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Oriented Diameter and Radius of Triangulations

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Abstract. Denote $\overrightarrow{diam}(G)$ ($\overrightarrow{rad}(G)$) as the minimum directed diameter (radius) among all orientations of the bridgeless graph G. Denote G as a triangulation with n vertices. Mondal, Parthiban and Rajasingh proved that $\overrightarrow{diam}(G) \le \frac{n}{2} + O(\sqrt{n})$. Ge, Liu and Wang improved it to $\frac{n}{2}$. In this paper, we first prove that $\overrightarrow{rad}(G) \le 2r$, where r is the radius of G. We also prove that for s-connected triangulation G, there exists an integer c such that $\overrightarrow{rad}(G) \le \frac{n}{s} + c$. As a corollary, we improve the upper bound of the oriented diameter of 5-connected triangulations to $\frac{2n}{5} + c$. Furthermore, we prove that under some connecting condition of the triangulation G, $\overrightarrow{rad}(G) \le r+1$, where r is the radius of G. Then for s-connected triangulation G under some connecting condition, there exists an integer c such that $\overrightarrow{rad}(G) \le \frac{n}{2s} + c$ and $\overrightarrow{diam}(G) \le \frac{n}{s} + c$, which are tight apart from a constant.

AMS subject classifications: 05C12,05C20

Key words: Oriented radius, Oriented diameter, Triangulation.

1 Introduction

In this paper, G is a simple undirected and connected graph. We define the *eccentricity* of any $x \in V(G)$, denote as $ex(x)$, is the maximum distance $d(y,x)$ for all $y \in V(G) - v$. Define the *diameter* (*radius*) of G, as the maximum (minimum) $ex(x)$ for all $x \in V(G)$. If $ex(x) = rad(G)$, we call x the *center* of G.

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We say a directed graph D is *strong* if for any two ordered vertex-pair (x,y), there exists a directed path from x to y. We use $\overrightarrow{G}$ to denote some strong orientation of G. Let $\theta(x,y)$ be the maximum value of directed distances from x to y, and from y to x in $\overrightarrow{G}$. We define the *eccentricity* of any $x\in V(\overrightarrow{G})$, denote as $ex(x)$, is the maximum distance $\theta(y,x)$ for all $y\in V(\overrightarrow{G})-v$. Define the *directed diameter* (*directed radius*) of $\overrightarrow{G}$, as the maximum (minimum) $ex(x)$ for all $x\in V(\overrightarrow{G})$. If $ex(x)=rad(\overrightarrow{G})$, we call x the *center* of $\overrightarrow{G}$.

Now we define the *oriented diameter* of G as the minimum directed diameter among all strong orientations $\overrightarrow{G}$ of G, and the *oriented radius* of G as the minimum directed radius among all strong orientations $\overrightarrow{G}$ of G.

In 1939, Robbins [16] proved the famous theorem that any connected graph G has a strong orientation if and only if G contains no cut-edge. Note that a cut-edge of G is an edge that $G-e$ is disconnected. Hence, by Robbins' theorem, we need only consider $\overrightarrow{diam}(G)$ ($\overrightarrow{rad}(G)$) for connected graphs with no cut edge. In 1978, Chvátal and Thomassen [5] proved that determining the exact value of oriented diameter (oriented radius) of any graph is NP-hard. Then any insight into the oriented diameter (oriented radius) is of interest.

Let $f(d)$ be the smallest value for which every bridgeless graph G with diameter d admits a strong orientation $\overrightarrow{G}$ such that $diam(\overrightarrow{G})\leq f(d)$. Chvátal and Thomassen [5] proved that $\frac{1}{2}d^2+d\leq f(d)\leq 2d^2+2d$. Recently, Babu et al. [2] improved the upper bound to $1.37d^2+6.97d$ which is smaller than $2d^2+2d$ when $d\geq 8$. For small diameter, Chvátal and Thomassen [5] proved that $f(2)=6$. Kwok et al. [12] proved that $9\leq f(3)\leq 11$. Wang and Chen [17] determined that $f(3)=9$. Babu et al. [2] proved that $f(4)\leq 21$.

Until now, the tight bound of $\overrightarrow{diam}(G)$ with given diameter is still not know. Furthermore, people tried to investigate tight upper bounds with given other graph parameters such as the domination number ([7], [11]), the minimum degree ([3], [15]), the maximum degree ([4], [6]) and so on.

It is also interesting to investigate the oriented diameter (oriented radius) for the special classes of graphs, such as chordal graphs [8], complete k-partite ($k\geq 2$) graphs ([9], [13]), sparse graphs, such as planar graphs. In 2021, Wang et al. [18] proved that for any maximal outerplanar graph G of order n, $\overrightarrow{diam}(G)\leq\lceil\frac{n}{2}\rceil$, except four maximal outerplanar graphs of order at most eight. A simple connected plane graph G is a *triangulation* if G admits a planar embedding such that all its boundaries of faces (including outer face) are triangles. In this paper, we investigate the oriented diameter and radius for the triangulation. In 2012, Ali, Dankelmann and Mukwembi [1] proved the following theorem.

Theorem 1.1. *(Ali, Dankelmann and Mukwembi [1]) If* G *is a triangulation with* n *vertices, then* $rad(G)\leq\frac{n}{6}+\frac{10}{3}$*. Furthermore, if* G *is 4-connected,* $rad(G)\leq\frac{n}{8}+\frac{19}{4}$*; if* G *is 5-connected,* $rad(G)\leq\frac{n}{10}+\frac{29}{5}$*. All these bounds are sharp, apart from a constant.*

By Theorem 1.1, there is an integer c such that $diam(G)\leq\frac{n}{s}+c$ for any s-connected triangulation G, $s\in\{3,4,5\}$. And this bound is tight apart from a constant as following. For $s=3$, let $n=3k+1$. We construct G_3 by first giving k triangles $x_iy_iz_ix_i$ for $1\leq i\leq k$, and then adding a new vertex z connecting to x_k,y_k,z_k, adding edges x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}, x_iy_{i+1}, y_iz_{i+1} and z_ix_{i+1} for $1\leq i\leq k-1$, as shown in Figure 1. For $s=4$, let $n=4k+1$. We construct G_4 by first giving k 4-cycles $w_ix_iy_iz_iw_i$ for $1\leq i\leq k$, and then adding a new vertex z connecting to w_k,x_k,y_k,z_k, adding edges w_iw_{i+1}, x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}, w_ix_{i+1}, x_iy_{i+1} y_iz_{i+1}, z_iw_{i+1} and w_1y_1 for $1\leq i\leq k-1$, as shown in Figure 2. For $s=5$, let $n=5k+1$. We construct G_5 by first giving k 5-cycles $v_iw_ix_iy_iz_iv_i$ for $1\leq i\leq k$, and then adding a new vertex z connecting to v_k,w_k,x_k,y_k,z_k, adding edges v_1x_1, v_1y_1, v_iv_{i+1}, w_iw_{i+1}, x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}, v_iw_{i+1}, w_ix_{i+1}, x_iy_{i+1}, y_iz_{i+1} and z_iv_{i+1} for $1\leq i\leq k-1$, as shown in Figure 3.

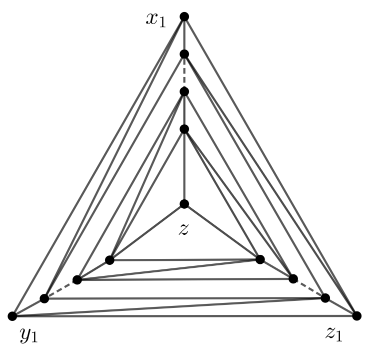

Figure 1: G_3

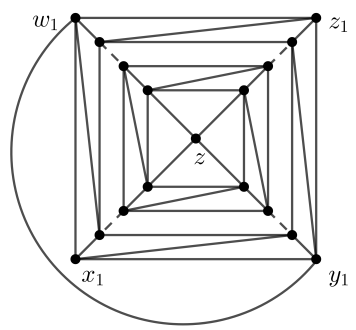

Figure 2: G_4

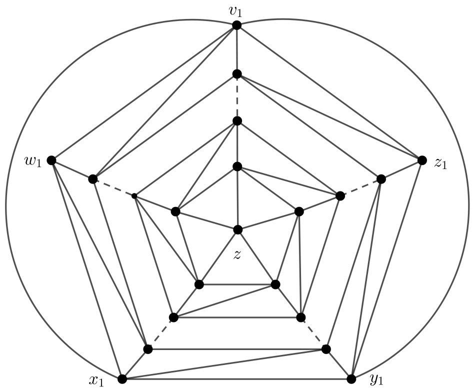

Figure 3: G_5

It is interesting to investigate the similar results of oriented diameter and oriented radius of triangulations. Recently, Mondal, Parthiban and Rajasingh [14] proved the following.

Theorem 1.2. *(Mondal, Parthiban and Rajasingh [14])* $\overrightarrow{diam}(G) \le \frac{n}{2} + O(\sqrt{n})$ *for all triangulations* G *with order* n.

Based on [18], Ge, Liu and Wang [10] proved that for 2-connected near triangulation G, $\overrightarrow{diam}(G) \le \lceil \frac{n}{2} \rceil$, except seven 2-connected near triangulations of order at most eight. As a corollary, they proved the following.

Theorem 1.3. *(Ge, Liu and Wang [10])* $\overrightarrow{diam}(G) \le \frac{n}{2}$ *for any triangulation* G *of order* n.

In this paper, we prove the following theorem.

Theorem 1.4. *Let* G *be a triangulation of order* n*, and* r *its radius. Then (1)* $\overrightarrow{rad}(G) \le 2r$*. (2)* $\overrightarrow{rad}(G) \le \frac{n}{3} + \frac{20}{3}$*; if* G *is 4-connected, then* $\overrightarrow{rad}(G) \le \frac{n}{4} + \frac{19}{2}$*; if* G *is 5-connected, then* $\overrightarrow{rad}(G) \le \frac{n}{5} + \frac{58}{5}$ *and* $\overrightarrow{diam}(G) \le \frac{2n}{5} + \frac{116}{5}$.

By Theorem 1.4, we improve the bound of 5-connected triangulations to $\frac{2n}{5} + \frac{116}{5}$. Furthermore, we obtain a tight upper bound of triangulations G under some conditions.

Theorem 1.5. *Let* G *be a triangulation of order* n*, and* r *its radius. Let* z *be its center. Suppose that for any* $v \in V(G)$ *with* $d(z,v) = d \ge 2$*, there exist two internal disjoint* (v,z)*-paths of length* d*, then (1)* $\overrightarrow{rad}(G) \le r+1$*. (2) for* s*-connected triangulation* G*, there is an integer* c *such that* $\overrightarrow{rad}(G) \le \frac{n}{2s} + c$ *and* $\overrightarrow{diam}(G) \le \frac{n}{s} + c$.

It is easy to check that the bounds in the second part of Theorem 1.5 are tight, apart from a constant by considering G_s, $s\in\{3,4,5\}$, as shown in Figures 1-3. The second part of Theorem 1.4 and Theorem 1.5 follow immediately by the first part, Theorem 1.1 and the fact $\overrightarrow{diam}(G)\leq 2\overrightarrow{rad}(G)$. So we only need to prove the first part of Theorem 1.4 and Theorem 1.5.

In the end of this section, we ask the following problem for further research.

Problem 1. For every s-connected triangulation G of order n, is there an integer c such that $\overrightarrow{rad}(G)\leq \frac{n}{2s}+c$ and $\overrightarrow{diam}(G)\leq \frac{n}{s}+c$?

In the rest of this paper, we first give a structure of triangulations in Section 2, and then prove Theorem 1.4 and Theorem 1.5 in Section 3 and Section 4, respectively.

2 A structure of triangulations

In this section, we will give a structure of the triangulation G, which help us to prove Theorem 1.4 and Theorem 1.5. Let z be the center of G. We will prove that all the vertices with distance i from z induce a disjoint union of connected outerplanar subgraphs. Firstly, we define some subgraphs of G. Suppose that we have embedded the triangulation G into the plane. Let C be a cycle of G, which separates the plane into two open sets, i.e. the interior set $Int(C)$ and exterior set $Ext(C)$. We say C is *extensible* if C has no chord in $Int(C)$ and $Int(C)$ is not a face of G. For convenience, we also say a single vertex is a extensible cycle. Since G is a triangulation, if C is extensible and $|C|\geq 3$, by the definition of "extensible", then every vertex in C has at least one neighbor in $Int(C)$, and there exists at least one vertex of G containing in $Int(C)$. Let $C:=v_0v_1\cdots v_{k-1}v_0$ be an extensible cycle. Note that for any vertex $v\in V(G)$, all the neighbors of v form a cycle. For any $v_i\in C$, let $\{u_{i_1},\ldots,u_{i_m}\}$ be the neighbors of v_i in $Int(C)$, $u_{i_1}\cdots u_{i_m}$ is a path in clockwise direction, and $v_{i-1}u_{i_1},u_{i_m}v_{i+1}\in E(G)$, where the above $i-1,i,i+1$ are written modulo k. Obviously, $u_{i_1}=u_{(i-1)_m}$ and $u_{i_m}=u_{(i+1)_1}$, Hence, all the paths $u_{i_1}\cdots u_{i_m}$ for all $i\in\{1,\ldots,k-1\}$ form a close walk W_c.

Proposition 2.1. *Let C and W_c be the above extensible cycle, and close walk, respectively. Then we have (1) O_c induced by the vertices of W_c is a connected outerplane subgraph. (2) for any $v\in O_c$, v has at least one neighbor in C. (3) For any $v\in C$, $N(v)\cap Int(C)$ßO_c.*

Proof. Since W_c is a close walk, O_c is connected. By the fact that each vertex in W_c has a neighbor in C, W_cß$Int(C)$, and G is a triangulation, O_c is a connected outerplane graph.

Since each vertex in W_c has at least one neighbor in C, and all the neighbors of C in $Int(C)$ belong to W_c, it is easy to see (2) and (3) hold for O_c. □

By Proposition 2.1 (1), if there exists an extensible cycle C in a triangulation G, we can find a corresponding outerplane subgraph O_c in $Int(C)$. Now we give an iteration to find some extensible cycles and their corresponding outerplanes in the triangulation G. Denote the center of G as z. By the definition of the extensible cycle, z is also an extensible cycle. We say that z lies on the level 0. Obviously, all the neighbors of z form a cycle C_z. We embed G in the plane such that all the vertices are inside C_z, except z and vertices in C_z. Note that $G[V(C_z)]$ is also a connected outerplane subgraph of G, then $G[V(C_z)]$ is the corresponding outerplane subgraph of the extensible cycle z. We say that all the vertices of $G[V(C_z)]$ and $G[V(C_z)]$ lie on level 1. Obviously, every vertex v with $d(v,z)=1$ lies in the outerplane subgraph $G[V(C_z)]$ on level 1.

Suppose that every vertex v with $d(v,z)=i\geq 1$ lies in some outerplane subgraph O on level i. For a connected outerplane subgraph O on level i, if there exists a cycle CßO that has no chord and is not a triangle boundary of G, then C is extensible. We say that all these extensible cycles C lie on level i. By Proposition 2.1 (1), for every extensible cycle C in level i, we can find a corresponding outerplane subgraph O_c of G. We say that all the vertices of all O_c and all O_c lie on level $i+1$. Then $d(z,u)=i+1$ for any $u\in O_c$. On the other hand, suppose $d(u,z)=i+1$ for some $u\in V(G)$. Note that all the vertices with distance i from z form a cut vertex set separating z and u, and all the vertices with distance i from z induce a disjoint union of connected outerplane subgraphs on level i. Then there must exist an extensible cycle C in some connected outerplane subgraph on level i separating z and u. Since $d(u,z)=i+1$, $u\in Int(C)$ and u is a neighbor of some vertex $v\in C$. By Proposition 2.1, $u\in O_c$, where O_c is the corresponding outerplane subgraph of C on level $i+1$. By the above arguments, every vertex u with $d(u,z)=i+1$ lies in some outerplane subgraph O on level $i+1$.

Such iteration must end since $rad(G)$ is limit. We can see that in level i, there are several vertex-disjoint connected outerplanar subgraphs containing all vertices with $d(z,v)=i$. And the largest level is $r=rad(G)$.

3 Proof of Theorem 1.4

We first introduce a useful lemma. Let U and V be two disjoint vertex sets of $V(G)$ such that for any $v\in V$, $N(v)\cap U\neq\emptyset$ and $G[V]$ contains no connected component of single vertex. Let F_V be a spanning forest of $G[V]$ and (V_1,V_2) be the bipartition of $V(F_V)$. We say an orientation as *U-V orientation* when we orient all the edges of

$[U,V_1]$ from U to V_1, $[V_1,V_2]$ from V_1 to V_2, and $[U,V_2]$ from V_2 to U.

Lemma 3.1. *(Kwok et al. [12]) Let U,V be the vertex sets as above, then under the U-V orientation, for any $v\in V$, $\theta(v,U)\leq 2$.*

Now we prove $\overrightarrow{rad}(G)\leq 2r$. By Proposition 2.1 , let C be an extensible cycle on level $i\geq 0$ and O_c its corresponding connected outerplane subgraph in level $i+1$, where $i<r$. If O_c has at least two vertices, by Proposition 2.1 (2), we give a U-V orientation to edges between such C and O_c, and edges in O_c, where $U=V(C)$ and $V=V(O_c)$. If O_c has just one vertex v, then $N(v)=V(C)$, and we orient edges between v and C in two ways. For other edges, we give arbitrary orientations. For any v with $d(v,z)=i\geq 1$, by the arguments in Section 2, v lies on some outerplane subgraph O_c on level i, and O_c is corresponding to an extensible cycle C on level $i-1$. By the above orientations and Lemma 3.1, we get $\theta(v,C)\leq 2$. Hence, for any vertex v with $d(v,z)=i\geq 1$, $\theta(v,z)\leq 2i\leq 2r$, and $\overrightarrow{rad}(G)\leq 2r$. □

4 Proof of Theorem 1.5

Since for any $v\in V(G)$ with $d(z,v)=i\geq 2$, there exist two internal disjoint (v,z)-paths of length i, we claim that any vertex $v\in O_c$ in level $i\geq 2$ has two neighbors in its corresponding extensible cycle C. Otherwise, by the constructions of C and O_c, all the neighbors of v that have distance $i-1$ from z must lie in its corresponding extensible cycle C. Hence, there exists at most one neighbor of v that has distance $i-1$ from z, a contradiction to the condition of Theorem 1.5. By this claim, we orient edges between any $v\in O_c$ in level $i\geq 2$ and its neighbors in its corresponding extensible cycle C in two ways. For $v\in O_c$ in level 1, we give a U-V orientation to edges between such O_c and $\{z\}$, and edges in O_c, where $U=\{z\}$ and $V=V(O_c)$. For other edges, we give arbitrary orientations. Under this orientations, for any vertex v in level $i\geq 1$, $\theta(v,z)\leq i+1\leq r+1$, and $\overrightarrow{rad}(G)\leq r+1$. □

Remark. We believe that after giving a detail analysis of the structure given in Section 2 and a complicated orientations of triangulation, one can improve the upper bounds of Theorem 1.4. But we think that the tight bound may need a new structure analysis and a more precise orientations. Furthermore, it is also interesting to investigate the oriented diameter with given diameter for maximal outerplanar graphs, 2-connected near triangulations and triangulations.

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