)|=|N_{s_1}|+|N_{s_2}|+|N_{s_{1,2}}|=2k-1$, we have $|N_{s_{1,2}}|=2$. Therefore, G belongs to class $\mathcal{H}_{19}$.

Case 3.2.3.2.2. s_1 is adjacent to u.

It follows from $g(uv)=0$ that $vs_1\notin E(G)$ and s_1 is not adjacent to L_1, which implies $M_1=M_4=\emptyset$. Let $e_{35}=us_1$ and let S_{35} be a vertex set satisfying Lemma 2.3 for e_{35}. Since $V_{\ge 3}(G)\subseteq\{u,s_1,s_2\}$, then by Lemma 2.5(1), we have $|S_{35}|=1$. As $uvs_2w_1s_1u$ is a cycle in which s_2 is the only possible cut vertex, we have $S_{35}=\{s_2\}$. Then $\omega(G-S_{35})=|N_{s_2}|+1=k$. Thus, $|N_{s_2}|=k-1$.

Suppose that $|N_{s_{1,2}}|\ge 2$. Note that $|D(e)|=|N_{s_1}|+|N_{s_2}|+|N_{s_{1,2}}|=2k-1$. Thus, $|N_{s_1}|\le k-2$. For the edge $e_{33}=s_2w_1$ and its corresponding vertex set S_{33}, since $V_{\ge 3}(G)-\{s_2,w_1\}\subseteq\{u,s_1\}$ and $\omega(G-\{u,s_1\})=|N_{s_1}|+|M_3|+1$, we have $|S_{33}|=1$ by Lemma 2.5(2). Let w_i be an arbitrary vertex in $N_{s_{1,2}}-w_1$. The cycle $s_1w_1s_2w_2s_1$ implies $S_{33}=\{s_1\}$. Then $\omega(G-S_{33})=|N_{s_1}|+1\le k-2+1=k-1$, which contradicts $\omega(G-S_{33})=k$. Therefore, $|N_{s_{1,2}}|=1$. Since $|D(e)|=|N_{s_1}|+|N_{s_2}|+|N_{s_{1,2}}|=2k-1$, we have $|N_{s_1}|=k-1$.

Suppose that $M_2\neq\emptyset$. Since $\{u,s_2\}$ is a cutset of G, we have

$$\omega(G-\{u,s_2\})=|M_3|+|M_2|+|N_{s_2}|+2=|M_3|+|M_2|+k+1\le 2k.$$

Combining this with $M_2\neq\emptyset$, we have $|M_3|\le k-2$. For the edge $e_{32}=vs_2$, since $V_{\ge 3}(G)-\{v,s_2\}\subseteq\{u,s_1\}$ and $\omega(G-\{u,s_1\})=|N_{s_1}|+|M_3|+1$, then by Lemma 2.5(2), we have $|S_{32}|=1$. Let x_1 be a vertex in M_2. As uvs_2x_1u is a cycle in which u is the only possible cut vertex, then $S_{32}=\{u\}$. Consequently, $\omega(G-S_{32})=|M_3|+1\le k-2+1=k-1$, which contradicts $\omega(G-S_{32})=k$. Hence, $M_2=\emptyset$. It follows that $1\le|M_3|\le k-1$. Therefore, G belongs to class $\mathcal{H}_{15}$.

Case 3.2.3.2.3. s_1 is adjacent to L_1.

Let x_2 be an arbitrary vertex in $N_{L_1}(s_1)$. Suppose that $vs_1\in E(G)$. Let $e_{36}=vs_1$ and let S_{36} be a vertex set satisfying Lemma 2.3 for e_{36}. Since vux_2s_1 and $vs_2w_1s_1$ are two v-s_1 paths in G, we have $|S_{36}|\ge 2$. Clearly, the neighbor set of each vertex of M_4 is $\{u,s_1,s_2\}$, which is contained in $N_G(v)\cup N_G(s_1)$. By Lemma 2.4, we have

$M_4 \cap S_{36} = \emptyset$. Then $S_{36} \subseteq V_{\geq 3}(G) \subseteq M_4 \cup \{u,v,s_1,s_2\}$. Therefore, $S_{36} = \{u,s_2\}$ and $\omega(G-S_{36}) = |M_3|+|N_{s_2}|+1 \leq k-1+k-1+1 = 2k-1$, which contradicts $w(G-S_{36}) \geq 2k$. Hence, $vs_1 \notin E(G)$.

Suppose that $M_4 \neq \emptyset$. Without loss of generality, Let x_3 be a vertex in M_4. Let $e_{37} = x_3s_2$ and let S_{37} be a vertex set satisfying Lemma 2.3 for e_{37}. Since x_3uvs_2 and $x_3s_1w_1s_2$ are two x_3-s_2 paths in G, we have $|S_{37}| \geq 2$. Since the neighbor set of each vertex of M_4 is $\{u,s_1,s_2\}$, then by Lemma 2.4, we have $M_4 \cap S_{37} = \emptyset$. Since $V_{\geq 3}(G) \subseteq M_4 \cup \{u,s_1,s_2\}$, we get $S_{37} = \{u,s_1\}$. Note that $|L_1| = |M_1|+|M_2|+|M_3|+|M_4| \leq k$. Then $\omega(G-S_{37}) = |N_{s_1}|+|M_3|+1+|M_1| \leq |N_{s_1}|+k-|M_2|-|M_4| \leq k-1+k-1+1 = 2k-1$, which contradicts $\omega(G-S_{37}) \geq 2k$. Hence, $M_4 = \emptyset$.

As s_1 is adjacent to L_1, we have $M_1 \neq \emptyset$ and $x_2 \in M_1$. Suppose that $M_2 \neq \emptyset$. Without loss of generality, let x_4 be a vertex in M_2. Let $e_{38} = x_4s_2$ and let S_{38} be a vertex set satisfying Lemma 2.3 for e_{38}. If $|S_{38}| \geq 2$, then since Lemma 2.4 ensures $S_{38} \subseteq V_{\geq 3}(G) \subseteq \{u,s_1,s_2\}$, we have $S_{38} = \{u,s_1\}$. Note that $|L_1| = |M_1|+|M_2|+|M_3| \leq k$. Thus, $|M_1|+|M_3| \leq k-1$ and $\omega(G-S_{38}) = |N_{s_1}|+|M_3|+|M_1|+1 \leq |N_{s_1}|+|L_1|-|M_2| \leq k-1+k-1+1 = 2k-1$, which contradicts $\omega(G-S_{38}) \geq 2k$. Thus, $|S_{38}| = 1$. The cycle uvs_2x_4u implies $S_{38} = \{u\}$. Then $\omega(G-S_{38}) = |M_3|+1 = k$. Thus, $|M_3| = k-1$. However, $|L_1| = |M_1|+|M_2|+|M_3| = |M_1|+|M_2|+k-1 \geq k+1$, which contradicts $|L_1| \leq k$. Hence, $M_2 = \emptyset$.

Suppose that $|M_1| \geq 2$. It follows from $|L_1| = |M_1|+|M_3| \leq k$ that $|M_3| \leq k-2$. Let x_i be a vertex in $M_1 - x_2$. Let $e_{39} = s_1x_2$ and let S_{39} be a vertex set satisfying Lemma 2.3 for e_{39}. If $|S_{39}| \geq 2$, then by Lemma 2.4, we have $S_{39} \subseteq V_{\geq 3}(G) \subseteq \{u,s_1,s_2\}$. Thus, $S_{39} = \{u,s_2\}$ and $\omega(G-S_{39}) = |M_3|+|N_{s_2}|+2 \leq k-2+k-1+2 = 2k-1$, which contradicts $w(G-S_{39}) \geq 2k$. Thus, $|S_{39}| = 1$. The cycle $ux_2s_1x_iu$ implies $S_{39} = \{u\}$, and so $\omega(G-S_{39}) = |M_3|+1 \leq k-1$, which contradicts $w(G-S_{39}) = k$. Therefore, $|M_1| = 1$.

Suppose that $|N_{s_{1,2}}| \geq 2$. It follows from $|D(e)| = |N_{s_{1,2}}|+|N_{s_1}|+|N_{s_2}| = 2k-1$ that $|N_{s_1}| \leq k-2$ or $|N_{s_2}| \leq k-2$. Without loss of generality, we assume that $|N_{s_1}| \leq k-2$. Let $w_2 \in N_{s_{1,2}} - w_1$. For the edge $e_{33} = s_2w_1$ and its corresponding vertex S_{33}, if $|S_{33}| \geq 2$, since $V_{\geq 3}(G) \subseteq \{u,s_1,s_2\}$, we conclude that $S_{33} = \{u,s_1\}$, and $\omega(G-S_{33}) = |M_3|+|N_{s_1}|+2 \leq k-1+k = 2k-1$, which contradicts $\omega(G-S_{33}) \geq 2k$. If $|S_{33}| = 1$, then since $s_1w_1s_2w_2s_1$ forms a cycle, we have $S_{33} = \{s_1\}$. Consequently, $\omega(G-S_{33}) = |N_{s_1}|+1 \leq k-2+1 = k-1$, which contradicts $\omega(G-S_{33}) = k$. Thus, $|N_{s_{1,2}}| = 1$.

It follows from $\tau(G) = \frac{1}{k}$ that $|M_3| \leq k-1$, $|N_{s_1}| \leq k-1$ and $|N_{s_2}| \leq k-1$. Since $|N_{s_{1,2}}| = 1$ and $|D(e)| = |N_{s_1}|+|N_{s_2}|+|N_{s_{1,2}}| = 2k-1$, we have $|N_{s_1}| = |N_{s_2}| = k-1$. Thus, G belongs to class $\mathcal{H}_{16}$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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