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Decay Estimates of Solutions for Incompressible Magneto-Micropolar Equations with Partial Dissipation

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Abstract. This paper concerns the Cauchy problem of three-dimensional incompressible magneto-micropolar equations with partially mixed velocity dissipation and magnetic diffusion. Under smallness assumption on initial data, we first establish the global existence of smooth solution, and then derive the long-time decay estimates for the solution. The proof is based on energy methods, and some new weighted energy functionals and bootstrap argument are introduced. We remark that, despite the lack of dissipation in certain directions, the solution exhibits power decay rates as time tends to infinity.

AMS subject classifications: 35A05, 35Q35, 76D03

Key words: Magneto-micropolar Fluid; Decay estimates; Partial dissipation

1 Introduction

The magneto-micropolar fluid equations describe the motion of small-scale ferromagnetic particle aggregates suspended in a viscous magneto-fluid, such as saline solutions, esters, or fluorocarbons, under an external magnetic field, and it is of great importance in both theoretical studies and practical applications; see, for example, [1]. Based on the dissipation mechanisms, the incompressible magneto-micropolar fluid equations are classified as either fully dissipative systems, where

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dissipation is uniform in all directions, or partially dissipative systems, where it is weak or absent in some directions. In this paper, we are concerned with the incompressible magneto-micropolar fluid system with mixed velocity dissipation and magnetic diffusion in dimension three. The mathematics model describing can be written as (cf. [6]):

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla p=A+b\cdot\nabla b+2\chi\nabla\times w, \\ \partial_t b+u\cdot\nabla b=B+b\cdot\nabla u, \\ \partial_t w+u\cdot\nabla w+4\chi w=\gamma\Delta w+2\chi\nabla\times u, \\ \nabla\cdot u=0,\nabla\cdot b=0, \end{cases} \tag{1.1}$$

where the functions $u=(u_1,u_2,u_3)$, $b=(b_1,b_2,b_3)$, and $w=(w_1,w_2,w_3)$ are the velocity field, magnetic field, and micro-rotation field, respectively. p is the pressure. γ and χ are the angular viscosities and spin viscosity, respectively. The functions

$$A=\begin{pmatrix} (\mu+\chi)\left(\partial^2_{x_2x_2}u_1+\partial^2_{x_3x_3}u_1\right) \\ (\mu+\chi)\left(\partial^2_{x_1x_1}u_2+\partial^2_{x_3x_3}u_2\right) \\ (\mu+\chi)\left(\partial^2_{x_1x_1}u_3+\partial^2_{x_2x_2}u_3\right) \end{pmatrix}, \quad B=\begin{pmatrix} \eta\left(\partial^2_{x_2x_2}b_1+\partial^2_{x_3x_3}b_1\right) \\ \eta\left(\partial^2_{x_1x_1}b_2+\partial^2_{x_3x_3}b_2\right) \\ \eta\left(\partial^2_{x_1x_1}b_3+\partial^2_{x_2x_2}b_3\right) \end{pmatrix},$$

where $\mu>0$ and $\chi>0$ are the kinematic viscosity coefficients, and $\eta>0$ is the magnetic diffusivity coefficient. We study the Cauchy problem of (1.1) with the given boundary conditions

$$(u,b,w)(x,t\geq 0)=(0,0,0) \quad \text{as}\, x\to\pm\infty \tag{1.2}$$

and

$$(u,b,w)(x,t=0)=(u_0,b_0,w_0)(x). \tag{1.3}$$

In recent years, substantial progress has been achieved in understanding the global regularity, well-posedness, and decay behavior of solutions for magneto-micropolar systems with partially dissipations. When all the velocity field, magnetic field, and micro-rotation field exhibit both horizontal and vertical dissipation, Wang-Wang [18] proved the global existence of smooth solutions under small initial assumption. Ma [13] proved the global existence of solutions of three-dimensional

Cauchy problem under anisotropic mixed dissipation. Wang-Li [17] discussed a large number of magneto-micropolar systems with mixed partial dissipation, and showed the global existence of classical solution and its stability in case when the initial data has small perturbations near equilliba. By energy methods and Blowup mechanism, Wang-Gu [16] proved the global existence of smooth solution with with finite initial energy. Zhong-Xu [21] established the local existence and uniqueness of strong solutions in both bounded domains and the whole space. Yan-Chen [22] analyzed the fully viscous incompressible magneto-micropolar system and present a new regularity criterion in terms of pressure term. Li-Shang [11] examined the fully viscous three-dimensional system with initial data belongs to $L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ and proved that the weak solutions possess the following decay rate

$$\|u(t)\|_{L^2}+\|w(t)\|_{L^2}+\|b(t)\|_{L^2}\le C(1+t)^{-3/4},$$

and additionally, under smallness assumption, the smooth solution satisfies

$$\|D^m u(t)\|_{L^2(\mathbb{R}^3)}+\|D^m w(t)\|_{L^2(\mathbb{R}^3)}+\|D^m b(t)\|_{L^2(\mathbb{R}^3)}\le C(1+t)^{-\frac{3}{4}-\frac{m}{2}}.$$

For other related results, we refer to [2–5, 9, 10, 12, 19, 20, 23] and references therein.

In this paper, we shall prove that (1.1)-(1.3) admits a unique global smooth solution, provided that the initial data are small in some Sobolev space. Furthermore, we obtain long-time decay rates for the solution.

Theorem 1.1. *Let* $s>\frac{3}{2}$ *and let the initial data* $(u_0,b_0,w_0)\in H^s(\mathbb{R}^3)$ *satisfy* $\nabla\cdot u_0=\nabla\cdot b_0=0$. *Assume that there is a small constant* $\delta>0$ *such that*

$$\|(u_0,b_0,w_0)\|_{H^s(\mathbb{R}^3)}<\delta. \tag{1.4}$$

Then the the problem (1.1)-(1.3) admits a unique global smooth solution (u,b,w) *which satisfies*

$$(u,b,w)\in L^\infty\left(0,T;H^s\left(\mathbb{R}^3\right)\right)\cap L^2\left(0,T;H^{s+1}\left(\mathbb{R}^3\right)\right),$$

and

$$\|(u,b,w)\|^2_{H^s(\mathbb{R}^3)}+\int_0^T\left(\frac{2}{3}\mu\|\nabla u\|^2_{H^s(\mathbb{R}^3)}d+\frac{2}{3}\eta\|\nabla b\|^2_{H^s(\mathbb{R}^3)}+2\gamma\|\nabla w\|^2_{H^s(\mathbb{R}^3)}\right)\mathrm{d}\tau\le C\delta^2, \tag{1.5}$$

where the constant C *depends on* μ,η,γ.

Theorem 1.2. *Let* $s^*\geq -2$ *and let the initial data* $(u_0,b_0,w_0)\in H^{s^*}(\mathbb{R}^3)$ *satisfy* $\nabla\cdot u_0=\nabla\cdot b_0=0$. *Assume that there is a some small* $\delta_1>0$ *such that*

$$\|(u_0,b_0,w_0)\|_{H^{s^*}(\mathbb{R}^3)}<\delta_1.$$

Then the problem (1.1)-(1.3) admits a global solution $(u,b,w)\in C\big([0,\infty);H^{s^*}(\mathbb{R}^3)\big)$ *which satisfies*

$$\|\big(\Lambda^k u,\Lambda^k b,\Lambda^k w\big)\|_{L^2(\mathbb{R}^3)}\leq C(1+t)^{-\frac{k}{2}-\frac{3}{4}},$$

where the constant C *depends on* μ,η,γ, *and* $k\geq -1$ *is an integer.*

Remark 1.1. We point out that the solutions still exhibited sharp power decay rates, although some dissipation lost in certain directions.

Remark 1.2. In the proof of the Thorem1.2, the idea is to use the weighted energy estimate to get the decays we want,namely

$$\begin{aligned}E_1(t)=&\sup_{0\leq\tau\leq t}\|\big(\Lambda^{-1}u(\tau),\Lambda^{-1}b(\tau),\Lambda^{-1}w(\tau)\big)\|_{L^2}^2+\frac{2}{3}\mu\int_0^t\|u(\tau)\|_{L^2}^2\mathrm{d}\tau\\&+\frac{2}{3}\eta\int_0^t\|b(\tau)\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t\|w(\tau)\|_{L^2}^2\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}E_2(t)=&\sup_{0\leq\tau\leq t}\|\big(\Lambda^{-2}u(\tau),\Lambda^{-2}b(\tau),\Lambda^{-2}w(\tau)\big)\|_{L^2}^2+\frac{2}{3}\mu\int_0^t\|\Lambda^{-1}u(\tau)\|_{L^2}^2\mathrm{d}\tau\\&+\frac{2}{3}\eta\int_0^t\|\Lambda^{-1}b(\tau)\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t\|\Lambda^{-1}w(\tau)\|_{L^2}^2\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}E_3(t)=&\sup_{0\leq\tau\leq t}(1+\tau)^{1/2}\|\big(\Lambda^{-1}u(\tau),\Lambda^{-1}b(\tau),\Lambda^{-1}w(\tau)\big)\|_{L^2}^2+\frac{2}{3}\mu\int_0^t(1+\tau)^{1/2}\|u(\tau)\|_{L^2}^2\mathrm{d}\tau\\&+\frac{2}{3}\eta\int_0^t(1+\tau)^{1/2}\|b(\tau)\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t(1+\tau)^{1/2}\|w(\tau)\|_{L^2}^2\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}E_4(t)=&\sup_{0\leq\tau\leq t}(1+\tau)^{3/2}\|(u(\tau),b(\tau),w(\tau))\|_{L^2}^2+\frac{2}{3}\mu\int_0^t(1+\tau)^{3/2}\|\nabla u(\tau)\|_{L^2}^2\mathrm{d}\tau\\&+\frac{2}{3}\eta\int_0^t(1+\tau)^{3/2}\|\nabla b(\tau)\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t(1+\tau)^{3/2}\|\nabla w(\tau)\|_{L^2}^2\mathrm{d}\tau.\end{aligned}$$

We comment on the proof of Theorem 1.1 and Theorem 1.2.

Since local existence result is known in [11], we complete the proof of Theorem 1.1 by using the *a priori* estimates and continuity method. To do this, we first prove Lemma 2.2, it not only reveals the intrinsic properties of the equations (1.1) but also greatly simplifies the proof. In Section 3, we are devoted to proving the needed *a priori* estimates. By considering the energy functional

$$
\begin{aligned}
E(t)=&\sup_{0\le\tau\le t}\|(u(\tau),b(\tau),w(\tau))\|_{H^s}^2+\frac{2}{3}\mu\int_0^t\|\nabla u(\tau)\|_{H^s}^2\mathrm{d}\tau\\
&+\frac{2}{3}\eta\int_0^t\|\nabla b(\tau)\|_{H^s}^2\mathrm{d}\tau+2\gamma\int_0^t\|\nabla w(\tau)\|_{H^s}^2\mathrm{d}\tau
\end{aligned}
$$

and making use of the equivalence of $\|(u,b,w)\|_{H^s}\sim\|(u,b,w)\|_{L^2}+\|\nabla^s(u,b,w)\|_{L^2}$, we can prove the inequality

$$
E(t)\le C_0E(0)+C_1E^{\frac{3}{2}}(t),\quad \forall t>0.
$$

By this and initial smallness assumption, we complete the proof via continuity method. The proof of Theorem 1.2 primarily employs the weighted energy method in combination with a mathematical induction argument. To obtain more precise decay estimates and to clarify the structure of the proof, we first introduce several new energy functionals and weighted energy functionals (see Remark 1.2 for details). Similar to the proof of Theorem 1.1, by synthesizing the four estimates from Propositions 4.1 and 4.2, it is evident that the weighted energy functional $\tilde{E}(t)$ remains uniformly bounded. As a result, we obtain decay estimates for the first-order negative derivatives of the solution (see Proposition 4.1), as well as for the basic energy (see Proposition 4.2). Finally, for the decay of higher-order derivatives, we combine the stability result provided in Theorem 1.1 with a mathematical induction argument to derive the desired conclusion.

Notations: We denote by $A\sim B$ if $c_1A\le B\le c_2A$ for some positive constants. Refer to [14], the operator $(-\Delta)^\alpha$ is defined via the Fourier transform $\widehat{(-\Delta)^\alpha f}(\xi)=|\xi|^{2\alpha}\hat{f}(\xi)$ for $\alpha\in(0,1)$, and in particular $\Lambda=(-\Delta)^{1/2}$. For simplicity we write

$$
\iiint_{\mathbb{R}^3}u(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3=\int_{\mathbb{R}^3}u(x)\mathrm{d}x.
$$

The norm identity

$$
\begin{aligned}
\|u\|_{H^s(\mathbb{R}^3)}+\|b\|_{H^s(\mathbb{R}^3)}+\|w\|_{H^s(\mathbb{R}^3)}=\|(u,b,w)\|_{H^s(\mathbb{R}^3)}&=\|(\Lambda^su,\Lambda^sb,\Lambda^sw)\|_{L^2(\mathbb{R}^3)}\\
&=\|(\Lambda^su,\Lambda^sb,\Lambda^sw)\|_{L^2},s\in\mathbb{Z}\cap[-2,\infty).
\end{aligned}
$$

The remainder of this paper is organized as follows. Section 2 introduces two fundamental lemmas: the commutator estimate and the dissipation lemma. Section 3 provides a detailed proof of Theorem 1.1, consisting of three parts: establishing basic and higher-order a priori estimates via propositions, and closing them through a bootstrap argument. Section 4 proves Theorem 1.2 through three propositions.

The first establishes a priori estimates for negative-order derivatives in three steps: estimating $(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)$, then $(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)$, and finally deriving decay rates via the weighted energy method. The second proposition gives the basic decay estimate, while the third, combining weighted energy with induction, yields decay estimates for higher-order derivatives. In the sequel, the capital letter $C>0$ symbols a generic constant which may vary from line to line.

2 Some useful lemmas

The first lemma is for the commutator estimates. The detailed proof can be find in [7,8].

Lemma 2.1. *Define* $[\Lambda^s,f\cdot\nabla]g:=\Lambda^s(f\cdot\nabla g)-f\cdot\nabla\Lambda^s g$. *Assume* $s>0$ *and* $1<r<\infty$, *with* $\frac{1}{r}=\frac{1}{q_1}+\frac{1}{p_1}=\frac{1}{q_2}+\frac{1}{p_2}$ *where* $q_1,p_2\in(1,\infty)$ *and* $p_1,q_2\in[1,\infty)$. *Then*

$$\|[\Lambda^s,f\cdot\nabla]g\|_{L^r}\leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}}+\|\Lambda^s f\|_{L^{p_2}}\|\nabla g\|_{L^{q_2}}),$$

and

$$\|\Lambda^s(fg)\|_{L^r}\leq C(\|g\|_{L^{p_1}}\|\Lambda^s f\|_{L^{q_1}}+\|\Lambda^s g\|_{L^{p_2}}\|f\|_{L^{q_2}}),$$

where the constant C *depends only on* r, q_1, q_2, p_1, *and* p_2.

Lemma 2.2. *Let* $h=(h_1,h_2,h_3)$ *satisfy* $\nabla\cdot h=0$. *Then, for any* $\alpha\in\mathbb{R}$, *the following inequality holds:*

$$\begin{aligned}\int_{\mathbb{R}^3}&(-\partial_{22}h_1\Lambda^{2\alpha}h_1-\partial_{33}h_1\Lambda^{2\alpha}h_1-\partial_{11}h_2\Lambda^{2\alpha}h_2-\partial_{33}h_2\Lambda^{2\alpha}h_2\\&-\partial_{22}h_3\Lambda^{2\alpha}h_3-\partial_{11}h_3\Lambda^{2\alpha}h_3)\mathrm{d}x\geq\frac{1}{3}\|\Lambda^{\alpha+1}h\|_{L^2}^2,\end{aligned}$$

where $h=u$ *or* $h=b$.

Proof. Set

$$\begin{aligned}\mathcal{M}=\int_{\mathbb{R}^3}\bigg(&-\partial_{22}h_1\Lambda^{2\alpha}h_1-\partial_{33}h_1\Lambda^{2\alpha}h_1-\partial_{11}h_2\Lambda^{2\alpha}h_2-\partial_{33}h_2\Lambda^{2\alpha}h_2\\&-\partial_{22}h_3\Lambda^{2\alpha}h_3-\partial_{11}h_3\Lambda^{2\alpha}h_3\bigg)\mathrm{d}x.\end{aligned}$$

Since $\int_{\mathbb{R}^3}-\partial_{22}h_1\Lambda^{2\alpha}h_1\mathrm{d}x=\int_{\mathbb{R}^3}\partial_{22}h_1\Delta\Lambda^{2\alpha-2}h_1\mathrm{d}x$, we express $\mathcal{M}$ as

$$\begin{aligned}\mathcal{M}=\int_{\mathbb{R}^3}&(\partial_{22}h_1\Delta\Lambda^{2\alpha-2}h_1+\partial_{33}h_1\Delta\Lambda^{2\alpha-2}h_1+\partial_{11}h_2\Delta\Lambda^{2\alpha-2}h_2+\partial_{33}h_2\Delta\Lambda^{2\alpha-2}h_2\\&+\partial_{22}h_3\Delta\Lambda^{2\alpha-2}h_3+\partial_{11}h_3\Delta\Lambda^{2\alpha-2}h_3)\mathrm{d}x\end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}\mathcal{M}=&\|\partial_2\Lambda^\alpha h_1\|_{L^2}^2+\|\partial_3\Lambda^\alpha h_1\|_{L^2}^2+\|\partial_1\Lambda^\alpha h_2\|_{L^2}^2+\|\partial_3\Lambda^\alpha h_2\|_{L^2}^2\\&+\|\partial_2\Lambda^\alpha h_3\|_{L^2}^2+\|\partial_1\Lambda^\alpha h_3\|_{L^2}^2\\=&\int_{\mathbb{R}^3}(\nabla\partial_2\Lambda^{\alpha-1}h_1\cdot\nabla\partial_2\Lambda^{\alpha-1}h_1+\nabla\partial_3\Lambda^{\alpha-1}h_1\cdot\nabla\partial_3\Lambda^{\alpha-1}h_1\\&+\nabla\partial_1\Lambda^{\alpha-1}h_2\cdot\nabla\partial_1\Lambda^{\alpha-1}h_2+\nabla\partial_3\Lambda^{\alpha-1}h_2\cdot\nabla\partial_3\Lambda^{\alpha-1}h_2\\&+\nabla\partial_2\Lambda^{\alpha-1}h_3\cdot\nabla\partial_2\Lambda^{\alpha-1}h_3+\nabla\partial_1\Lambda^{\alpha-1}h_3\cdot\nabla\partial_1\Lambda^{\alpha-1}h_3)dx\\=&\int_{\mathbb{R}^3}(\left|\partial_{12}\Lambda^{\alpha-1}h_1\right|^2+\left|\partial_{22}\Lambda^{\alpha-1}h_1\right|^2+\left|\partial_{32}\Lambda^{\alpha-1}h_1\right|^2+\left|\partial_{13}\Lambda^{\alpha-1}h_1\right|^2+\left|\partial_{23}\Lambda^{\alpha-1}h_1\right|^2\\&+\left|\partial_{33}\Lambda^{\alpha-1}h_1\right|^2+\left|\partial_{11}\Lambda^{\alpha-1}h_2\right|^2+\left|\partial_{21}\Lambda^{\alpha-1}h_2\right|^2+\left|\partial_{31}\Lambda^{\alpha-1}h_2\right|^2+\left|\partial_{13}\Lambda^{\alpha-1}h_2\right|^2\\&+\left|\partial_{23}\Lambda^{\alpha-1}h_2\right|^2+\left|\partial_{33}\Lambda^{\alpha-1}h_2\right|^2+\left|\partial_{11}\Lambda^{\alpha-1}h_3\right|^2+\left|\partial_{21}\Lambda^{\alpha-1}h_3\right|^2+\left|\partial_{31}\Lambda^{\alpha-1}h_3\right|^2\\&+\left|\partial_{12}\Lambda^{\alpha-1}h_3\right|^2+\left|\partial_{22}\Lambda^{\alpha-1}h_3\right|^2+\left|\partial_{32}\Lambda^{\alpha-1}h_3\right|^2)dx\end{aligned}$$

Using the incompressibility condition and the Cauchy inequality, we deduce

$$\begin{aligned}\mathcal{M}\geq&\int_{\mathbb{R}^3}\sum_{i,j,m=1}^{3}\left(\left|\partial_{ii}\Lambda^{\alpha-1}h_1\right|^2+\left|\partial_{jj}\Lambda^{\alpha-1}h_2\right|^2+\left|\partial_{mm}\Lambda^{\alpha-1}h_3\right|^2\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\\geq&\frac{1}{3}\int_{\mathbb{R}^3}\left(\left|\Delta\Lambda^{\alpha-1}h_1\right|^2+\left|\Delta\Lambda^{\alpha-1}h_2\right|^2+\left|\Delta\Lambda^{\alpha-1}h_3\right|^2\right)\mathrm{d}x\\=&\frac{1}{3}\left\|\Lambda^{\alpha+1}h\right\|_{L^2}^2.\end{aligned}$$

The proof is completed. □

3 Proof of Theorem 1.1

Since the uniqueness and local-in-time existence results are known (see, for example, the appendix of [11]), our main task in this section is to derive the a priori estimates.

Proposition 3.1. *Let the initial data* $(u_0,b_0,w_0)\in(L^2(\mathbb{R}^3))^3$. *Then, for any* $t>0$, *the solution to the three-dimensional magneto-micropolar fluid system (1.1) satisfies the following estimate*

$$\|(u,b,w)\|_{L^2}^2+\frac{2}{3}\mu\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\|\nabla b\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t\|\nabla w\|_{L^2}^2\mathrm{d}\tau\leq\|(u_0,b_0,w_0)\|_{L^2}^2.$$

Proof of Proposition 3.1. Taking the inner product of system (1.1) with (u,b,w), and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b,w)\|_{L^2}^2+4\chi\|w\|_{L^2}^2+\gamma\|\nabla w\|_{L^2}^2+(\mu+\chi)\big(\|\partial_2u_1\|_{L^2}^2+\|\partial_3u_1\|_{L^2}^2\\
&\quad+\|\partial_1u_2\|_{L^2}^2+\|\partial_3u_2\|_{L^2}^2+\|\partial_2u_3\|_{L^2}^2+\|\partial_3u_3\|_{L^2}^2\big)+\eta\big(\|\partial_2b_1\|_{L^2}^2+\|\partial_3b_1\|_{L^2}^2\\
&\quad+\|\partial_1b_2\|_{L^2}^2+\|\partial_3b_2\|_{L^2}^2+\|\partial_2b_3\|_{L^2}^2+\|\partial_3b_3\|_{L^2}^2\big)\\
&=\int(b\cdot\nabla b+2\chi\nabla\times w-u\cdot\nabla u)\cdot u\mathrm{d}x+\int(b\cdot\nabla u-u\cdot\nabla b)\cdot b\mathrm{d}x\\
&\quad+\int(2\chi\nabla\times u-u\cdot\nabla w)\cdot w\mathrm{d}x-\int\nabla p\cdot u.
\end{aligned}\tag{3.1}$$

By the incompressibility conditions $\nabla\cdot u=\nabla\cdot b=0$, we obtain the following identities

$$\begin{aligned}
&\int u\cdot\nabla u\cdot u\mathrm{d}x=0,\int u\cdot\nabla b\cdot b\mathrm{d}x=0,\int u\cdot\nabla w\cdot w\mathrm{d}x=0,\\
&\int\nabla p\cdot u\doteq 0,\int b\cdot\nabla b\cdot u\mathrm{d}x+\int b\cdot\nabla u\cdot b\mathrm{d}x=0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int 2\chi\nabla\times w\cdot u+2\chi\nabla\times u\cdot w\mathrm{d}x=\int 4\chi\nabla\times u\cdot w\mathrm{d}x\\
&\leq 4\chi\int((\partial_2u_3-\partial_3u_2)-(\partial_1u_3-\partial_3u_1)+(\partial_1u_2-\partial_2u_1))w\mathrm{d}x\\
&\leq\chi\big(\|\partial_2u_3\|_{L^2}^2+\|\partial_3u_2\|_{L^2}^2+\|\partial_1u_3\|_{L^2}^2+\|\partial_3u_1\|_{L^2}^2+\|\partial_1u_2\|_{L^2}^2\\
&\quad+\|\partial_2u_1\|_{L^2}^2\big)+4\chi\|w\|_{L^2}^2.
\end{aligned}$$

By combining the above estimates, Lemma 2.2, and (3.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b,w)\|_{L^2}^2+\frac{2}{3}\mu\|\nabla u\|_{L^2}^2+\frac{2}{3}\eta\|\nabla b\|_{L^2}^2+2\gamma\|\nabla w\|_{L^2}^2\leq 0.\tag{3.2}$$

Integrating both sides of (3.2) from 0 to $t(\forall t>0)$ gives

$$\|(u,b,w)\|_{L^2}^2+\frac{2}{3}\mu\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\|\nabla b\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t\|\nabla w\|_{L^2}^2\mathrm{d}\tau\leq C\delta^2.\tag{3.3}$$

□

Proposition 3.2. *Suppose the initial assumptions of Theorem1.1 hold, the system of equations (1.1) admits a global smooth solution* (u,b,w)*. For any* $t>0$*, the following inequality holds:*

$$\begin{aligned}&\|(u,b,w)(\tau)\|_{H^s}^2+\frac{2}{3}\mu\int_0^t\|\nabla u(\tau)\|_{H^s}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\|\nabla b(\tau)\|_{H^s}^2\mathrm{d}\tau+2\gamma\int_0^t\|\nabla w(\tau)\|_{H^s}^2\mathrm{d}\tau\\&\le C\|(u_0,b_0,w_0)\|_{H^s}^2+\int_0^t\|(u,b,w)\|_{H^s}\|(\nabla u,\nabla b,\nabla w)\|_{H^s}^2\mathrm{d}\tau,\end{aligned}$$

where $C(\mu,\eta,\gamma)$ is a constant.

Proof of Proposition 3.2. By applying the operator Λ^s to the first three equations of the system (1.1), and taking the L^2 inner product with $(\Lambda^s u,\Lambda^s b,\Lambda^s w)$, one obtains:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda^s u,\Lambda^s b,\Lambda^s w)\|_{L^2}^2+(\mu+\chi)\big(\|\partial_2\Lambda^s u_1\|_{L^2}^2+\|\partial_3\Lambda^s u_1\|_{L^2}^2+\|\partial_1\Lambda^s u_2\|_{L^2}^2+\|\partial_3\Lambda^s u_2\|_{L^2}^2\\&\quad+\|\partial_2\Lambda^s u_3\|_{L^2}^2+\|\partial_1\Lambda^s u_3\|_{L^2}^2\big)+\eta\big(\|\partial_2\Lambda^s b_1\|_{L^2}^2+\|\partial_3\Lambda^s b_1\|_{L^2}^2+\|\partial_1\Lambda^s b_2\|_{L^2}^2\\&\quad+\|\partial_3\Lambda^s b_2\|_{L^2}^2+\|\partial_2\Lambda^s b_3\|_{L^2}^2+\|\partial_1\Lambda^s b_3\|_{L^2}^2\big)+4\chi\|\Lambda^s w\|_{L^2}^2+\gamma\left\|\Lambda^{s+1}w\right\|_{L^2}^2\\&=\int[\Lambda^s,b\cdot\nabla]b\cdot\Lambda^s u\mathrm{d}x+\int 2\chi\nabla\times\Lambda^s w\cdot\Lambda^s u\mathrm{d}x-\int[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u\mathrm{d}x\\&\quad+\int[\Lambda^s,b\cdot\nabla]u\cdot\Lambda^s b\mathrm{d}x-\int[\Lambda^s,u\cdot\nabla]b\cdot\Lambda^s b\mathrm{d}x+\int 2\chi\nabla\times\Lambda^s u\cdot\Lambda^s w\mathrm{d}x\\&\quad-\int[\Lambda^s,u\cdot\nabla]w\cdot\Lambda^s w\mathrm{d}x:=\sum_{i=1}^{8}I_i.\end{aligned}\tag{3.4}$$

This utilizes the following fact

$$\begin{aligned}&\int\Lambda^s u\cdot\nabla\Lambda^s u\cdot\Lambda^s u\mathrm{d}x=0,\int\Lambda^s u\cdot\nabla\Lambda^s b\cdot\Lambda^s b\mathrm{d}x=0,\int\Lambda^s\nabla p\cdot\Lambda^s u\mathrm{d}x=0,\\&\int\Lambda^s b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x+\int\Lambda^s b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x=0.\end{aligned}$$

First, estimate I_1, by applying Lemma 2.1, we obtain

$$\begin{aligned}|I_1|=\left|\int[\Lambda^s,b\cdot\nabla]b\cdot\Lambda^s u\mathrm{d}x\right|&\le\|[\Lambda^s,b\cdot\nabla]b\|_{L^2}\left\|\Lambda^{s-1}\nabla u\right\|_{L^2}\\&\le C\|\nabla b\|_{L^\infty}\|\Lambda^s b\|_{L^2}\left\|\Lambda^{s-1}\nabla u\right\|_{L^2},\end{aligned}$$

Similarly,

$$\begin{aligned}&|I_3+I_4+I_5+I_7|\\&\le C(\|\nabla b\|_{L^\infty}\|\Lambda^s u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2})\left\|\Lambda^{s-1}\nabla b\right\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}\left\|\Lambda^{s-1}\nabla u\right\|_{L^2}\\&\quad+C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+\|\nabla w\|_{L^\infty}\|\Lambda^s u\|_{L^2})\left\|\Lambda^{s-1}\nabla w\right\|_{L^2}.\end{aligned}$$

Finally, we estimate the remaining two terms. Since $\int 2\chi\Lambda^s\nabla\times w\cdot\Lambda^s udx=\int 2\chi\nabla\times\Lambda^s u\cdot\Lambda^s wdx$, we then apply Young's inequality to obtain

$$\begin{aligned}|I_2+I_6|=&4\chi\left|\int\nabla\times\Lambda^s u\cdot\Lambda^s w\mathrm{d}x\right|\\ \le&\chi\big(\|\partial_2\Lambda^s u_3\|_{L^2}^2+\|\partial_3\Lambda^s u_2\|_{L^2}^2+\|\partial_1\Lambda^s u_3\|_{L^2}^2+\|\partial_3\Lambda^s u_1\|_{L^2}^2+\|\partial_1\Lambda^s u_2\|_{L^2}^2\\ &+\|\partial_2\Lambda^s u_1\|_{L^2}^2\big)+4\chi\|\Lambda^s w\|_{L^2}^2.\end{aligned}$$

Similar to the proof of Proposition 3.1, by combining the above estimates, applying $\|f\|_{L^\infty}\le C\|f\|_{H^s}(s>\frac{3}{2})$ and using Young's inequality, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda^s u,\Lambda^s b,\Lambda^s w)\|_{L^2}^2+\frac{2}{3}\mu\left\|\Lambda^{s+1}u\right\|_{L^2}^2+\frac{2}{3}\eta\left\|\Lambda^{s+1}b\right\|_{L^2}^2+2\gamma\left\|\Lambda^{s+1}w\right\|_{L^2}^2\\ \le&C\|\Lambda^s u\|_{L^2}\left(\|\nabla u\|_{H^s}^2+\|\nabla b\|_{H^s}^2+\|\nabla w\|_{H^s}^2\right)+C\|\Lambda^s b\|_{L^2}\|\nabla u\|_{H^s}\|\nabla b\|_{H^s}\\ &+C\|\Lambda^s w\|_{L^2}\|\nabla u\|_{H^s}\|\nabla w\|_{H^s}\\ \le&C(\|\Lambda^s u\|_{L^2}+\|\Lambda^s b\|_{L^2}+\|\Lambda^s w\|_{L^2})\left(\|\nabla u\|_{H^s}^2+\|\nabla b\|_{H^s}^2+\|\nabla w\|_{H^s}^2\right).\end{aligned}\tag{3.5}$$

Then, by adding (3.3) and (3.5), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b,w)\|_{H^s}^2+\frac{2}{3}\mu\|\nabla u\|_{H^s}^2+\frac{2}{3}\eta\|\nabla b\|_{H^s}^2+2\gamma\|\nabla w\|_{H^s}^2\\ \le&C(\|u\|_{H^s}+\|b\|_{H^s}+\|w\|_{H^s})\left(\|\nabla u\|_{H^s}^2+\|\nabla b\|_{H^s}^2+\|\nabla w\|_{H^s}^2\right).\end{aligned}\tag{3.6}$$

By integrating (3.6) with respect to time over the interval $[0,t]$, we obtain

$$\begin{aligned}&\|(u,b,w)\|_{H^s}^2+\frac{2}{3}\mu\int_0^t\|\nabla u\|_{H^s}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\|\nabla b\|_{H^s}^2\mathrm{d}\tau+2\gamma\int_0^t\|\nabla w\|_{H^s}^2\mathrm{d}\tau\\ \le&C\|(u_0,b_0,w_0)\|_{H^s}^2+C\int_0^t(\|(u,b,w)\|_{H^s})\left(\|(\nabla u,\nabla b,\nabla w)\|_{H^s}^2\right)\mathrm{d}\tau.\end{aligned}\tag{3.7}$$

Finally, by combining the assumption on the initial values, we obtain

$$E(t)\le C_0E(0)+C_1E^{\frac{3}{2}}(t).$$

Now, we begin the proof of the uniform boundedness of the a priori estimates. First, assume that $E(t)$ is bounded. For a constant $C_2>0$, we have

$$E(t)\le M:=\frac{1}{4{C_2}^2}.\tag{3.8}$$

Next, we prove that if we choose a sufficiently small $\delta>0$ such that

$$E(0)=\|(u_0,b_0,w_0)\|_{H^s}^2\le\delta^2\le\delta_0^2:=\frac{1}{16C_0C_2^2},\tag{3.9}$$

then $E(t)$ actually has an even smaller upper bound, i.e.,

$$E(t)\le\frac{M}{2}.$$

Furthermore, by the bootstrapping principle [15], it can be asserted that the solution to the system exists globally at any time.

By combining the a priori estimates and the upper bound condition (3.8), we obtain

$$E(t)\leq C_0E(0)+C_1E(t)^{\frac{1}{2}}E(t)\leq C_0E(0)+\frac{1}{2}E(t), \tag{3.10}$$

Substituting (3.9) into (3.10), we deduce that

$$E(t)\leq 2\delta^2\leq\frac{1}{8C_2^2}=\frac{1}{2}M.$$

At this point, by the bootstrapping principle, we conclude that for any $t>0$, we have

$$\begin{aligned}&\|(u,b,w)(\tau)\|_{H^s}^2+\frac{2}{3}\mu\int_0^t\|\nabla u(\tau)\|_{H^s}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\|\nabla b(\tau)\|_{H^s}^2\mathrm{d}\tau+2\gamma\int_0^t\|\nabla w(\tau)\|_{H^s}^2\mathrm{d}\tau\\&\leq C\delta^2.\end{aligned}$$

This means that the solution is globally bounded for all time $t>0$, as shown by (1.5).

This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

This section is primarily devoted to establishing the decay properties of the solution. By employing the interpolation inequality, it suffices to prove the decay rates stated in Theorem 1.2 for integer values of k. In view of the stability result in Theorem 1.1, it is no longer necessary to perform the basic L^2 estimates and higher-order estimates. However, for the decay estimates of $(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)$ in Proposition 4.1, we consider the a priori estimates involving negative derivatives, $E_1(t)$ and $E_2(t)$, as well as the energy estimate $E_3(t)$ obtained via the weighted energy method. To facilitate the closure of these estimates, we incorporate the weighted energy estimate $E_4(t)$ from Proposition 4.2 and define the total energy functional $\tilde{E}(t)=E_1(t)+E_2(t)+E_3(t)+E_4(t)$ (the definitions of $E_i(t)$, $i=1,2,3,4$, can be found in Remark1.2).

By combining the a priori estimates obtained in the previous two propositions (namely, equations (4.8), (4.14), (4.21), and (4.26)), we derive the following energy inequality:

$$\tilde{E}(t)\leq C\tilde{E}(0)+C\tilde{E}^{\frac{3}{2}}(t).$$

By employing a bootstrapping argument, we deduce that there exists a sufficiently small constant $\delta>0$ such that the initial energy satisfies $\tilde{E}(0)<\delta^2$, which implies that $\tilde{E}(t)$ remains uniformly bounded in time. We are now in a position to begin the proof of Theorem1.2.

Proposition 4.1. *Suppose the initial assumptions of Theorem1.2 hold. If there exists a sufficiently small constant* $\delta_1>0$ *such that*

$$\left\|(\Lambda^{-i}u_0,\Lambda^{-i}b_0,\Lambda^{-i}w_0)\right\|_{L^2}<\delta_1,$$

then the system(1.1) *admits a global smooth solution* $(u,b,w)\in C([0,\infty);L^2(\mathbb{R}^3))$, *which satisfies*

$$\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\le C(1+t)^{-\frac{1}{4}},$$

where $i=1,2$, *and* $C(\mu,\eta,\gamma)$ *is a constant.*

Proof of Proposition 4.1. **Step I:** Estimates for $(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)$

Similar to Proposition 3.2, multiplying the system (1.1) by $(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2+(\mu+\chi)\Big(\left\|\partial_2\Lambda^{-1}u_1\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}u_1\right\|_{L^2}^2\\
&\quad+\left\|\partial_1\Lambda^{-1}u_2\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^{-1}u_3\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-1}u_3\right\|_{L^2}^2\Big)\\
&\quad+\eta\Big(\left\|\partial_2\Lambda^{-1}b_1\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}b_1\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-1}b_2\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}b_2\right\|_{L^2}^2\\
&\quad+\left\|\partial_2\Lambda^{-1}b_3\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-1}b_3\right\|_{L^2}^2\Big)+4\chi\left\|\Lambda^{-1}w\right\|_{L^2}^2+\gamma\|w\|_{L^2}^2\\
&=\int b\cdot\nabla b\cdot\Lambda^{-2}u\mathrm{d}x+\int 2\chi\nabla\times w\cdot\Lambda^{-2}u\mathrm{d}x-\int u\cdot\nabla u\cdot\Lambda^{-2}u\mathrm{d}x\\
&\quad+\int b\cdot\nabla u\cdot\Lambda^{-2}b\mathrm{d}x-\int u\cdot\nabla b\cdot\Lambda^{-2}b\mathrm{d}x+\int 2\chi\nabla\times u\cdot\Lambda^{-2}w\mathrm{d}x\\
&\quad-\int u\cdot\nabla w\cdot\Lambda^{-2}w\mathrm{d}x-\int\nabla p\cdot\Lambda^{-2}u\mathrm{d}x=:\sum_{i=1}^{8}K_i.
\end{aligned}\tag{4.1}$$

First, by the incompressibility condition, we easily obtain $K_8=0$. For K_1, applying the Gagliardo-Nirenberg inequality and the Young's inequality, we obtain

$$\begin{aligned}
|K_1|&=\left|\int b\cdot\nabla b\cdot\Lambda^{-2}u\mathrm{d}x\right|=\left|\int\Lambda^{-1}\nabla(b\otimes b)\cdot\Lambda^{-1}u\mathrm{d}x\right|\\
&\le\|b\otimes b\|_{L^2}\left\|\Lambda^{-1}u\right\|_{L^2}\le C\left\|\Lambda^{-1}u\right\|_{L^2}\|b\|_{L^4}^2\\
&\le C\left\|\Lambda^{-1}u\right\|_{L^2}\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\\
&\le C\left\|\Lambda^{-1}u\right\|_{L^2}\left(\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right).
\end{aligned}\tag{4.2}$$

In a manner similar to the estimate for K_1, we obtain

$$|K_3|\leq C\left\|\Lambda^{-1}u\right\|_{L^2}\left(\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right). \tag{4.3}$$

Next, we estimate K_4, and derive the following bound

$$\begin{aligned}|K_4|=&\left|\int \Lambda^{-1}\nabla(b\otimes u)\cdot\Lambda^{-1}b\mathrm{d}x\right|\leq\|b\otimes u\|_{L^2}\left\|\Lambda^{-1}b\right\|_{L^2}\\ &\leq C\|b\|_{L^4}\|u\|_{L^4}\left\|\Lambda^{-1}b\right\|_{L^2}\leq C\|b\|_{L^2}^{\frac{1}{4}}\|\nabla b\|_{L^2}^{\frac{3}{4}}\|u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{3}{4}}\left\|\Lambda^{-1}b\right\|_{L^2}\\ &\leq C\left\|\Lambda^{-1}b\right\|_{L^2}\left(\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}+\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\right)\\ &\leq C\left\|\Lambda^{-1}b\right\|_{L^2}\left(\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right).\end{aligned} \tag{4.4}$$

Similarly,

$$\begin{aligned}|K_5+K_7|\leq& C\left(\left\|\Lambda^{-1}b\right\|_{L^2}+\left\|\Lambda^{-1}w\right\|_{L^2}\right)\\ &\times\left(\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|w\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right).\end{aligned} \tag{4.5}$$

Finally, we estimate K_2 and K_6.

$$\begin{aligned}|K_2+K_6|=&\left|4\chi\int\nabla\times\Lambda^{-1}u\cdot\Lambda^{-1}w\mathrm{d}x\right|\\ &\leq\chi\Big(\left\|\partial_2\Lambda^{-1}u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}u_2\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-1}u_3\right\|_{L^2}^2\\ &\quad+\left\|\partial_3\Lambda^{-1}u_1\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-1}u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^{-1}u_1\right\|_{L^2}^2\Big)+4\chi\left\|\Lambda^{-1}w\right\|_{L^2}^2.\end{aligned} \tag{4.6}$$

By substituting (4.2) through (4.6) into (4.1) and combining with Lemma 2.2, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2+\frac{2}{3}\mu\|u\|_{L^2}^2+\frac{2}{3}\eta\|b\|_{L^2}^2+2\gamma\|w\|_{L^2}^2\\ &\leq C\left(\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\right)\left(\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2+\|(u,b,w)\|_{L^2}^2\right).\end{aligned}$$

Integrating the above inequality over the time interval $[0,t]$, we obtain

$$\begin{aligned}&\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2+\frac{2}{3}\mu\int_0^t\|u\|_{L^2}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\|b\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t\|w\|_{L^2}^2\mathrm{d}\tau\\ &\leq C\left\|(\Lambda^{-1}u_0,\Lambda^{-1}b_0,\Lambda^{-1}w_0)\right\|_{L^2}^2+C\int_0^t\left(\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\right)\\ &\quad\times\left(\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2+\|(u,b,w)\|_{L^2}^2\right)\mathrm{d}\tau.\end{aligned} \tag{4.7}$$

Therefore,

$$E_1(t)\le C_3E_1(0)+C_4E_1^{\frac{3}{2}}(t)+C_5E_1^{\frac{1}{2}}(t)E(t), \tag{4.8}$$

where C_3, C_4, and C_5 are positive constants.

Step II: Estimates for $(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)$

To estimate $E_2(t)$, we apply the operator Λ^{-2} to system(1.1), multiply the result by $(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)$, and integrate over $\mathbb{R}^3$, obtaining the following

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)\right\|_{L^2}^2+(\mu+\chi)\bigg(\left\|\partial_2\Lambda^{-2}u_1\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-2}u_1\right\|_{L^2}^2\\
&\quad+\left\|\partial_1\Lambda^{-2}u_2\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-2}u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^{-2}u_3\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-2}u_3\right\|_{L^2}^2\bigg)\\
&\quad+\eta\bigg(\left\|\partial_2\Lambda^{-2}b_1\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-2}b_1\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-2}b_2\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-2}b_2\right\|_{L^2}^2\\
&\quad+\left\|\partial_2\Lambda^{-2}b_3\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-2}b_3\right\|_{L^2}^2\bigg)+4\chi\left\|\Lambda^{-2}w\right\|_{L^2}^2+\gamma\left\|\Lambda^{-1}w\right\|_{L^2}^2 \qquad (4.9)\\
&=\int\Lambda^{-2}(b\cdot\nabla b)\cdot\Lambda^{-2}u\mathrm{d}x+\int 2\chi\nabla\times\Lambda^{-2}w\cdot\Lambda^{-2}u\mathrm{d}x-\int\Lambda^{-2}\nabla p\cdot\Lambda^{-2}u\mathrm{d}x\\
&\quad+\int\Lambda^{-2}(b\cdot\nabla u)\cdot\Lambda^{-2}b\mathrm{d}x-\int\Lambda^{-2}(u\cdot\nabla b)\cdot\Lambda^{-2}b\mathrm{d}x-\int\Lambda^{-2}(u\cdot\nabla u)\cdot\Lambda^{-2}u\mathrm{d}x\\
&\quad+\int 2\chi\nabla\times\Lambda^{-2}u\cdot\Lambda^{-2}w\mathrm{d}x-\int\Lambda^{-2}(u\cdot\nabla w)\cdot\Lambda^{-2}w\mathrm{d}x=:\sum_{i=1}^{8}N_i.
\end{aligned}$$

First, it is straightforward to verify that $N_3=0$. We now proceed to estimate N_1. By employing the Hardy–Littlewood–Sobolev inequality, the interpolation inequality, the Gagliardo–Nirenberg inequality, Hölder's inequality, and Young's inequality, we derive the following estimate

$$\begin{aligned}
|N_1|&=\left|\int\Lambda^{-2}(b\cdot\nabla b)\cdot\Lambda^{-2}u\mathrm{d}x\right|=\left|\int\Lambda^{-2}\nabla(b\otimes b)\cdot\Lambda^{-2}u\mathrm{d}x\right|\\
&\le\left\|\Lambda^{-1}(b\otimes b)\right\|_{L^2}\left\|\Lambda^{-2}u\right\|_{L^2}\le C\left\|\Lambda^{-2}u\right\|_{L^2}\|b\otimes b\|_{L^{\frac{6}{5}}}\\
&\le C\left\|\Lambda^{-2}u\right\|_{L^2}\|b\otimes b\|_{L^1}^{\frac{2}{3}}\|b\otimes b\|_{L^2}^{\frac{1}{3}}\\
&\le C\left\|\Lambda^{-2}u\right\|_{L^2}(\|b\otimes b\|_{L^1}+\|b\otimes b\|_{L^2})\\
&\le C\left\|\Lambda^{-2}u\right\|_{L^2}\left(\|b\|_{L^2}^2+\|b\|_{L^4}^2\right)\\
&\le C\left\|\Lambda^{-2}u\right\|_{L^2}\left(\|b\|_{L^2}^2+\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\right)\\
&\le C\left\|\Lambda^{-2}u\right\|_{L^2}\left(\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right).
\end{aligned} \tag{4.10}$$

Similarly,

$$
\begin{aligned}
|N_4+N_5+N_6+N_8| \le\, & C\left(\left\|(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)\right\|_{L^2}\right)\\
& \times\left(\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|w\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right).
\end{aligned}
\tag{4.11}
$$

Finally, we estimate N_2 and N_7, and derive

$$
\begin{aligned}
|N_2+N_7| \le \chi\bigg(& \left\|\partial_2\Lambda^{-2}u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-2}u_2\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-2}u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-2}u_1\right\|_{L^2}^2\\
& +\left\|\partial_1\Lambda^{-2}u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^{-2}u_1\right\|_{L^2}^2\bigg)+4\chi\left\|\Lambda^{-2}w\right\|_{L^2}^2.
\end{aligned}
\tag{4.12}
$$

Combining (4.9)-(4.12), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)\right\|_{L^2}^2+\frac{2}{3}\mu\left\|\Lambda^{-1}u\right\|_{L^2}^2+\frac{2}{3}\eta\left\|\Lambda^{-1}b\right\|_{L^2}^2+2\gamma\left\|\Lambda^{-1}w\right\|_{L^2}^2\\
&\quad\le C\left(\left\|(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)\right\|_{L^2}\right)\left(\|(u,b,w)\|_{L^2}^2+\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2\right).
\end{aligned}
$$

Integrating the above inequality over the time interval $[0,t]$, we obtain

$$
\begin{aligned}
&\left\|(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)\right\|_{L^2}^2+\frac{2}{3}\mu\int_0^t\left\|\Lambda^{-1}u\right\|_{L^2}^2\mathrm{d}\tau+\frac{2}{3}\eta\int_0^t\left\|\Lambda^{-1}b\right\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t\left\|\Lambda^{-1}w\right\|_{L^2}^2\mathrm{d}\tau\\
&\le\left\|(\Lambda^{-2}u_0,\Lambda^{-2}b_0,\Lambda^{-2}w_0)\right\|_{L^2}^2\\
&\quad+C\int_0^t\left\|(\Lambda^{-2}u,\Lambda^{-2}b,\Lambda^{-2}w)\right\|_{L^2}\left(\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2+\|(u,b,w)\|_{L^2}^2\right)\mathrm{d}\tau.
\end{aligned}
\tag{4.13}
$$

Further, we obtain

$$
E_2(t)\le C_6E_2(0)+C_7E_2^{\frac{1}{2}}(t)E_1(t)+C_8E_2^{\frac{1}{2}}(t)E(t),
\tag{4.14}
$$

where C_6, C_7, and C_8 are positive constants.

Step III: Weighted energy estimate for $(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)$

Applying the operator Λ^{-1} to system(1.1), taking the L^2 inner product over $\mathbb{R}^3$ with $(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)$, and multiplying the resulting expression by the time weight

$(1+t)^{1/2}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2+(\mu+\chi)(1+t)^{\frac{1}{2}}\Big(\|\partial_2\Lambda^{-1}u_1\|_{L^2}^2+\|\partial_3\Lambda^{-1}u_1\|_{L^2}^2\\&\quad+\|\partial_1\Lambda^{-1}u_2\|_{L^2}^2+\|\partial_3\Lambda^{-1}u_2\|_{L^2}^2+\|\partial_2\Lambda^{-1}u_3\|_{L^2}^2+\|\partial_1\Lambda^{-1}u_3\|_{L^2}^2\Big)\\&\quad+\eta(1+t)^{\frac{1}{2}}\Big(\|\partial_2\Lambda^{-1}b_1\|_{L^2}^2+\|\partial_3\Lambda^{-1}b_1\|_{L^2}^2\\&\quad+\|\partial_1\Lambda^{-1}b_2\|_{L^2}^2+\|\partial_3\Lambda^{-1}b_2\|_{L^2}^2+\|\partial_2\Lambda^{-1}b_3\|_{L^2}^2+\|\partial_1\Lambda^{-1}b_3\|_{L^2}^2\Big)\\&\quad+4\chi(1+t)^{\frac{1}{2}}\|\Lambda^{-1}w\|_{L^2}^2+\gamma(1+t)^{\frac{1}{2}}\|w\|_{L^2}^2=\sum_{i=1}^{9}J_i.\end{aligned}\tag{4.15}$$

Here,

$$\begin{aligned}&J_1=\frac{1}{4}(1+t)^{-\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2, J_2=(1+t)^{\frac{1}{2}}\int\Lambda^{-1}(b\cdot\nabla b)\cdot\Lambda^{-1}u\mathrm{d}x,\\&J_3=-(1+t)^{\frac{1}{2}}\int\Lambda^{-1}(u\cdot\nabla u)\cdot\Lambda^{-1}u\mathrm{d}x, J_4=-(1+t)^{\frac{1}{2}}\int\nabla\Lambda^{-1}p\cdot\Lambda^{-1}u\mathrm{d}x,\\&J_5=(1+t)^{\frac{1}{2}}\int2\chi\nabla\times\Lambda^{-1}w\cdot\Lambda^{-1}u\mathrm{d}x, J_6=-(1+t)^{\frac{1}{2}}\int\Lambda^{-1}(u\cdot\nabla b)\cdot\Lambda^{-1}b\mathrm{d}x,\\&J_7=(1+t)^{\frac{1}{2}}\int\Lambda^{-1}(b\cdot\nabla u)\cdot\Lambda^{-1}b\mathrm{d}x, J_8=(1+t)^{\frac{1}{2}}\int2\chi\nabla\times\Lambda^{-1}u\cdot\Lambda^{-1}w\mathrm{d}x,\\&J_9=-(1+t)^{\frac{1}{2}}\int\Lambda^{-1}(u\cdot\nabla w)\cdot\Lambda^{-1}w\mathrm{d}x.\end{aligned}$$

It is straightforward to verify that $J_4=0$. By integrating J_1 in time over the interval $[0,t]$, we obtain

$$\int_0^t J_1\mathrm{d}\tau\le\frac{1}{4}\int_0^t\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2\mathrm{d}\tau\le\frac{1}{4}E_2(t).\tag{4.16}$$

We now estimate J_5 and J_8. By applying Young's inequality, we obtain

$$\begin{aligned}|J_5+J_8|\le{}&\chi(1+t)^{\frac{1}{2}}\Big(\left\|\partial_2\Lambda^{-1}u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}u_2\right\|_{L^2}^2+\left\|\partial_1\Lambda^{-1}u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^{-1}u_1\right\|_{L^2}^2\\&+\left\|\partial_1\Lambda^{-1}u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^{-1}u_1\right\|_{L^2}^2\Big)+4\chi(1+t)^{\frac{1}{2}}\left\|\Lambda^{-1}w\right\|_{L^2}^2.\end{aligned}\tag{4.17}$$

Following a similar procedure as in (4.2), we obtain

$$|J_2+J_3+J_6+J_7+J_9|\le C(1+t)^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^{\frac{3}{2}}\|(u,b,w)\|_{L^2}^{\frac{1}{2}}.\tag{4.18}$$

By integrating (4.18) in time over the interval $[0,t]$, we obtain

$$\begin{aligned}
&\int_0^t |J_2+J_3+J_6+J_7+J_9|\mathrm{d}\tau\\
\le& C\int_0^t \left((1+\tau)^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^{\frac{3}{2}}\ \|(u,b,w)\|_{L^2}^{\frac{1}{2}}\right)\mathrm{d}\tau\\
\le& C\sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{8}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\int_0^t\left((1+\tau)^{\frac{1}{8}}\|(u,b,w)\|_{L^2}^{\frac{1}{2}}\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^{\frac{3}{2}}\right)\mathrm{d}\tau\\
\le& C\sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{8}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\int_0^t\left((1+\tau)^{\frac{1}{2}}\|(u,b,w)\|_{L^2}^2+\|(\nabla u,\nabla b,\nabla w)\|_{L^2}^2\right)\mathrm{d}\tau\\
\le& C\left(E_3^{\frac{3}{2}}(t)+E_3^{\frac{1}{2}}(t)E(t)\right).
\end{aligned}\tag{4.19}$$

On the other hand, combining (4.15), (4.17), and (4.18), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2+\frac{2}{3}\mu(1+t)^{\frac{1}{2}}\|u\|_{L^2}^2+\frac{2}{3}\eta(1+t)^{\frac{1}{2}}\|b\|_{L^2}^2+2\gamma(1+t)^{\frac{1}{2}}\|w\|_{L^2}^2\\
&\quad\le C(1+t)^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\|(u,b,w)\|_{L^2}^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^{\frac{3}{2}}\\
&\qquad+\frac{1}{4}(1+t)^{-\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}^2.
\end{aligned}\tag{4.20}$$

Finally, integrating (4.20) over the interval $[0,t]$ and combining it with (4.16) and (4.19), we obtain

$$\begin{aligned}
&\sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{2}}\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)(\tau)\right\|_{L^2}^2+\frac{2}{3}\mu\int_0^t(1+\tau)^{\frac{1}{2}}\|u(\tau)\|_{L^2}^2\mathrm{d}\tau\\
&\quad+\frac{2}{3}\eta\int_0^t(1+\tau)^{\frac{1}{2}}\|b(\tau)\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t(1+\tau)^{\frac{1}{2}}\|w(\tau)\|_{L^2}^2\mathrm{d}\tau\\
&\le CE_3(0)+C\left(E_3^{\frac{3}{2}}(t)+E_3^{\frac{1}{2}}(t)E(t)\right)+\frac{1}{4}E_2(t).
\end{aligned}$$

which yields

$$E_3(t)\le CE_3(0)+C\left(E_3^{\frac{3}{2}}(t)+E_3^{\frac{1}{2}}(t)E(t)\right)+\frac{1}{4}E_2(t),\tag{4.21}$$

At this stage, utilizing the uniform boundedness of the energy functional $\tilde{E}(t)$, we derive

$$\left\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\right\|_{L^2}\le C(1+t)^{-\frac{1}{4}}.$$

□

Proposition 4.2. *Suppose the initial assumptions of Theorem1.2 hold. If there exists a sufficiently small constant* $\delta_2>0$ *such that*

$$\|(u_0,b_0,w_0)\|_{L^2}<\delta_2,$$

then the system (1.1) *has a global smooth solution* $(u,b,w)\in C([0,\infty);L^2(\mathbb{R}^3))$ *that satisfies*

$$\|(u,b,w)\|_{L^2}\le C(1+t)^{-\frac{3}{4}},$$

where $C=C(\mu,\eta,\gamma)$ *is a constant.*

Proof of Proposition 4.2. The equation (1.1) is inner-multiplied in L^2 on $\mathbb{R}^3$ and then multiplied by the time weight $(1+t)^{\frac{3}{2}}$, which gives the result:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{\frac{3}{2}}\|(u,b,w)\|_{L^2}^2+(\mu+\chi)(1+t)^{\frac{1}{2}}\big(\|\partial_2u_1\|_{L^2}^2+\|\partial_3u_1\|_{L^2}^2+\|\partial_1u_2\|_{L^2}^2\\&\quad+\|\partial_3u_2\|_{L^2}^2+\|\partial_2u_3\|_{L^2}^2+\|\partial_1u_3\|_{L^2}^2\big)+\eta(1+t)^{\frac{1}{2}}\big(\|\partial_2b_1\|_{L^2}^2+\|\partial_3b_1\|_{L^2}^2+\|\partial_1b_2\|_{L^2}^2\\&\quad+\|\partial_3b_2\|_{L^2}^2+\|\partial_2b_3\|_{L^2}^2+\|\partial_1b_3\|_{L^2}^2\big)+4\chi(1+t)^{\frac{3}{2}}\|w\|_{L^2}^2+\gamma(1+t)^{\frac{3}{2}}\|\nabla w\|_{L^2}^2\\&=\frac{3}{4}(1+t)^{\frac{1}{2}}\|(u,b,w)\|_{L^2}^2+(1+t)^{\frac{3}{2}}\int b\cdot\nabla b\cdot u\mathrm{d}x-(1+t)^{\frac{3}{2}}\int u\cdot\nabla u\cdot u\mathrm{d}x\\&\quad-(1+t)^{\frac{3}{2}}\int\nabla p\cdot u\mathrm{d}x+(1+t)^{\frac{3}{2}}\int2\chi\nabla\times w\cdot u\mathrm{d}x+(1+t)^{\frac{3}{2}}\int b\cdot\nabla u\cdot b\mathrm{d}x\\&\quad-(1+t)^{\frac{3}{2}}\int u\cdot\nabla b\cdot b\mathrm{d}x+(1+t)^{\frac{3}{2}}\int2\chi\nabla\times u\cdot w\mathrm{d}x-(1+t)^{\frac{3}{2}}\int u\cdot\nabla w\cdot w\mathrm{d}x=\sum_{i=1}^{9}L_i.\end{aligned}\tag{4.22}$$

Note that, due to the incompressibility condition $\nabla\cdot u=\nabla\cdot b=0$, we obtain

$$\begin{aligned}&L_2+L_6=(1+t)^{\frac{3}{2}}\int b\cdot\nabla b\cdot u\mathrm{d}x+(1+t)^{\frac{3}{2}}\int b\cdot\nabla u\cdot b\mathrm{d}x=0;\\&L_3=-(1+t)^{\frac{3}{2}}\int u\cdot\nabla u\cdot u\mathrm{d}x=0;L_4=-(1+t)^{\frac{3}{2}}\int\nabla p\cdot u\mathrm{d}x=0;\\&L_7=-(1+t)^{\frac{3}{2}}\int u\cdot\nabla b\cdot b\mathrm{d}x=0;L_9=-(1+t)^{\frac{3}{2}}\int u\cdot\nabla w\cdot w\mathrm{d}x=0.\end{aligned}\tag{4.23}$$

For L_5 and L_8, by using the identity $\int\nabla\times w\cdot u,dx=\int\nabla\times u\cdot w,dx$ and applying Young's inequality, we obtain

$$\begin{aligned}L_5+L_8\le{}&\chi(1+t)^{\frac{3}{2}}\big(\|\partial_2u_3\|_{L^2}^2+\|\partial_3u_2\|_{L^2}^2+\|\partial_1u_3\|_{L^2}^2\\&+\|\partial_3u_1\|_{L^2}^2+\|\partial_1u_2\|_{L^2}^2+\|\partial_2u_1\|_{L^2}^2\big)+4\chi(1+t)^{\frac{3}{2}}\|w\|_{L^2}^2.\end{aligned}\tag{4.24}$$

For L_1, integrating with respect to time over $[0,t]$, we obtain

$$\int_0^tL_1\mathrm{d}\tau=\frac{3}{4}\int_0^t(1+\tau)^{\frac{1}{2}}\|(u,b,w)\|_{L^2}^2d\tau\le\frac{3}{4}E_3(t).$$

Moreover, combining (4.22) with (4.23) and (4.24), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{\frac{3}{2}}\|(u,b,w)\|_{L^2}^2+\frac{2}{3}\mu(1+t)^{\frac{3}{2}}\|\nabla u\|_{L^2}^2+\frac{2}{3}\eta(1+t)^{\frac{3}{2}}\|\nabla b\|_{L^2}^2+2\gamma(1+t)^{\frac{3}{2}}\|\nabla w\|_{L^2}^2\\&\quad\le C(1+t)^{\frac{1}{2}}\|(u,b,w)\|_{L^2}^2.\end{aligned}\tag{4.25}$$

Integrating (4.25) over the interval $[0,t]$, we obtain

$$E_4(t)\le CE_4(0)+\frac{3}{4}E_3(t).\tag{4.26}$$

Using the uniform boundedness of $\tilde{E}(t)$ and the initial bound $E_4(0)$, we conclude that $E_4(t)$ remains uniformly bounded for all $t>0$. As a result, we further obtain

$$\|(u,b,w)\|_{L^2}\le C(1+t)^{-\frac{3}{4}}.$$

□

Proposition 4.3. *Under the initial assumptions of Theorem1.2, if there exists a sufficiently small constant* $\delta_3>0$ *such that*

$$\left\|(\Lambda^{s^*}u_0,\Lambda^{s^*}b_0,\Lambda^{s^*}w_0)\right\|_{L^2}<\delta_3,$$

then the system (1.1) *admits a global smooth solution* $(u,b,w)\in C([0,\infty);H^{s^*}(\mathbb{R}^3))$ *satisfying*

$$\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}\le C(1+t)^{-\frac{k}{2}-\frac{3}{4}},$$

where $C=C(\mu,\eta,\gamma)$ *is a constant and* $k\in\mathbb{R}$ *with* $k\ge -1$.

Proof of Proposition 4.3. We now proceed by mathematical induction to prove it. First, Propositions 4.1 and 4.2 have established the decay rates for $\|(\Lambda^{-1}u,\Lambda^{-1}b,\Lambda^{-1}w)\|_{L^2}$ and $\|(u,b,w)\|_{L^2}$. Assume that the solution (u,b,w) has derivatives up to order $k-1$, and suppose that the following statement holds:

$$\begin{aligned}&(1+t)^{(k-1)+\frac{3}{2}}\left\|\left(\Lambda^{k-1}u,\Lambda^{k-1}b,\Lambda^{k-1}w\right)\right\|_{L^2}^2+\frac{2}{3}\mu\int_0^t(1+\tau)^{(k-1)+\frac{3}{2}}\left\|\Lambda^k u\right\|_{L^2}^2\mathrm{d}\tau\\&\quad+\frac{2}{3}\eta\int_0^t(1+\tau)^{(k-1)+\frac{3}{2}}\left\|\Lambda^k b\right\|_{L^2}^2\mathrm{d}\tau+2\gamma\int_0^t(1+\tau)^{(k-1)+\frac{3}{2}}\left\|\Lambda^k w\right\|_{L^2}^2\mathrm{d}\tau\le C\delta_3^2.\end{aligned}\tag{4.27}$$

To prove that the estimate holds when the solution (u,b,w) possesses derivatives up to order k, namely,

$$\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}\le C(1+t)^{-\frac{k}{2}-\frac{3}{4}},$$

we apply the operator Λ^k to the system (1.1), multiply the resulting equations by $(\Lambda^k u,\Lambda^k b,\Lambda^k w)$ respectively, take the L^2 inner product over $\mathbb{R}^3$, and then multiply by the time weight $(1+t)^{k+\frac{3}{2}}$. This yields:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k(u,b,w)\right\|_{L^2}^2+(\mu+\chi)(1+t)^{k+\frac{3}{2}}\Big(\left\|\partial_2\Lambda^k u_1\right\|_{L^2}^2+\left\|\partial_3\Lambda^k u_1\right\|_{L^2}^2+\left\|\partial_1\Lambda^k u_2\right\|_{L^2}^2\\
&\quad+\left\|\partial_3\Lambda^k u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^k u_3\right\|_{L^2}^2+\left\|\partial_1\Lambda^k u_3\right\|_{L^2}^2\Big)+\eta(1+t)^{k+\frac{3}{2}}\Big(\left\|\partial_2\Lambda^k b_1\right\|_{L^2}^2\\
&\quad+\left\|\partial_3\Lambda^k b_1\right\|_{L^2}^2+\left\|\partial_1\Lambda^k b_2\right\|_{L^2}^2+\left\|\partial_3\Lambda^k b_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^k b_3\right\|_{L^2}^2+\left\|\partial_1\Lambda^k b_3\right\|_{L^2}^2\Big)\\
&\quad+\gamma(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}w\right\|_{L^2}^2+4\chi(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k w\right\|_{L^2}^2=:\sum_{i=1}^{9}M_i,
\end{aligned}\tag{4.28}$$

where

$$\begin{aligned}
&M_1=\frac{1}{2}\left(k+\frac{3}{2}\right)(1+t)^{k+\frac{1}{2}}\left\|\Lambda^k(u,b,w)\right\|_{L^2}^2, M_2=(1+t)^{k+\frac{3}{2}}\int\Lambda^k(b\cdot\nabla b)\cdot\Lambda^k u\mathrm{d}x,\\
&M_3=-(1+t)^{k+\frac{3}{2}}\int\Lambda^k(u\cdot\nabla u)\cdot\Lambda^k u\mathrm{d}x, M_4=-(1+t)^{k+\frac{3}{2}}\int\nabla p\cdot\Lambda^{2k}u\mathrm{d}x,\\
&M_5=(1+t)^{k+\frac{3}{2}}\int 2\chi\nabla\times w\cdot\Lambda^{2k}u\mathrm{d}x, M_6=(1+t)^{k+\frac{3}{2}}\int\Lambda^k(b\cdot\nabla u)\cdot\Lambda^k b\mathrm{d}x,\\
&M_7=-(1+t)^{k+\frac{3}{2}}\int\Lambda^k(u\cdot\nabla b)\cdot\Lambda^k b\mathrm{d}x, M_8=-(1+t)^{k+\frac{3}{2}}\int\Lambda^k(u\cdot\nabla w)\cdot\Lambda^k w\mathrm{d}x,\\
&M_9=(1+t)^{k+\frac{3}{2}}\int 2\chi\nabla\times u\cdot\Lambda^{2k}w\mathrm{d}x.
\end{aligned}$$

Integrating M_1 over the time interval $[0,t]$ and applying the induction hypothesis (4.27), we obtain

$$\int_0^t M_1\mathrm{d}\tau\le C\int_0^t\left((1+\tau)^{k+\frac{1}{2}}\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}^2\right)\mathrm{d}\tau\le C.$$

By the incompressibility condition, it follows directly that $M_4=0$. For the estimates of M_5 and M_9, by applying Young's inequality, we obtain

$$\begin{aligned}
|M_5+M_9|&=(1+t)^{k+\frac{3}{2}}\left|\int 2\chi\Lambda^k\nabla\times w\cdot\Lambda^k u+2\chi\nabla\times\Lambda^k u\cdot\Lambda^k w\mathrm{d}x\right|\\
&\le\chi(1+t)^{k+\frac{3}{2}}\Big(\left\|\partial_2\Lambda^k u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^k u_2\right\|_{L^2}^2+\left\|\partial_1\Lambda^k u_3\right\|_{L^2}^2+\left\|\partial_3\Lambda^k u_1\right\|_{L^2}^2\\
&\quad+\left\|\partial_1\Lambda^k u_2\right\|_{L^2}^2+\left\|\partial_2\Lambda^k u_1\right\|_{L^2}^2\Big)+4\chi(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k w\right\|_{L^2}^2.
\end{aligned}\tag{4.29}$$

For M_2, by applying the Gagliardo–Nirenberg inequality, Hölder's inequality, and Young's inequality, we obtain

$$
\begin{aligned}
|M_2| &= \left|(1+t)^{k+\frac{3}{2}}\int \Lambda^k(b\cdot\nabla b)\cdot\Lambda^k u\,\mathrm{d}x\right| = \left|(1+t)^{k+\frac{3}{2}}\int \Lambda^k(b\otimes b)\cdot\Lambda^k\nabla u\,\mathrm{d}x\right| \\
&\le (1+t)^{k+\frac{3}{2}}\left\|\Lambda^k(b\otimes b)\right\|_{L^2}\left\|\Lambda^{k+1}u\right\|_{L^2} \\
&\le \frac{\mu}{12}(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}u\right\|_{L^2}^2 + C(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k b\right\|_{L^4}^2\|b\|_{L^4}^2 \\
&\le \frac{\mu}{12}(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}u\right\|_{L^2}^2 + C(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}b\right\|_{L^2}^{\frac{3}{2}}\left\|\Lambda^k b\right\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}} \\
&\le (1+t)^{k+\frac{3}{2}}\left(\frac{\mu}{12}\left\|\Lambda^{k+1}u\right\|_{L^2}^2 + \frac{\eta}{12}\left\|\Lambda^{k+1}b\right\|_{L^2}^2\right) + C(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k b\right\|_{L^2}^2\|b\|_{L^2}^2.
\end{aligned}
\tag{4.30}
$$

Similar to M_2, we can estimate M_3, M_6, M_7, and M_8 together as follows

$$
\begin{aligned}
&|M_3+M_6+M_7+M_8| \\
&\le C(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k u\right\|_{L^2}^2\left(\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|w\|_{L^2}^2\right) + C(1+t)^{k+\frac{3}{2}}\left\|\Lambda^k b\right\|_{L^2}^2\|u\|_{L^2}^2 \\
&\quad + \frac{\mu}{12}(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}u\right\|_{L^2}^2 + \frac{\eta}{12}(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}b\right\|_{L^2}^2 + \frac{\gamma}{2}(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}w\right\|_{L^2}^2.
\end{aligned}
\tag{4.31}
$$

By applying Lemma2.2 and combining (4.28), (4.29), (4.30), and (4.31), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{k+\frac{3}{2}}\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}^2 + \frac{1}{3}\mu(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}u\right\|_{L^2}^2 + \frac{1}{3}\eta(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}b\right\|_{L^2}^2 \\
&\quad + \gamma(1+t)^{k+\frac{3}{2}}\left\|\Lambda^{k+1}w\right\|_{L^2}^2 \\
&\le C(1+t)^{k+\frac{3}{2}}\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}^2\|(u,b,w)\|_{L^2}^2 + \left(k+\frac{3}{2}\right)(1+t)^{k+\frac{1}{2}}\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}^2.
\end{aligned}
$$

Based on Theorem1.1, the uniform boundedness of the energy functional $\tilde{E}(t)$, and the initial value assumptions, together with Gronwall's inequality, we obtain

$$
\begin{aligned}
&(1+t)^{k+\frac{3}{2}}\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}^2 \\
\le\, & C\bigg(\left\|(\Lambda^k u(0),\Lambda^k b(0),\Lambda^k u(0))\right\|_{L^2}^2 \\
&+\left(k+\frac{3}{2}\right)\int_0^t (1+\tau)^{k+\frac{1}{2}}\left\|(\Lambda^k u(\tau),\Lambda^k b(\tau),\Lambda^k w(\tau))\right\|_{L^2}^2\mathrm{d}\tau\bigg) \\
&\times\exp\left\{C\int_0^t\|(u(\tau),b(\tau),w(\tau))\|_{L^2}^2\mathrm{d}\tau\right\} \\
\le\, & C\exp\{CE_1(t)\}(\delta_3^2+C)\le C.
\end{aligned}
$$

Furthermore, we obtain

$$\left\|(\Lambda^k u,\Lambda^k b,\Lambda^k w)\right\|_{L^2}\leq C(1+t)^{-\frac{k}{2}-\frac{3}{4}},$$

Thus, by mathematical induction, we conclude that for any $k\in\mathbb{R}$ and $k\geq -1$, the above decay estimates hold. Therefore, the proof of Theorem1.2 is complete. □

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